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Modal analysis of wave motion in inhomogeneous waveguides which are modelled by FEM

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This paper presents a simple computing procedure for the analysis of the wave motion in infinite layered waveguides via the analysis of the propagating wave modes. Waveguides may have irregular inclusions, which yields complicated reflections of waves, and an analytical solution is practically not feasible. The section of the waveguide, where we want to analyze the displacements and stress waves, is modelled by finite elements using standard programs for FEM. The external problem is solved as an internal one, while the radiation conditions are satisfied exactly. The procedure only some simple mathematical manipulations and is performed in the frequency domain. It yields exact results and a clear insight into the propagating wave modes. The results of the first presented numerical example are compared to the exact ones, while in the second example the foundation represents an irregularity in the waveguide composed of two layers

1. Introduction

In infinite waveguides the waves propagate only in the direction away from the source of excitation. When there are inhomogeneities and/or variation of the boundary conditions present in a section of the waveguide, the solution of the wave equation, which requires the fulfilment of the radiation conditions, is feasible only numerically [1, 2]. A variety of methods are available for this purpose. If we glance over them, we could describe them roughly as follows. Boundary element methods satisfy radiation conditions, but are not simple to apply for complex cases, see for example [3, 4]. By finite-element methods the radiation conditions are satisfied only by using special elements on the fictive boundary, or by evaluating certain computational phases analytically, see for instance [4–6]. Operator methods require the implementation of special operators on the fictive boundary, for instance [7–12]. A special approach using an original definition of Sommerfeld conditions is presented by [13]. We can classify the available methods superficially and briefly as being either a great deal sophisticated and in certain cases exact, or simple and considerably approximate. Unfortunately, a simple and exact method, which in addition yields the results that are easily understood and used by engineers, is not available.

We are demonstrating a computing procedure, performed in the frequency domain, which yields exact results for the wave motion in layered waveguides, while the case of homogeneous wave guide is presented in [14]. The procedure is performed in conjunction with the finite-element-method. Therefore, the modelling of a complex waveguide can be carried out by the aid of the standard FEM computer routines. The basic underlying theory is well known, and practically requires only the knowledge of the standard graduate study in engineering.

2. Basics and the outline of the computing procedure

When we are analysing wave motion in an infinite waveguide by the aid of finite elements, we can model only a finite section. The lateral boundaries of such a section are the so-called fictive boundaries, Fig. 1, and the displacements of waves are governed by Eq. (1).

$$(\mathbf{K}-\omega^2\mathbf{M})\left\{\begin{array}{c}\mathbf{u}_0\\ \mathbf{u}\\ \mathbf{u}_r\end{array}\right\}=\left\{\begin{array}{c}\mathbf{p}_0\\ \mathbf{0}\\ \mathbf{p}_r\end{array}\right\}. \tag{1}$$

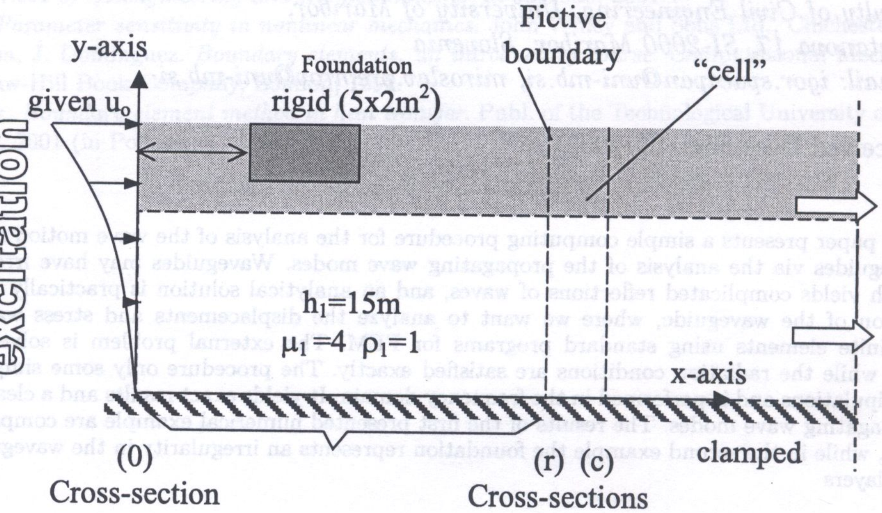

Fig. 1. Disposition of the layers, the foundation, and cell

In this formulation of the wave motion we used the co-called dynamic stiffness matrix of the section to get a system of linear equations. **K** and **M** are the stiffness and the mass matrices, respectively, while $\mathbf{u}_0$, $\mathbf{u}$ and $\mathbf{u}_r$ are known nodal displacements of the excitation, the unknown displacements of internal nodes and the radiating nodal displacements on the fictive boundary, respectively. The nodal forces **P** are marked analogous to displacements. Equation (1) is easily solved on displacements **u** by some simple mathematical manipulations, providing we know the radiating displacements. It is worth noting that in such a case we have a correctly posed problem, where all needed boundary conditions are known. Clearly, in the computing approach we have to compute the radiating displacements as they are not known in advance. This is done as follows. The radiating displacements and stresses are first presented by the superposition of only the radiating wave modes, Eq. (2). These wave modes propagate only outwards of the section, which is suggested by the sign +, and they are denoted by $\Psi^+_{u,k}$ and $\Psi^+_{P,k}$ for the displacements and nodal forces on the fictive boundary, respectively. It is worth noting that their typical characteristic is that each of them propagates with a constant velocity, which formally applies also to standing waves, occurring below the cut-off frequency, as well. They are computed first. Their amplitudes a_k, which may be complex, are of cause also not known in advance. Substituting Eq. (2) into Eq. (1) we get a modified equation where amplitudes represent some of the unknowns. These amplitudes are also termed modal or weighting factors.

$$\left\{\begin{array}{c}\mathbf{u}_r\\ \mathbf{P}_r\end{array}\right\}=\left[\left\{\begin{array}{c}\Psi^+_{u,k}\\ \Psi^+_{P,k}\end{array}\right\}\right]\{a_k\}. \tag{2}$$

Wave modes are computed from the FE discretized "cell", which is adjacent to the fictive boundary, Fig. 1, by using the advantage of two facts. The first is that wave modes preserve their shape during propagation [3]. This yields Eq. (3), where indexes f and c stand for the values on the first cross-section of the cell, the fictive boundary, and for the values on the second lateral one, respectively. λ_k are distinct factors of proportionality, and N is the number of nodal points on the cross-section. There are altogether $2N$ wave modes and eigenvalues because the radiating is in general in both directions.

$$\left\{ \begin{array}{c} \Psi_{u,k} \\ \Psi_{P,k} \end{array} \right\}_c = \lambda_k \left\{ \begin{array}{c} \Psi_{u,k} \\ \Psi_{P,k} \end{array} \right\}_f, \qquad k = 1, 2 \ldots 2N. \tag{3}$$

The second fact follows from the uniqueness of the wave equation solution: the displacements and nodal forces on both lateral surfaces of the cell are interrelated by the transfer matrix $\mathbf{T}$, Eq. (4).

$$\left\{ \begin{array}{c} \Psi_{u,k} \\ \Psi_{P,k} \end{array} \right\}_c = \mathbf{T} \left\{ \begin{array}{c} \Psi_{u,k} \\ \Psi_{P,k} \end{array} \right\}_c, \qquad k = 1, 2 \ldots 2N \tag{4}$$

This transfer matrix $\mathbf{T}$ is computed either from the dynamic stiffness, or from the dynamic flexibility matrix of the cell. Both facts together yield the equation of the eigenvalue problem, Eq. (5), which has distinct solutions representing the waves modes, and the belonging eigenvalues $\lambda_k, k = 1, 2 \ldots 2N$,

$$(\mathbf{T} - \lambda \mathbf{I}) \left\{ \begin{array}{c} \mathbf{u} \\ \mathbf{P} \end{array} \right\} = \mathbf{0}. \tag{5}$$

However, the solution yields all the wave modes, but those propagating in the positive direction, radiating modes, are easily distinguished from the other ones. They have either the negative value of the imaginary part of the belonging eigenvalue, or they have diminishing amplitudes represented by the real eigenvalues, which are less than a unit.

The practical course of the computing approach of the complete analysis may be performed stepwise and in several ways. However, the computation of the wave modes is always a detached computing step and is performed first. Substituting Eq. (2) into the Eq. (1) the unknowns are the internal displacements $\mathbf{u}$ and modal factors $\mathbf{a}$ of radiation displacements on the fictive boundary. In addition, the unknowns are also either $\mathbf{u}_0$, or $\mathbf{P}_0$, which depends on whether the excitation is given by displacements or by nodal forces. Such a system of equations may be solved directly to yield entire displacements filed, or indirectly by using transfer function of the segment, which yields radiating displacements. Specifically, in the later case we first get the modal factors, and then by Eq. (2) the radiating displacements and forces on the fictive boundary. Eventually, having all boundary conditions of the segment, the displacement field is solved in a standard way.

3. Numerical examples

We are considering, for the sake of simpler analytic verification of the numerical results, only 2-D case for antiplane shear wave motion, yet the approach is valid for the case of general wave motion in parallel waveguides. For two-dimensional case the displacements in a layered soil are governed in the frequency domain by the wave equation, Eq. (6), of course with distinct wave numbers k for different layers.

$$\nabla^2 u + k^2 u = 0. \tag{6}$$

The analyzed case is symbolically presented in Fig. 1. The soil consists of two layers over rigid subsoil. The characteristic data are in the figure. The contact between layers and the sub-soil is considered as firm. Excitation is given by displacements, distributed according to quadratic parabola, with the amplitude a unit, as suggested in Fig. 1. The frequency of excitation is a unit. The fictive boundary is 25 from the excitation cross-section, which makes the considered section 25 meters long. The width of the cell is 0.33333 meters and is chosen arbitrarily. Finite elements are simple linear ones. The mesh has 40 × 50 nodal points.

First, we consider the case where there is no foundation and excavation in order to compare the analytical results to the numerical ones. The analytical solution for the displacement field is in the form:

$$u(x,y) = \sum_{n=1}^{\infty} a_n \Psi_n(y).e^{(-z_n.x)}. \tag{7}$$

$\Psi_n(y)$ and a_n are the wave modes and their weighting factors, respectively. These weighting factors are represent the complex amplitudes of the wave modes. The z_n are the roots of the equation:

$$\begin{aligned} f(z) = \mu_2\sqrt{k_2^2+z^2}\,\sin(h_1\sqrt{k_1^2+z^2}).\sin(h_2\sqrt{k_2^2+z^2}) \\ +\mu_1\sqrt{k_1^2+z^2}\,\cos(h_1\sqrt{k_1^2+z^2}).\cos(h_2\sqrt{k_2^2+z^2}) = 0\,, \end{aligned} \tag{8}$$

k_1 and k_2 are the wave numbers of the lower and the top layer, respectively. This equation originates from the continuity conditions on the contact surface between the two layers. The relation between the roots z_n and the eigenvalues λ_n in Eq. (3) is

$$\lambda_n = e^{(-z_n \Delta L)}, \tag{9}$$

where ΔL is the width of the cell. Some of the results for the case considered are presented in Fig. 2 demonstrating very good resemblance between the first five analytical and numerical values. All eigenvalues that belong to radiating standing wave modes, the diminishing ones, are situated on the real axis between zero and one. The propagating outgoing wave modes have negative imaginary parts. Eigenvalues are marked by numbers, separately for the standing and the propagating wave modes. These numbers occur also in Fig. 3 in order to see to which eigenvalue the wave-mode belongs. Other values, which are real, belong to standing waves, which decay very rapidly. Closer consideration shows, that the accuracy of these waves has utterly negligible role on the results of the radiating conditions on the fictitious boundary.

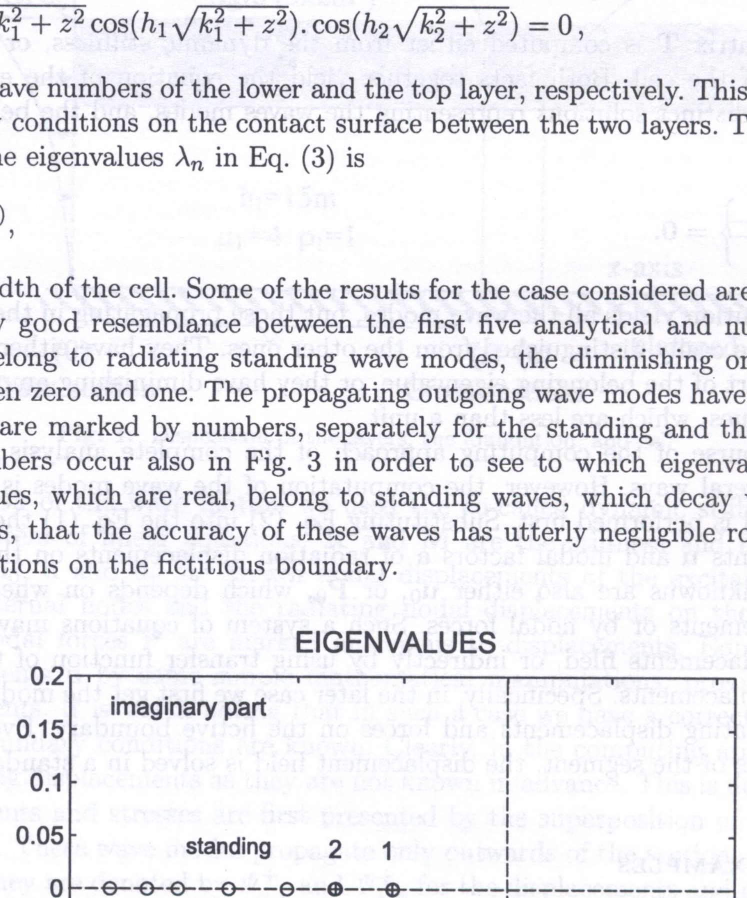

Fig. 2. Eigenvalues belonging to radiating wave modes

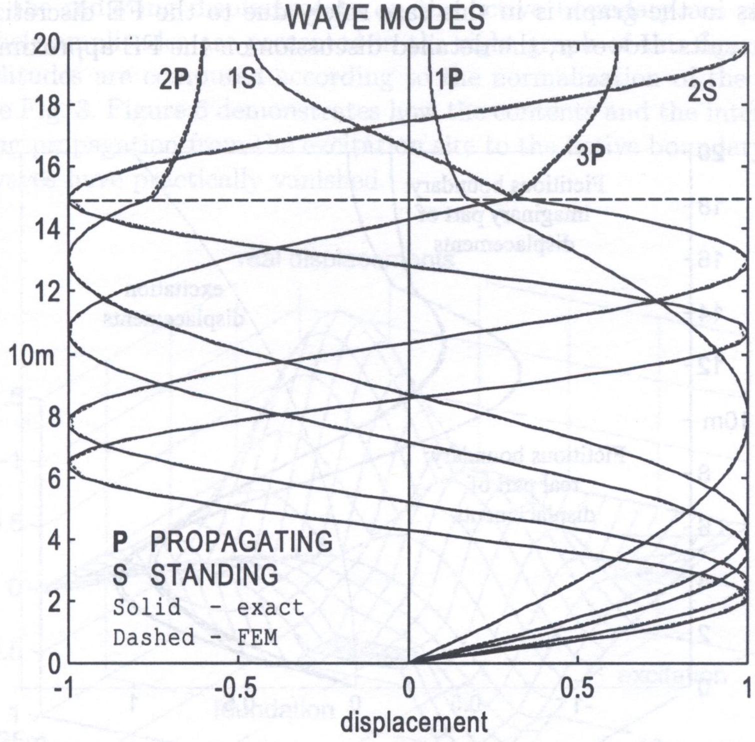

Fig. 3. The three propagating and the first two standing wave modes

The analytical formulas for the wave modes, regarding the notations and the data according to Fig. 1, are presented by Eq. (10) and (11).

$$\Psi_n(y) = 2i.\sin(y\sqrt{k_1^2 + z_n^2}), \qquad 0 \le y \le h_1, \tag{10}$$

$$\Psi_n(y) = 2i\,\frac{\sin(h_1\sqrt{k_1^2 + z_n^2})}{\cos(h_2\sqrt{k_2^2 + z_n^2})}\cos\left[(y - h_1 - h_2)\sqrt{k_2^2 + z_n^2}\right], \qquad h_1 \le y \le h_1 + h_2. \tag{11}$$

It is important to note that wave modes can be normalized in various ways, that is, multiplied by different constants. Consequently, the amplitudes in Eq. (7) depend on the normalization used and, of course, on the given excitation. First five displacement wave modes computed analytically and numerically are presented in Fig. 3. They are normalized to yield extreme displacement a real unit, which serves for clearer graphical presentation.

These wave modes are used to compute their amplitudes a_n in order to match the given excitation. In our case we solved the Eq. (12).

$$u_0(x = 0, y) = \frac{1}{h_1 + h_2} y^2 = \sum_{n=1}^{\infty} a_n \Psi_n(y), \qquad 0 \le y \le h_1 + h_2. \tag{12}$$

Clearly, only finite number of wave modes can be considered. The number of wave modes depend on the number of mesh nodes in the discretised model. There are 40 wave modes in our case. On the other hand, we have computed only twelve wave modes for the analytical results, but it already yields very high accuracy of analytically computed radiating displacements on the fictitious boundary – the estimated accuracy is certainly better then 0.1 percent, as higher modes vanish very rapidly. There will be no observable improvement in the graphical presentation of the analytical results when more wave modes are employed. The results of the radiating displacements on the fictitious boundary are presented in Fig. 4. The observable discrepancy between the analytical and the numerical results in the graph is in this case solely due to the FE discretization. Using finer mesh improves the results. However, the detailed discussion on the FE approximations exceeds the aim of this paper.

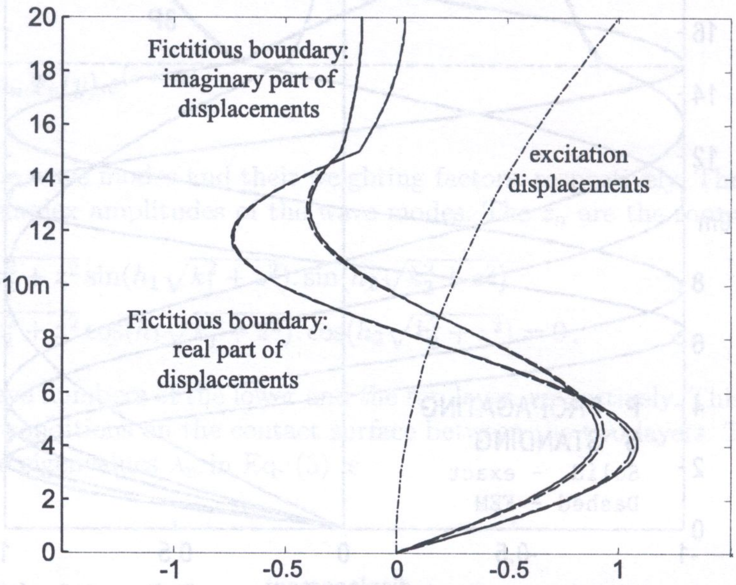

Fig. 4. Excitation displacements and displacements on the fictive boundary for the case when there is no foundation and no excavation. Analytical solutions are presented by solid lines, and numerical by the dashed ones

The first twelve dominant roots z_n of the Eq. (8) and the weighting factors a_n in Eq. (7), that yield analytical results presented in the Fig. 4, are displayed in Table 1. Note that the first three roots are imaginary and belong to the propagating wave modes whilst the others belong to standing ones.

Table 1. The roots of the Eq. (8), and the weighting factors (complex amplitudes) of the first twelve wave modes normalized as presented by Eqs. (10) and (11)

n	root z_n	weighting f. a_n
1	0.4591i	−0.1934I
2	0.3453I	+0.3336I
3	0.2376i	−0.2661I
4	0.3440	+0.0925i
5	0.5399	−0.0460I
6	0.7003	+0.0143i
7	0.9171	−0.0015I
8	1.1168	+0.0019i
9	1.2390	−0.0033I
10	1.3945	+0.0035i
11	1.5939	−0.0029I
12	1.7776	+0.0033i

The entire displacement field of the foundation-soil interaction, which is the second example, is presented in Fig. 5. It is computed using mesh having 20 × 25 nodal points. The excitation displacements and the radiating displacements on the fictive boundary are presented in the left graph in Fig. 6. Their amplitudes are presented in the right graph of this figure. It is neccesary to note that the amplitudes are computed according to the normalization of the wave modes, which are presented in the Fig. 3. Figure 6 demonstrates how the contents and the intensity of wave modes have changed during propagation from the excitation site to the fictive boundary. It is worth noting that all standing waves have practically vanished.

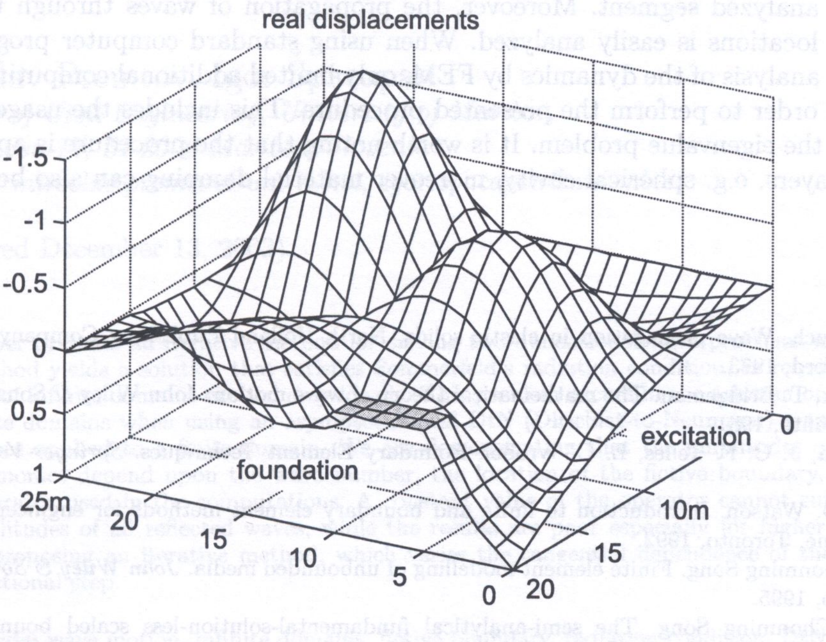

Fig. 5. Displacements field in the analyzed section of the waveguide with the foundation

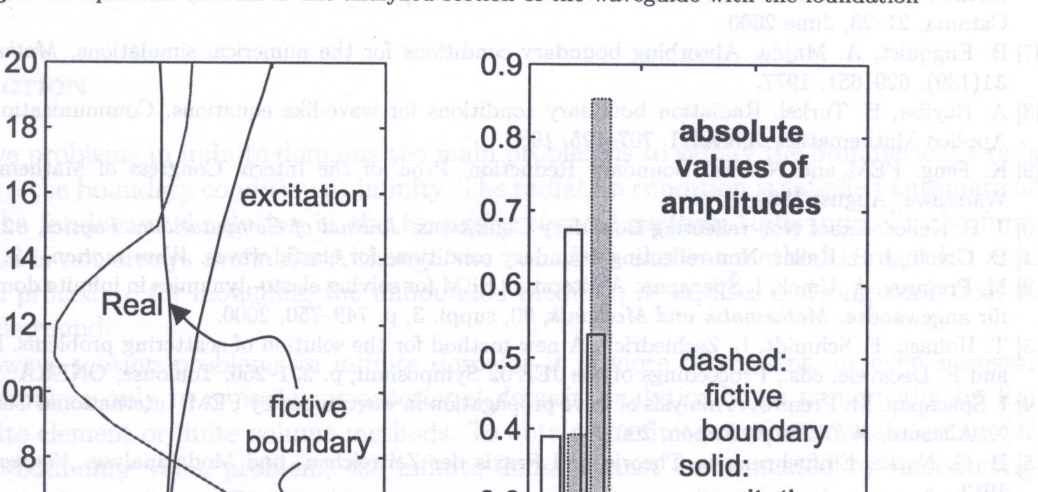

Fig. 6. The case with the foundation: the displacements of the excitation and of the fictive boundary, left figure. The weighting factors, right figure

4. Concluding remarks

The presented computing procedure is in principle much like the well-known modal analysis applied to a finite structure [15]. Of course, in our procedure we have propagating waves, which are computed in the above presented way. The computing approach yields wave modes and their amplitudes of the excitation, and of the waves passing the fictive boundary. This information yields the possibility to study effectively the influence of various kinds of excitation on the distribution and the intensity of waves in the analyzed segment. Moreover, the propagation of waves through the segment and towards distant locations is easily analyzed. When using standard computer programs, which are available for the analysis of the dynamics by FEM, only limited additional computing manipulations are necessary in order to perform the presented procedure. This includes the usage of the standard routine to solve the eigenvalue problem. It is worth noting that the procedure is applicable to other cases than the layers, e.g. spherical cavity, moreover material damping can also be encountered.

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