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The compliance approach for analyzing bimaterial interface cracks

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A numerical method is presented for analyzing the mixed mode interface crack between two dissimilar isotropic materials. A simple and efficient solution procedure is developed based on the finite element method and the compliance approach in conjunction with the fundamental relations in fracture mechanics. The procedure makes it possible to separate the Mode I and Mode II stress intensity factors K_{I} and K_{II} respectively for an interfacial crack in bi-material media under different loading conditions. The strain energy release rate is first computed, then using the compliance method and the known auxiliary solutions, the values for K_{I} and K_{II} are evaluated. The procedure is investigated for different crack extensions. The formulations used for computing the strain energy release rate and the stress intensity factors are presented. The method converges to accurate solutions for small crack extensions. A numerical example is presented to demonstrate the accuracy of the proposed model.

Keywords: bi-material interface crack, strain energy release rate, stress intensity factors, finite element analysis, compliance approach

1. Introduction

The study of an interface crack between two dissimilar isotropic elastic materials has gained considerable attention in the field of fracture mechanics. The problem represents an idealization of two different materials joined together with a crack developing along the interface due to faulty joining techniques. This problem was studied earlier by Williams [14]. Analytical solutions to evaluate the stress intensity factors and to find the values of displacements and stresses near the crack tip were formulated by Erdogan [3], Rice and Sih [11, 12,] and England [2]. The solutions show that both the opening mode stress intensity factor (K_{I}) and the sliding mode stress intensity factor (K_{II}) are present in a single mode loading which is different than the homogeneous case when only one stress intensity factor was involved. These solutions, however, beside being asymptotic were also for specific geometries and loadings. Thus, it was necessary to develop numerical procedures to obtain complete solutions for more general configurations of loading and geometries. The J-integral method was used by several researchers [6–8] in conjunction with the finite element method to evaluate the stress intensity factors in a continuum containing an interface crack. Hong and Stern [9] formulated a path independent integral for calculating the stress intensity factors in a continuum enclosing an interface crack. Smelser [13] obtained the stress intensity factors from crack flanges displacement data using finite elements. Hamoush et al. used the crack closure integral approach [4] and a stiffness derivative technique [7] for the evaluation of K_{I} and K_{II}.

In this paper, an alternate method is developed to separate the K_{I} and K_{II} stress intensity factors for the composite sheet shown in Fig. 1. The method is based on the compliance approach and the finite element method to determine the total energy release rate. By imposing known auxiliary solutions, the stress intensity factors can be successfully computed with a high degree of accuracy.

A similar approach was used by the author [5] to compute the stress intensity factors for single mode loadings and also to separate K_I, K_II for a homogeneous sheet subjected to a mix-mode loading.

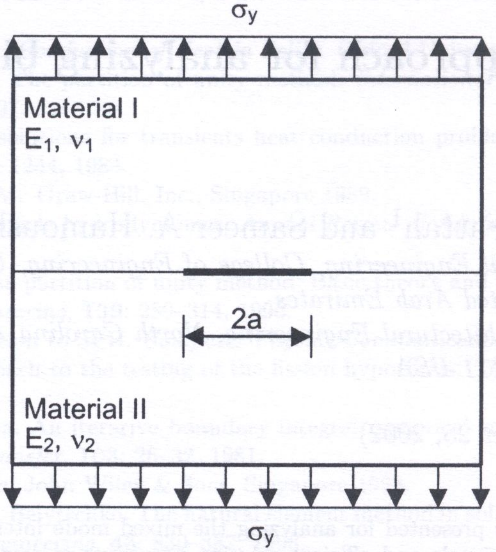

Fig. 1. The bi-material interface crack problem

2. Formulation

The strain energy release rate can be evaluated as proposed by Irwin [10]. When a crack of length $2a$ extends to a new length $2a + 2\Delta a$, the strain energy release rate G is defined as follows:

$$G = \lim_{\Delta a \to 0} \frac{\Delta U}{\Delta A}\,, \tag{1}$$

where ΔU is the change in potential energy and ΔA is the newly formed area for the extended part of the crack. The change in potential energy is calculated in terms of the applied loads and the displacements by the following:

$$\Delta U = \frac{1}{2}\,\Sigma(P_i)\,(\Delta u_i), \tag{2}$$

Δu_i are the changes in displacements in the direction of the applied loads P_i as the crack length increases from $2a$ to $2a+2\Delta a$. Substituting for ΔU and the new formed crack area, Eq. (1) becomes:

$$G = \frac{1}{8\Delta ab}\,\Sigma(P_i)\,(\Delta u_i)\,. \tag{3}$$

In Eq. (3), b is the thickness of the composite sheet enclosing the crack. Thus, with the knowledge of the crack extension Δa, the applied loads P_i and the numerical values for Δu_i, the strain energy release rate G can be computed.

The relationship between the computed G and the stress intensity factors K_I and K_II can be obtained as proposed by Irwin [10]. The analytical expression for the strain energy release rate G can be expressed as follows [4]:

$$G = \alpha\left[K_\mathrm{I}^2 + K_\mathrm{II}^2\right], \tag{4}$$

where α is a constant that depends on the properties of the two materials that form the composite sheet and is evaluated as follows:

$$\alpha = \frac{1}{16}\left[\frac{(k_1+1)}{\mu_1} + \frac{(k_2+1)}{\mu_2}\right]. \tag{5}$$

The constants k_1 and k_2 are given by Eq. (A4) in the Appendix and μ_1, μ_2 are the shear moduli for the two materials forming the composite sheet.

Thus, for the loading Case 1 in Fig. 2, which also represents the original problem under study, Eqs. (3) and (4) become:

$$G^{(1)} = \frac{1}{8\Delta ab}\,\Sigma(P_i)^{(1)}\,(\Delta u_i)^{(1)}, \tag{6}$$

$$G^{(1)} = \alpha\left[K_{\mathrm{I}}^{(1)^2} + K_{\mathrm{II}}^{(1)^2}\right], \tag{7}$$

where $(\Delta u_i)^{(1)}$ are the changes in the displacements in the direction of the applied loads $(P_i)^{(1)}$. Equation (6) makes it possible to compute numerically the value of $G^{(1)}$ and from Eq. (7) the sum of the squares of the stress intensity factors can be evaluated. The two equations do not give enough information to separate K_{I} and K_{II}. The separation, however, is possible by superposition. In Fig. 2, when the independent state of equilibrium (Case 2) is superimposed over the original problem (Case 1), the result will be the combined state (Case 1, 2).

For Case 2, Eqs. (3) and (4) are written:

$$G^{(2)} = \frac{1}{8\Delta ab}\,\Sigma(P_i)^{(2)}\,(\Delta u_i)^{(2)}, \tag{8}$$

$$G^{(2)} = \alpha\left[K_{\mathrm{I}}^{(2)^2} + K_{\mathrm{II}}^{(2)^2}\right], \tag{9}$$

where $(\Delta u_i)^{(2)}$ are the changes in the displacements in the direction of the applied loads $(P_i)^{(2)}$ for Case (2). Similarly, for the combined Case (1, 2), the equations are written:

$$G^{(1,2)} = \frac{1}{8\Delta ab}\,\Sigma(P_i)^{(1,2)}\,(\Delta u_i)^{(1,2)}, \tag{10}$$

$$G^{(1,2)} = \alpha\left[K_{\mathrm{I}}^{(1,2)^2} + K_{\mathrm{II}}^{(1,2)^2}\right] \tag{11}$$

also, $(\Delta u_i)^{(1,2)}$ are the changes in the displacements in the direction of the applied loads $(P_i)^{(1,2)}$ for Case (1, 2).

This last equation can be written:

$$\begin{aligned} G^{(1,2)} &= \alpha\left[\left(K_{\mathrm{I}}^{(1)} + K_{\mathrm{I}}^{(2)}\right)^2 + \left(K_{\mathrm{II}}^{(1)} + K_{\mathrm{II}}^{2}\right)^2\right] \\ &= \alpha\left[K_{\mathrm{I}}^{(1)^2} + K_{\mathrm{II}}^{(1)^2} + 2K_{\mathrm{I}}^{(1)}K_{\mathrm{I}}^{(2)} + 2K_{\mathrm{II}}^{(1)}K_{\mathrm{II}}^{(1)} + K_{\mathrm{I}}^{(2)^2} + K_{\mathrm{II}}^{(2)^2}\right]. \end{aligned} \tag{12}$$

In order to separate the Mode I stress intensity factor, we need to employ a known auxiliary solution. According to Rice and Sih [12], the Stress Intensity Factors for a semi-infinite crack problem with isolated forces P and Q located close to the crack tip are:

$$K_{\mathrm{I}} = \frac{1}{\pi}\sqrt{\frac{2}{a}}\left[P\cos(\varepsilon\log a) + Q\sin(\varepsilon\log a)\right], \tag{13}$$

$$K_{\mathrm{II}} = \frac{1}{\pi}\sqrt{\frac{2}{a}}\left[Q\cos(\varepsilon\log a) - P\sin(\varepsilon\log a)\right], \tag{14}$$

where a is half the crack length and ε is the bimaterial constant given in Eq. (A3).

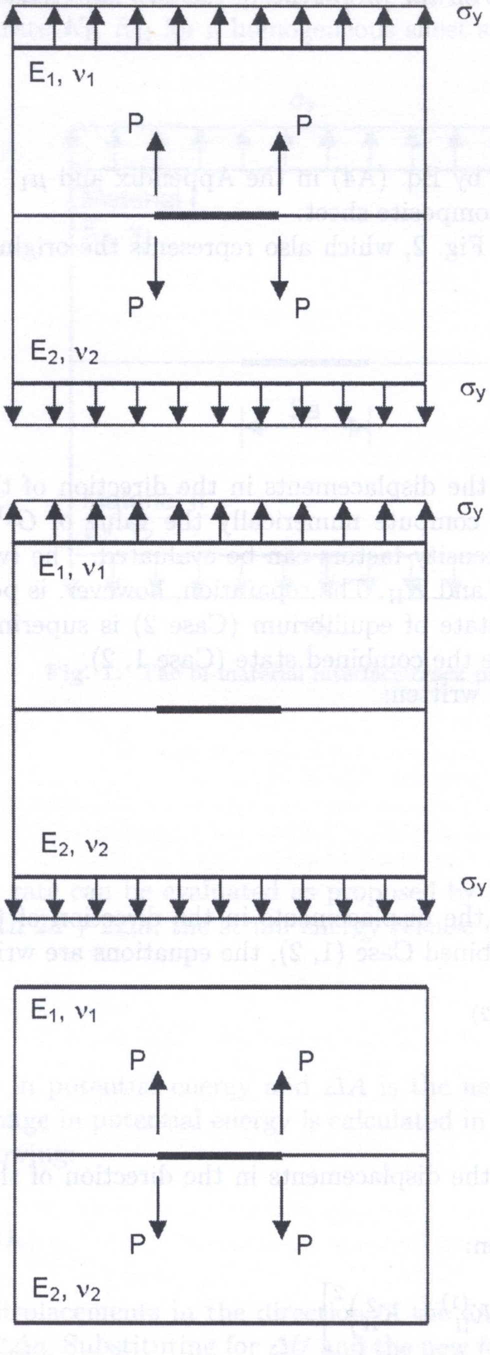

Fig. 2. Separation of the mode I stress intensity factor

By assigning $K_{\mathrm{I}} = 1.0$ and $K_{\mathrm{II}} = 0$, the values of the forces P and Q can be computed by solving Eqs. (13) and (14). The value of the resulting sliding force Q is too small compared to the opening force P and therefore it may be neglected.

For Case 2, by assigning the values of $K_{\mathrm{I}}^{(2)} = 1$ and $K_{\mathrm{II}}^{(2)} = 0$, Eq. (12) becomes:

$$G^{(1,2)} = \alpha \left[K_{\mathrm{I}}^{(1)^2} + K_{\mathrm{II}}^{(1)^2} + 2K_{\mathrm{I}}^{(1)} + 1 \right] \tag{15}$$

which can be simplified into the following form

$$G^{(1,2)} = G^{(1)} + 2\alpha K_{\mathrm{I}}^{(1)} + \alpha. \tag{16}$$

Rearranging Eq. (16), the value of the stress intensity factor ($K_{\rm I} = K_{\rm I}^{(1)}$) can be computed as follows:

$$K_{\rm I}^{(1)} = \frac{1}{2\alpha}\Big[G^{(1,2)} - G^{(1)} - \alpha\Big]. \tag{17}$$

The numerical values for the strain energy release rate are computed from the applied nodal loads and the resulting nodal displacements. The values of $G^{(1,2)}$ and $G^{(1)}$ are computed numerically for Case (1, 2) and Case (1) respectively. The value of the bimaterial constant α is given by Eq. (5). After the separation of $K_{\rm I}$, the other stress intensity factor $K_{\rm II}$ can be computed from Eq. (7).

Note that from Eq. (10), the relationship between the Strain Energy Release Rates for all three cases can be expressed as follows:

$$\begin{aligned}
G^{(1,2)} &= \frac{1}{8\Delta ab}\Sigma\left(P_i^{(1)} + P_i^{(2)}\right)\left(\Delta u_i^{(1)} + \Delta u_i^{(2)}\right) \\
&= \frac{1}{8\Delta ab}\left(\Sigma P_i^{(1)}\Delta u_i^{(1)} + \Sigma P_i^{(2)}\Delta u_i^{(2)}\right) + \frac{1}{8\Delta ab}\left(\Sigma P_i^{(1)}\Delta u_i^{(2)} + \Sigma P_i^{(2)}\Delta u_i^{(1)}\right) \\
&= G^{(1)} + G^{(2)} + \frac{1}{8\Delta ab}\left(\Sigma P_i^{(1)}\Delta u_i^{(2)} + \Sigma P_i^{(2)}\Delta u_i^{(1)}\right).
\end{aligned} \tag{18}$$

It can be noted that the same approach can be deployed to find the $K_{\rm II}$ factor by assigning $K_{\rm II} = 1.0$ and $K_{\rm I} = 0$.

This solution technique was used within a finite element model as shown the following sections.

3. Numerical investigation

The procedure described in the previous section has been incorporated into a conventional finite element code. The accuracy of the current approach is demonstrated by solving the problem shown in Fig. 1 that has a known closed-form asymptotic solution available in the literature [11]. The problem shown represents an interface crack between two semi-infinite plates of two dissimilar materials. The composite sheet is subjected to a uniform tensile stress of unit value applied at infinity as shown in Fig. 2. The finite element discretization of the plate is shown in Fig. 3. The ratio $2w/2a$ is taken to be 10. Previous research [5] has shown that this ratio is suitable for modeling this type of problems. Four node elements are used in the analysis with an aspect ratio below 5. The finite element mesh was optimized based on previous studies [4, 6, 7] and it is noted that the finite element solution converges when a total of 424 elements and 482 nodes are used in the analysis. A much finer mesh

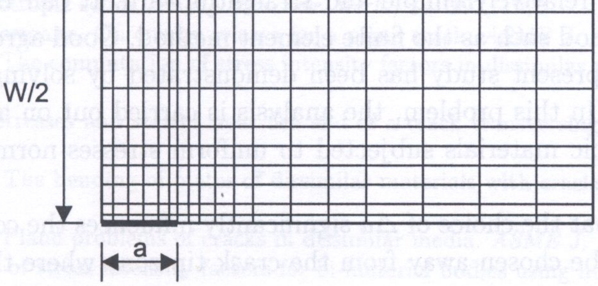

Fig. 3. One fourth of the finite element mesh

is used at the crack tip zones. The exact analytical solution for the problem to determine the stress intensity factors was developed by Rice and Sih [11] and is given by Eqs. (A1) and (A2) in the Appendix.

The ratio of the numerical stress intensity factor ($K_{\mathrm{I\ num}}$) obtained by the given model and the exact stress intensity factor ($K_{\mathrm{I\ exact}}$) given by the analytical solution given in Appendix is shown in Fig. 4. The figure gives the plot of the K_{I} ratios against the crack extensions $\Delta a/a$ for different values of E_1/E_2. Note that for $E_1/E_2 = 10$, the solution converges to the exact value when the crack extension is equal to about 0.012 a, where a is half the crack length. For higher values of E_1/E_2, the convergence occurs at bigger crack extensions. For all the computed values for K_{I}, the maximum error was less than 5%. After separating the K_{I} factor, the K_{II} factor can be computed from Eq. (7).

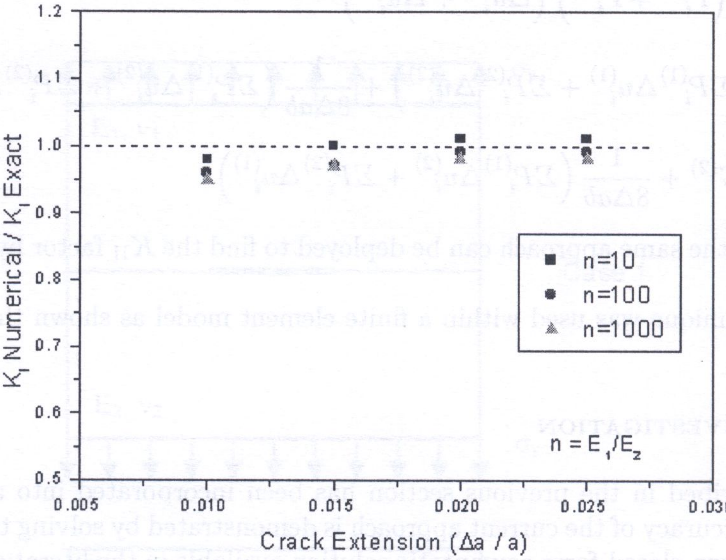

Fig. 4. The normalized stress intensity factor K_{I} for different E_1/E_2 ratios

4. Conclusion

A method of analysis based on the compliance approach and the fundamental relationships in fracture mechanics has been proposed for analyzing the bi-material interface crack problem. The analysis is formulated by imposing two independent states of equilibrium in conjunction with the compliance method.

The present model is relatively simple and straightforward. It can be conveniently conducted using any numerical method such as the finite element method. Good agreement between analytical exact solutions and the present study has been demonstrated by solving the classical bi-material interface crack problem. In this problem, the analysis is carried out on a composite sheet made of two homogeneous isotropic materials subjected to uniform stresses normal to the direction of the interface crack.

It can be concluded that the choice of Δa significantly influences the convergence of the solution. The value of Δa should be chosen away from the crack tip zone where the stress oscillations associated with interface cracks occur. It appears that the ratio of $\Delta a/a$ should be selected within the range of 0.012 to 0.02 in order for the method to converge to the exact solution within acceptable accuracy.

APPENDIX

For a bi-material interface crack in a composite sheet subjected to a uniform tensile stress σ_y normal to the crack direction, the stress intensity factors are expressed as follows:

$$K_{\mathrm{I}} = \frac{1}{\cosh \pi\varepsilon}\{\sigma_y[\cos(\varepsilon \log 2a) + 2\varepsilon \sin(\varepsilon \log 2a)]\}\sqrt{\pi a}, \tag{A1}$$

$$K_{\mathrm{II}} = \frac{1}{\cosh \pi\varepsilon}\{\sigma_y[\sin(\varepsilon \log 2a) + 2\varepsilon \cos(\varepsilon \log 2a)]\}\sqrt{\pi a}. \tag{A2}$$

The bimaterial constant ε in Eqs. (A1) and (A2) is computed as follows:

$$\varepsilon = \frac{1}{2\pi} \ln \left[\frac{\dfrac{k_1}{\mu_1} + \dfrac{1}{\mu_2}}{\dfrac{k_2}{\mu_2} + \dfrac{1}{\mu_1}} \right], \tag{A3}$$

where μ_1 and μ_2 are the shear moduli of the two materials forming the composite sheet enclosing the bi-material interface crack.

Rice and Sih [12] defined the stress intensity factors $K = K_{\mathrm{I}} - iK_{\mathrm{II}}$ as

$$K_{\mathrm{I}} - iK_{\mathrm{II}} = 2(2\pi)^{1/2} e^{\pi\varepsilon} \lim_{z \to a} (z-a)^{1/2} (z-a)^{i\varepsilon} \Phi_1(z). \tag{A4}$$

The constant k is evaluated as follows:

$$\begin{aligned} k_i &= (3 - 4\nu_i) && \text{for plane strain,} \\ k_i &= (3 - 4\nu_i)/(1 + \nu_i) && \text{for plane stress,} \end{aligned} \tag{A5}$$

where ν_i is Poisson's ratio.

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