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Trefftz polynomials reciprocity based boundary element formulations for elastodynamics

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In this paper Trefftz polynomials are used for the BEM (Boundary Element Method) based on the reciprocity relations. BEM provides a powerful tool for the calculation of dynamic structural response in the frequency and time domains. Field equations of motion and boundary conditions are cast into boundary integral equations (BIE), which are discretized only on the boundary [1]. Trefftz polynomials or other non-singular (e.g. harmonic), Trefftz functions [2] (i.e. functions satisfying all governing differential equations but not the boundary conditions) used in the Betti's reciprocity relations lead to corresponding BIE that do not contain any (weak, strong, hyper) singularities. Fundamental solutions are not needed and evaluation of the field variables inside the domain is simpler.

1. Introduction

Trefftz was the first person who performed a BEM calculation (in 1917 he calculated numerically the value of the contraction coefficient of a round jet issuing from an infinite tank, i.e., he solved a nonlinear free surface problem). His method is based on the use of a complete set of solutions instead of using a fundamental solution [3].

The equivalent procedure in elasticity is to express u (e.g. the displacement field) as a series of complete functions satisfying Lame–Navier's equation (1) with coefficients which need to be numerically or analytically determined through utilization of the boundary conditions. The functions satisfying the Lame–Navier's equations are called Trefftz functions [3].

Our formulation rests on Betti's reciprocity theorem and the Trefftz (T-)functions present the reciprocal states of the body. As it is well known in the BEM formulations the reciprocity theorem is used to relate the displacements and tractions of the known (Trefftz) states of the body to its unknown states (which we seek).

2. Generation of Trefftz polynomial functions

In this part we will describe how the 4-th order Trefftz polynomials can be obtained analytically or numerically in the frequency domain for 2D isotropic solids as an example. The Einstein notation is used below: with summation on repeated indices, and partial derivatives with respect to Cartesian coordinates defined by indices after the comma. Using the kinematic relations, equilibrium equations and Hooke's law [1] one can obtain a complete system of governing equations of motions for isotropic, homogeneous linear elastic bodies in terms of displacements as

$$\mu u_{i,jj} + (\lambda + \mu)\, u_{j,ij} - \rho \frac{\partial^2 u}{\partial t^2} = -\rho b_i, \qquad (1)$$

where λ and μ are the two material parameters known as Lamé's constants, ρ is the material density and b_i are components of prescribed body forces. The displacement field u_i can be written in exponential form [1]

$$u_i = \bar{u}_i e^{-i\omega t}, \tag{2}$$

where the time derivative is

$$\frac{\partial^2 u_i}{\partial t^2} = \omega^2 \bar{u}_i e^{-i\omega t}, \tag{3}$$

ω is the circular frequency and $\bar{u}_i$ is the approximated displacement field evaluated using the T-functions. i in the exponent denotes the imaginary unit.

For time harmonic problems the Eq. (1) without body forces becomes

$$\mu \bar{u}_{i,jj} + (\lambda + \mu)\, \bar{u}_{j,ij} - \rho \omega^2 \bar{u}_i = 0. \tag{4}$$

The T-functions defining the reciprocal state of the body have to satisfy static equilibrium in the absence of body forces. For 2D isotropic solids they have the form

$$\begin{aligned} &(\lambda + 2\mu) u_{1,11} + \mu\left(u_{1,22} + u_{2,21}\right) + \lambda u_{2,21} = 0, \\ &(\lambda + 2\mu) u_{2,22} + \mu\left(u_{2,11} + u_{1,12}\right) + \lambda u_{1,12} = 0. \end{aligned} \tag{5}$$

In order to explain the derivation of T-polynomials, we will suppose the displacement field so to be in the form of polynomials containing 4-th order terms only, e.g.

$$p_4 = \left[\; x^4 \quad x^3 y \quad x^2 y^2 \quad x y^3 \quad y^4 \;\right]. \tag{6}$$

The displacement field can be written in the form

$$\left\{ \begin{matrix} u_1 \\ u_2 \end{matrix} \right\} = \left[\begin{matrix} x^4 & y^4 & 0 & 0 \\ 0 & 0 & y^4 & x^4 \end{matrix} \right] \{\mathbf{a}\} + \left[\begin{matrix} x^3 y & x^2 y^2 & x y^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & x y^3 & x^2 y^2 & x^3 y \end{matrix} \right] \{\mathbf{b}\}, \tag{7}$$

or, in matrix notation

$$\{\mathbf{u}\} = [\mathbf{A}(x, y)]\{\mathbf{a}\} + [\mathbf{B}(x, y)]\{\mathbf{b}\}, \tag{8}$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors of unknown coefficients. The second derivatives of the displacement field are

$$\left\{ \begin{matrix} u_{1,11} \\ u_{1,12} \\ u_{1,22} \end{matrix} \right\} = \left[\begin{matrix} 12x^2 & 0 \\ 0 & 0 \\ 0 & 12y^2 \end{matrix} \right] \left\{ \begin{matrix} a_1 \\ a_2 \end{matrix} \right\} + \left[\begin{matrix} 6xy & 2y^2 & 0 \\ 3x^2 & 4xy & 3y^2 \\ 0 & 2x^2 & 6xy \end{matrix} \right] \left\{ \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \right\} \tag{9}$$

and

$$\left\{ \begin{matrix} u_{2,11} \\ u_{2,12} \\ u_{2,22} \end{matrix} \right\} = \left[\begin{matrix} 0 & 12y^2 \\ 0 & 0 \\ 12x^2 & 0 \end{matrix} \right] \left\{ \begin{matrix} a_3 \\ a_4 \end{matrix} \right\} + \left[\begin{matrix} 0 & 2y^2 & 6xy \\ 3y^2 & 4xy & 3x^2 \\ 6xy & 2x^2 & 0 \end{matrix} \right] \left\{ \begin{matrix} b_4 \\ b_5 \\ b_6 \end{matrix} \right\}. \tag{10}$$

The derivatives obtained are set into equilibrium equations (5) and the following relation between the coefficients $\mathbf{a}$ and $\mathbf{b}$ is obtained

$$[\mathbf{M}(x, y]\{\mathbf{b}\} + [\mathbf{N}(x, y]\{\mathbf{a}\} = \{\mathbf{0}\} \tag{11}$$

from which the coefficients $\mathbf{b}$ are to be evaluated analytically or numerically. Thus the dimensions of the matrices $\mathbf{M}$ and $\mathbf{N}$ are (2×6) and (2×4), respectively. For the numerical evaluation, Eq. (11)

has to be expressed at three different points, which do not lie on a line (in order to get a non-singular matrix **M**) and one obtains

$$\{\mathbf{b}\} = -[\bar{\mathbf{M}}]^{-1}[\bar{\mathbf{N}}]\{\mathbf{a}\}. \tag{12}$$

The bar denotes the matrices at the 3 different points, thus having 6 rows now. The T-polynomial displacements of the 4-th order are defined using Eqs. (12) and (8)

$$\{\mathbf{u}\} = \Big([\mathbf{A}(x,y)] - [\mathbf{B}(x,y)][\bar{\mathbf{M}}]^{-1}[\bar{\mathbf{N}}]\Big)\{\mathbf{a}\} = [\mathbf{U}(x,y)]\{\mathbf{a}\}. \tag{13}$$

Each column of [U] introduces a T-displacement function. In 2D problems we obtain $(2n+1)$ and for 3D $(n+1)^2$ T-functions where n is the polynomial order [2]. This means that, if we assume to use T-polynomials of 4-th order ($n=4$) we will have 9 T-functions for each displacement component in 2D. We remark that it is necessary to use a series of complete polynomials involving each from the order zero up to order n.

If the problem (4) has to be solved, then the displacements $\bar{u}_i(x)$ and tractions $\bar{t}_i(x)$ will be related by Betti's reciprocity equations.

3. Multi-domain Reciprocity Based Formulation

With increasing complexity of the problem to be solved, it is necessary to use higher orders of T-polynomials, and both their complexity and computation time increases. For this reason, it is more efficient to decompose the whole domain into sub-domains.

The T-tractions on the boundaries with the outer normal **n** corresponding to T-displacements are

$$T_i = S_{ij} n_j, \tag{14}$$

where T-stresses S_{ij} can be found from the T-displacements by

$$S_{ij} = \mu\left(U_{i,j} + U_{j,i}\right) + \lambda\delta_{ij}U_{k,k}. \tag{15}$$

If the problem (4) has to be solved, then the displacements $\bar{u}_i(x)$ and tractions $\bar{t}_i(x)$ will be related by Betti's reciprocity equations for each sub-domain (element).

$$\int_{\Gamma e} T_i(x)\,\bar{u}_i(x)\,\mathrm{d}\Gamma(\mathrm{x}) = \int_{\Gamma e} U_i(x)\,\bar{t}_i(x)\,\mathrm{d}\Gamma(\mathrm{x}) - \int_{\Omega e} \rho\omega^2 U_i(x)\,\bar{u}_i(x)\,\mathrm{d}\Omega(\mathrm{x}). \tag{16}$$

where Γe and Ωe denote the sub-domain boundaries and volume, respectively. This equation is known also as the dual reciprocity approach and was first presented by Nardini and Brebia [4].

The boundary displacements and tractions can be expressed by their nodal values and the corresponding shape functions

$$u_i(\xi) = N_u^{(j)}(\xi)\,d_i^{(j)}, \tag{17}$$

$$t_i(\xi) = N_t^{(j)}(\xi)\,q_i^{(j)}. \tag{18}$$

N_t and N_u are the shape functions well known from FEM theory. The upper index (j) denotes the element nodal points. This leads to the matrix form of Eq. (16)

$$(\mathbf{T} + \omega^2\mathbf{F})\mathbf{d}^e = \mathbf{U}\mathbf{q}^e. \tag{19}$$

Elements of matrices **T**, **F** and **U** are obtained by numerical integration using the Gauss quadrature nodal points ξ on the element boundaries and the quadrature nodal points η in the element volume as in Eqs. (20) to (22), where w denotes corresponding weights.

$$T_{kl} = \int_{\Gamma e} T^{(k)}(x(\xi))\, N_u^{(l)}(\xi)\, \mathrm{d}\Gamma = \sum_j T^{(k)}\left(x\left(\xi^{(j)}\right)\right) N_u^{(l)}\left(\xi^{(j)}\right) J\left(\xi^{(j)}\right) w^{(j)}\,, \tag{20}$$

$$U_{kl} = \int_{\Gamma e} U^{(k)}(x(\xi))\, N_t^{(l)}(\xi)\, \mathrm{d}\Gamma = \sum_j U^{(k)}\left(x\left(\xi^{(j)}\right)\right) N_t^{(l)}\left(\xi^{(j)}\right) J\left(\xi^{(j)}\right) w^{(j)}\,, \tag{21}$$

$$F_{kl} = \rho \int_{\Omega e} U^{(k)}(x(\eta)) N_u^{(l)}(\eta) \mathrm{d}\Omega = \rho \sum_i U^{(k)}(x(\eta^{(i)})) N_u^{l}(\eta^{(i)}) J(\eta^{(i)}) w^{(i)}\,. \tag{22}$$

We assume that the whole domain will be decomposed into sub-domains (elements) and displacements between sub-domains will be compatible. The tractions, however, will not be codiffusive between the elements, and inter-element equilibrium and natural boundary conditions will be satisfied only in a weak (integral) sense using a variational formulation:

$$\int_{\Gamma t} \delta\mathbf{u}^T (\mathbf{t} - \bar{\mathbf{t}})\, \mathrm{d}\Gamma + \int_{\Gamma i} \delta\mathbf{u}^T (\mathbf{t}^A - \mathbf{t}^B)\, \mathrm{d}\Gamma = \int_{\Gamma t} \delta\mathbf{u}^T \mathbf{t}\, \mathrm{d}\Gamma - \int_{\Gamma t} \delta\mathbf{u}^T \bar{\mathbf{t}}\, \mathrm{d}\Gamma = 0\,, \tag{23}$$

where Γi denotes inter-element and Γt denotes the element boundary with prescribed tractions denoted by a bar, respectively. **u** and **t** are displacement and traction vectors, respectively. Variables $\mathbf{t}^A$ and $\mathbf{t}^B$ denote traction vectors on neighbouring elements.

Equation (23) in the discretized form is

$$\sum_e \sum_j \sum_l N_u^{(k)}\left(\xi^{(j)}\right) N_t^{(l)}\left(\xi^{(j)}\right) J\left(\xi^{(j)}\right) w^{(j)} \mathbf{q}^{(j)} = \sum_e \sum_i N_u^{(k)}\left(\xi^{(i)}\right) \bar{\mathbf{t}}\left(\xi^{(i)}\right) J\left(\xi^{(i)}\right) w^{(i)} \tag{24}$$

with summation over all elements e. And the matrix form of Eq. (24) is

$$\sum_e \mathbf{H}^e \mathbf{q}^e = \sum_e \mathbf{p}^e \tag{25}$$

where **q** and **p** denote nodal tractions and equivalent surface loads, respectively. Substituting the boundary nodal displacements for the nodal tractions of a sub-domain from Eq. (19) into Eq. (25) one can write

$$\sum_e \mathbf{H}^e \mathbf{U}^{-1} (\mathbf{T} + \omega^2 \mathbf{F}) \mathbf{d}^e = \sum_e \mathbf{p}^e \tag{26}$$

or

$$(\mathbf{K} + \omega^2 \mathbf{M})\mathbf{d} = \mathbf{p}\,, \tag{27}$$

where **K** and **M** are global stiffness and mass matrices, respectively.

We have to note that in Eq. (26), there is the inverse of matrix **U**, which is not square, in the general case. And so, the solution of (19) is performed in the least squares sense.

If the boundary tractions are zero, we have to solve the free vibration problem (24) as is known from FEM.

4. Numerical example

Two cantilevers of 1 m and 4 m height and 20 m length were examined for this well known free vibration problem with modulus of elasticity $E = 2.1e11$ Pa, Poisson ratio $\mu = 0.3$, and density $\rho = 7800$ kg/m^3. In the first part the free vibration problem for the Free-Free case was computed. In the second part the solution of the cantilever with Clamped-Free supports for both cases is given.

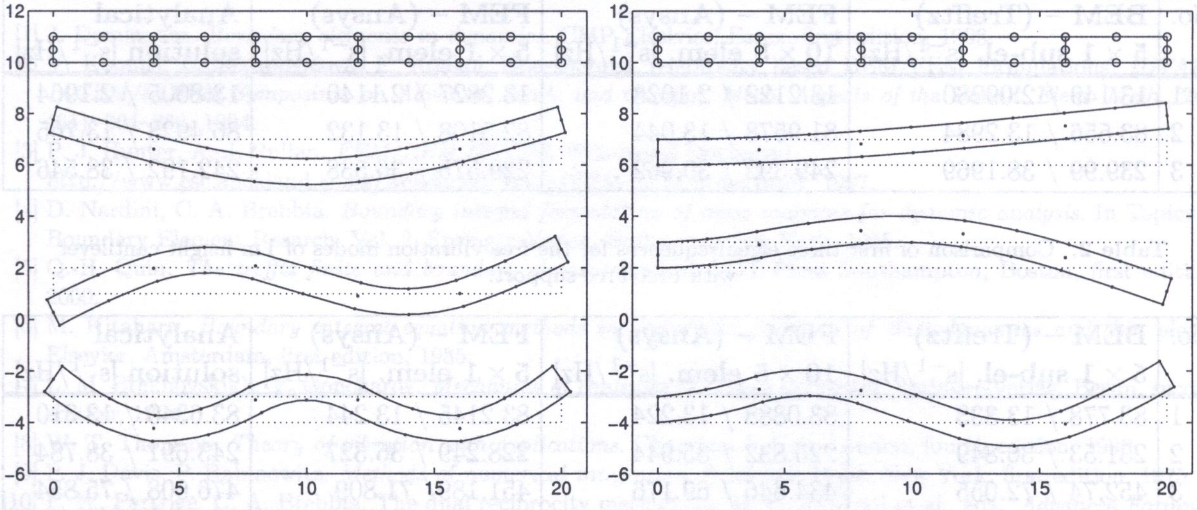

Fig. 1. First three mode shapes (from top to bottom) for the vibration modes of the 1m high cantilever with Free-Free (on the left), and Clamped-Free (on the right).

The domain was decomposed into five sub-domains with four quadratic boundary elements (5 quadratic T-elements) for each sub-domain. According to the number of degrees of freedom per sub-domain in this computation the Trefftz polynomials of 6-th order were used. The same problem was solved by FEM using the Ansys software with second degree two-dimensional elements (eight-node PLANE183 element) with 10×5 and 5×1 elements for comparison each problem, see Fig. 2.

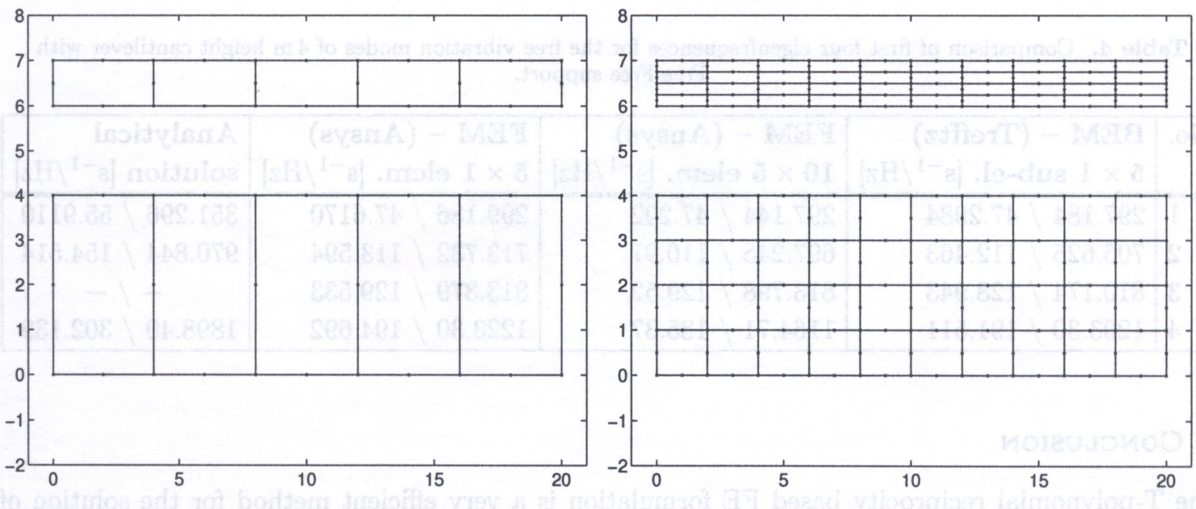

Fig. 2. Domain description of the cantilevers analyzed by shape and sub-elements (on the left) and FEM elements (on the right).

Results for the free vibration problem of the cantilever with 1 m height are given in Table 1 for the Clamped-Free support and in Table 2 for the Free-Free support. The results obtained correspond well with analytical solutions obtained by simplified 1D solutions based on Euler's differential equation, which excludes the shear deformation and hence tends to give greater frequences particularly for the deeper beams.

Table 1. Comparison of first three eigenfrequences the free vibration modes of 1 m height cantilever with Clamped-Free support (in both the circular frequency and the frequency in Hz).

No.	**BEM – (Trefftz)** **5 × 1 sub-el.** [s^{-1}/Hz]	**FEM – (Ansys)** **10 × 5 elem.** [s^{-1}/Hz]	**FEM – (Ansys)** **5 × 1 elem.** [s^{-1}/Hz]	**Analytical** **solution** [s^{-1}/Hz]
1	13.149 / 2.09280	13.2122 / 2.1028	13.2827 / 2.1140	13.8005 / 2.1964
2	83.556 / 13.2984	81.9578 / 13.044	82.5108 / 13.132	86.4923 / 13.765
3	239.99 / 38.1969	249.392 / 39.962	229.575 / 36.538	242.192 / 38.546

Table 2. Comparison of first three eigenfrequences for the free vibration modes of 1 m height cantilever with Free-Free support.

No.	**BEM – (Trefftz)** **5 × 1 sub-el.** [s^{-1}/Hz]	**FEM – (Ansys)** **10 × 5 elem.** [s^{-1}/Hz]	**FEM – (Ansys)** **5 × 1 elem.** [s^{-1}/Hz]	**Analytical** **solution** [s^{-1}/Hz]
1	83.778 / 13.333	83.0888 / 13.224	83.2145 / 13.244	83.6340 / 13.310
2	231.53 / 36.849	225.832 / 35.944	228.249 / 36.327	243.691 / 38.784
3	452.74 / 72.055	434.646 / 69.176	451.189/ 71.809	476.608 / 75.854

Table 3. Comparison of first four eigenfrequences for the free vibration modes of 4 m height cantilever with Clamped-Free support.

No.	**BEM – (Trefftz)** **5 × 1 sub-el.** [s^{-1}/Hz]	**FEM – (Ansys)** **10 × 5 elem.** [s^{-1}/Hz]	**FEM – (Ansys)** **5 × 1 elem.** [s^{-1}/Hz]	**Analytical** **solution** [s^{-1}/Hz]
1	51.0370 / 8.1228	51.2494 / 8.1566	51.7860 / 8.2420	55.2018 / 8.7856
2	277.011 / 44.087	277.013 / 44.088	283.026 / 45.045	345.969 / 55.065
3	406.640 / 64.718	408.495 / 65.014	408.696 / 65.046	– / –
4	669.450 / 106.54	661.996 / 105.36	684.490 / 108.94	968.771/ 154.184

Table 4. Comparison of first four eigenfrequences for the free vibration modes of 4 m height cantilever with Free-Free support.

No.	**BEM – (Trefftz)** **5 × 1 sub-el.** [s^{-1}/Hz]	**FEM – (Ansys)** **10 × 5 elem.** [s^{-1}/Hz]	**FEM – (Ansys)** **5 × 1 elem.** [s^{-1}/Hz]	**Analytical** **solution** [s^{-1}/Hz]
1	297.184 / 47.2984	297.144 / 47.292	299.186 / 47.6170	351.296 / 55.9110
2	706.625 / 112.463	697.245 / 110.97	713.732 / 113.594	970.844 / 154.514
3	810.174 / 128.943	813.798 / 129.52	813.879 / 129.533	– / –
4	1203.30 / 191.511	1164.71 / 185.37	1223.30 / 194.692	1898.40 / 302.139

5. Conclusion

The T-polynomial reciprocity based FE formulation is a very efficient method for the solution of dynamic problems. The stiffness matrix is formulated by solving non-singular integral equations over the element boundaries. The mass matrix in the present formulation is computed by integration over the element volume using the same reciprocal field variables as those used for the stiffnes matrix. The numerical examples show good acuracy of the models. The Trefftz elements get comparable accuracy to displacement FEM with a lower number of elements.

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