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A different strategy for coupling of finite and boundary element methods

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In this study, a different coupling strategy is used for the coupling of finite element (FE) and boundary element (BE) methods. In the literature, the coupling is done by transforming nodal forces into nodal tractions using distribution matrix at the interface line. In this study, however, the stress-traction equilibrium is used at the interface line for coupling of both methods. A finite and boundary element program is written using FORTRAN 95 and ordinary and developed coupling methods are adapted to this program. The results of both methods are compared with each other, ANSYS, FE, BE and analytical solution whenever possible. It has been seen that the developed method supply more efficient results against the ordinary method.

Keywords: Coupling, FEM, BEM, distribution matrix, stress-traction equilibrium

1. Introduction

The main purpose of coupling of finite element method (FEM) and boundary element method (BEM) is to use the advantages of both methods for the solution of various engineering problems. Both methods have some advantages for certain applications. While the BEM gives better results for the surface type problems like contact problems, FEM is more effective technique for the domain type problems. Zienkiewicz *et al.* [1] are one of the frontier of coupling process. They discussed the coupling in a general context. Later, Kelly *et al.* [2] have proposed a method obtaining the symmetric stiffness matrices of the BE region which satisfy the equilibrium equation and applied the method to a number of field problems in fluid mechanics. Furthermore Felippa [3] used the coupling methods for a three dimensional structure submerged in an acoustic fluid. Beer [4] developed a general coupled BE and FE program in 1983. A number of special elements was included for the analysis of shell structures and joint/interfaces. Same author analyzed the unbounded problems in elastostatics by coupling method [5]. He claimed that the BEM is very efficient for unbounded problems in elastostatics. Costabel [6] used a symmetric method for coupling of FE and BE. He [7] also extended his work to an elastoplastic interface problem using coupling process. Furthermore, Guzman [8] carried on some works on the combined BEM and FEM for elastostatics focusing on axisymmetric cases. Fusco [9] proposed a unified formulation of the FEM and BEMs using energy methods for structural problems. Ohtsu [10] studied a coupling analysis in two dimensional elastodynamics and elastostatics. He suggested that the BEM preferably applicable to problems in the homogeneous and infinite domain, while the FEM was versatile for problems in the inhomogeneous and finite domain. Z. Z. Chen *et al.*, [11] developed a coupling procedure for the Hybrid/Mixed FE and BE using a different discretization schemes. Grannell [12] also studied on the simplified hybrid methods for coupling. In his study, the hybrid FEM was characterized by the use of elements, which were non-conforming. Most of the publications on coupling use the distribution matrix idea. However, a new approach based upon stress-traction equilibrium has been derived and used in this study.

2. The distribution matrix

During the coupling process, the nodal forces at the interface line of FE domain must be converted into the nodal tractions. This operation can be done using simple energy considerations. The work done over an interface element in terms of nodal forces and in terms of tractions can be used to derive distribution matrix as follows;

$$\mathbf{F} = \int_{S_e} \mathbf{N}^t \mathbf{N} ds\mathbf{t} = \mathbf{M}\mathbf{t} \tag{1}$$

where the matrix $\mathbf{M} = \int_{S_e} \mathbf{N}^t \mathbf{N} ds$ is called as distribution matrix [1, 8] and N is a one dimensional shape function for the interface line. It is used to transform the nodal forces into the nodal tractions at the interface line during the coupling process or reverse.

3. Coupling process by subregional technique

Although the subregion is a technique in BEMs [13], coupling process may be thought as a subregional technique due to the division of problem domain into the two subdomains as BE and FE. Stress-traction equilibrium approach also considered as a subregion technique due to existing BE and FE subregions and the compatibility conditions of the interface line of two regions which is similar to the compatibility conditions of subregion technique.

When the stress-traction equilibrium approach is used, imaginary constraints must be considered at the interface line of the FE and BE regions. There are three types of boundary conditions. The first one includes the normal and tangential constraints at the interface line as shown in Fig. 1a. The second one includes normal constraints at the interface line as shown in Fig. 1b. The last one includes only tangential constraints to the interface as shown in Fig. 1c. The type of the problem determines the type of imaginary boundary constraints. The case studies have been selected considering these different boundary constraints. The imaginary constraints can be seen explicitly in Fig. 1.

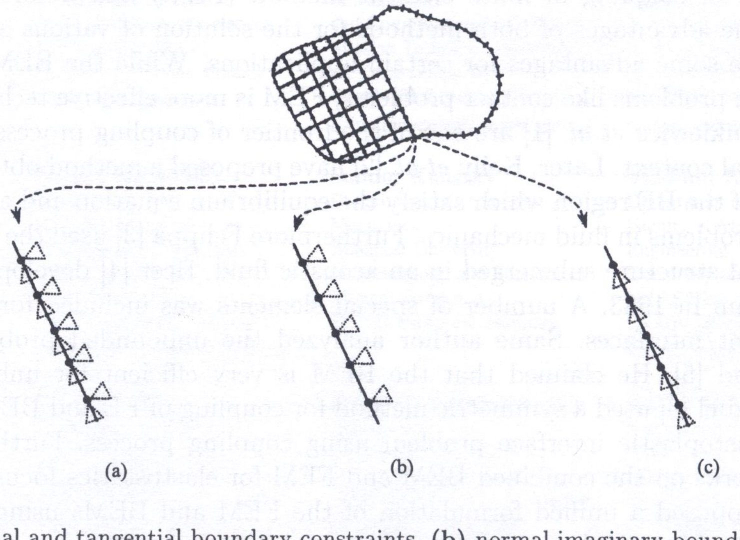

Fig. 1. (a) normal and tangential boundary constraints, (b) normal imaginary boundary constraints, (c) tangential imaginary boundary constraints

3.1. Distribution matrix approach

The general form of FE equation can be written as follows:

$$\mathbf{F} = \mathbf{K}\,\mathbf{u}. \tag{2}$$

It can be rewritten in a new form including FE domain sub-matrix and FE interface sub-matrix as follows:

$$\begin{vmatrix}\mathbf{F}_{FE}\\ \mathbf{F}^*_{FE_i}\end{vmatrix} = \begin{vmatrix}\mathbf{K}_{FE} & \mathbf{K}_{FE_i}\end{vmatrix}\begin{vmatrix}\mathbf{u}_{FE}\\ \mathbf{u}^*_{FE_i}\end{vmatrix} \tag{3}$$

where

$\mathbf{F}_{FE}$ – force vector at the FE domain,
$\mathbf{F}^*_{FE_i}$ – imaginary internal reaction force vector at the interface line,
$\mathbf{U}_{FE}$ – displacement vector at the FE domain,
$\mathbf{U}^*_{FE_i}$ – imaginary displacement vector at the interface line.

The above equation may be solved for imaginary reaction force vector, $\mathbf{F}^*_{FE_i}$.

After finding the imaginary reaction force vector at the interface line, the distribution matrix, **M**, may be used to transform it into the imaginary traction vector. So the Eq. (1) is written as follows:

$$\mathbf{F}^*_{FE_i} = \mathbf{M}\mathbf{t}^*_{FE_i}\,. \tag{4}$$

Equation (3) can be rewritten in a new form which is similar to general BE equation form using the Eq. (4);

$$\begin{vmatrix}\mathbf{I} & \mathbf{M}_{FE_i}\end{vmatrix}\begin{vmatrix}\mathbf{F}_{FE}\\ \mathbf{t}^*_{FE_i}\end{vmatrix} = \begin{vmatrix}\mathbf{K}_{FE} & \mathbf{K}_{FE_i}\end{vmatrix}\begin{vmatrix}\mathbf{u}_{FE}\\ \mathbf{u}^*_{FE_i}\end{vmatrix} \tag{5}$$

where

$\mathbf{I}$ – unit matrix,
$\mathbf{M}_{FE_i}$ – distribution matrix at the interface line,
$\mathbf{F}_{FE}$ – force vector at the FE domain,
$\mathbf{t}^*_{FE}$ – imaginary traction vector at the interface line.

The general BE equation is as follows;

$$\mathbf{Hu} = \mathbf{Gt}\,. \tag{6}$$

It can also be rewritten including BE domain sub-matrix and interface BE sub-matrix as follows:

$$\begin{vmatrix}\mathbf{H}_{BE_i} & \mathbf{H}_{BE}\end{vmatrix}\begin{vmatrix}\mathbf{u}_{BE_i}\\ \mathbf{u}_{BE}\end{vmatrix} = \begin{vmatrix}\mathbf{G}_{BE_i} & \mathbf{G}_{BE}\end{vmatrix}\begin{vmatrix}\mathbf{t}^*_{BE_i}\\ \mathbf{t}_{BE}\end{vmatrix}, \tag{7}$$

where

$\mathbf{t}^*_{BE_i}$ – imaginary traction vector,
$\mathbf{u}_{BE_i}$ – real displacement vector.

The traction equilibrium must be satisfied for coupling purpose at the interface line as follows:

$$\mathbf{t}^*_{FE_i} = -\mathbf{t}^*_{BE_i}\,. \tag{8}$$

Then the Eq. (6) may be solved for real displacement vector, $\mathbf{u}_{\mathrm{BE_i}}$. After finding the real displacement vector at the interface line, the displacement continuity requirement can be satisfied for coupling purpose.

$$\mathbf{u}_{FE_i} = \mathbf{u}_{BE_i} = \mathbf{u}_i\,. \tag{9}$$

As a result, the general coupling equation can be written using Eqs. (5) and (7);

$$\begin{vmatrix}\mathbf{K}_{FE} & \mathbf{K}_{FE_i} & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_{BE_i} & \mathbf{H}_{BE}\end{vmatrix}\begin{vmatrix}\mathbf{u}_{FE}\\ \mathbf{u}_i\\ \mathbf{u}_{BE}\end{vmatrix} = \begin{vmatrix}\mathbf{I} & \mathbf{M}_{FE_i} & \mathbf{0}\\ \mathbf{0} & -\mathbf{G}_{BE_i} & \mathbf{G}_{BE}\end{vmatrix}\begin{vmatrix}\mathbf{F}_{FE}\\ \mathbf{t}_i\\ \mathbf{t}_{BE}\end{vmatrix}. \tag{10}$$

Due to the formulation of Eq. (10), the only interface part of FE force vector has been converted to tractions so the remaining part has been kept as original force vector.

3.2. Stress-traction equilibrium approach

In the FE domain, the stress and displacement relationships can be written as follows:

$$\sigma = \mathbf{DB}\delta. \tag{11}$$

The imaginary stress components can be found writing the above equation in terms of interface and domain sub-matrices and considering the imaginary constraints at the interface line as shown in Fig. 2.

$$\begin{vmatrix} \sigma^*_{FEi} \\ \sigma_{FE} \end{vmatrix} = \left|(\mathbf{DB})_{FEi}\, (\mathbf{DB})_{FE}\right| \begin{vmatrix} \delta^*_{FEi} \\ \delta_{FE} \end{vmatrix}. \tag{12}$$

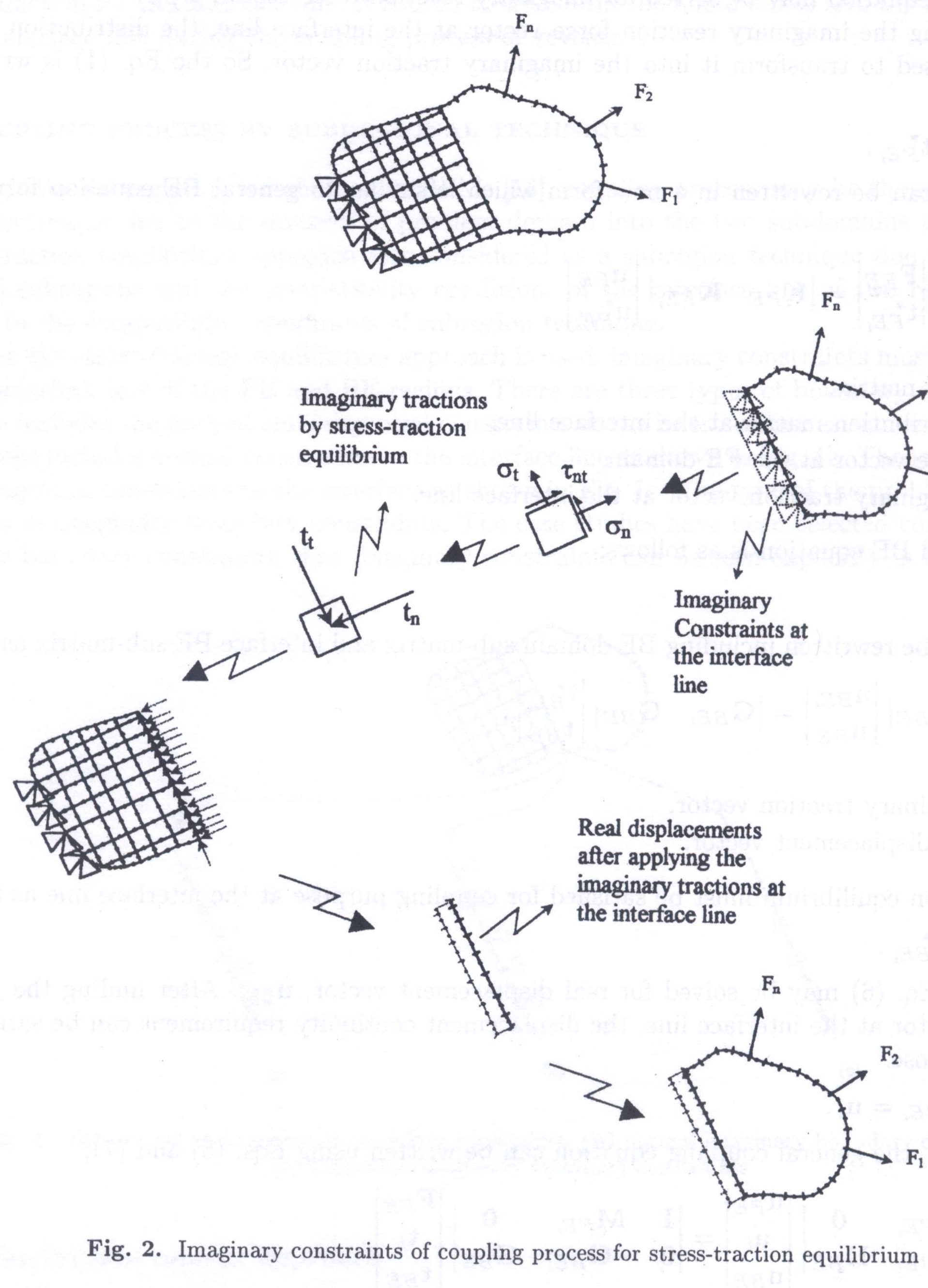

Fig. 2. Imaginary constraints of coupling process for stress-traction equilibrium

Then these stress components can be transformed into tractions using stress-tractions equilibrium equations as follows:

$$t^*_{xi} = (\sigma^*_{xi} l + \tau^*_{xyi} m) * th, \tag{13}$$

$$t^*_{yi} = (\sigma^*_{yi} m + \tau^*_{yxi} l) * th. \tag{14}$$

Where l, m are directional cosines of the interface line and th is the thickness of the member. After finding the imaginary tractions at the interface line, the general BE equation can be used to find the real interface displacements. Then these displacements can be used as an initial value for the FE region. So the correct stresses and forces can be found in the FE region.

$$\left|\mathbf{H}_{BE_i} \quad \mathbf{H}_{BE}\right| \left|\begin{matrix}\mathbf{u}_{BE_i} \\ \mathbf{u}_{BE}\end{matrix}\right| = \left|\mathbf{G}_{BE_i} \quad \mathbf{G}_{BE}\right| \left|\begin{matrix}\mathbf{t}^*_{BE_i} \\ \mathbf{t}_{BE}\end{matrix}\right|. \tag{15}$$

One of the crucial points of this approach is absent of one dimensional shape functions in the formulation due to different formulation from the distribution matrix approach. So the stress-traction equilibrium approach offers more stable algorithm than the distribution matrix approach.

4. Case studies

A few cases have been designed for the validations of the developed approaches. DMC is referring to coupling with distribution matrix and STC is referring to coupling with stress-traction equilibrium.

4.1. Axially loaded bar

This is a simple plane stress problem. A thin axially loaded aluminum bar has been used for the validation ($E = 70$ GPa; $\nu = 0.3$). The dimensions are shown in the Fig. 3a. Although the theory works very well with quadratic elements, linear elements are used for the FE, BE and coupling models (Figs. 3b, 3c and 3d) to show the reliability of the developed approach. Axial stress and displacement are exactly same for FEM, BEM, ANSYS5.6, analytical solution and the coupling approaches as seen in the Figs. 4 and 5.

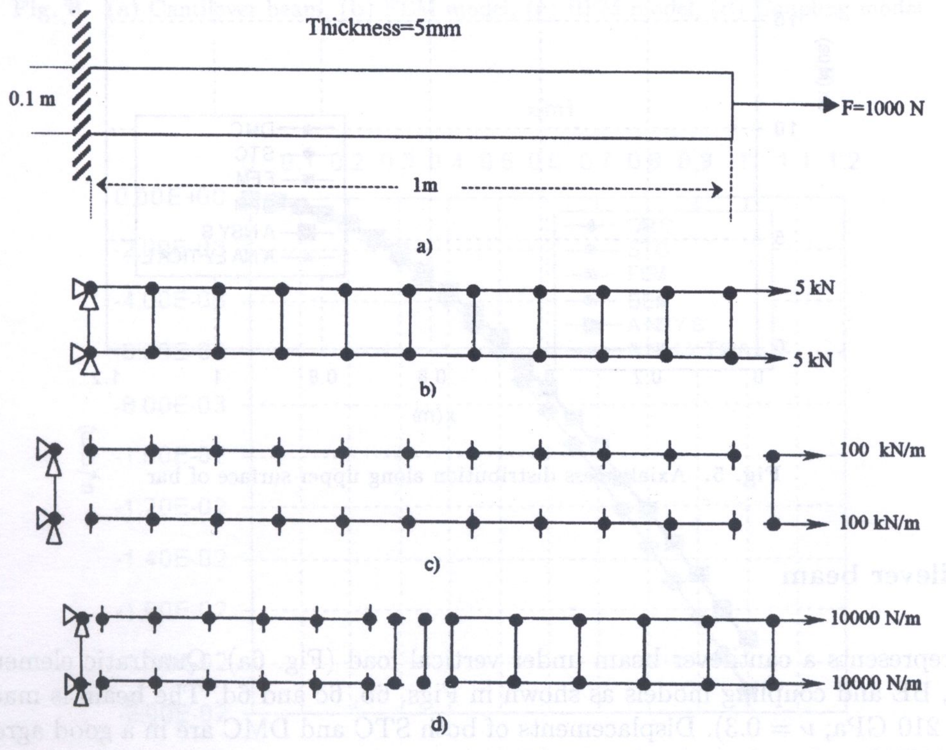

Fig. 3. (a) Axially loaded bar, (b) FEM model, (c) BEM model, (d) Coupling model

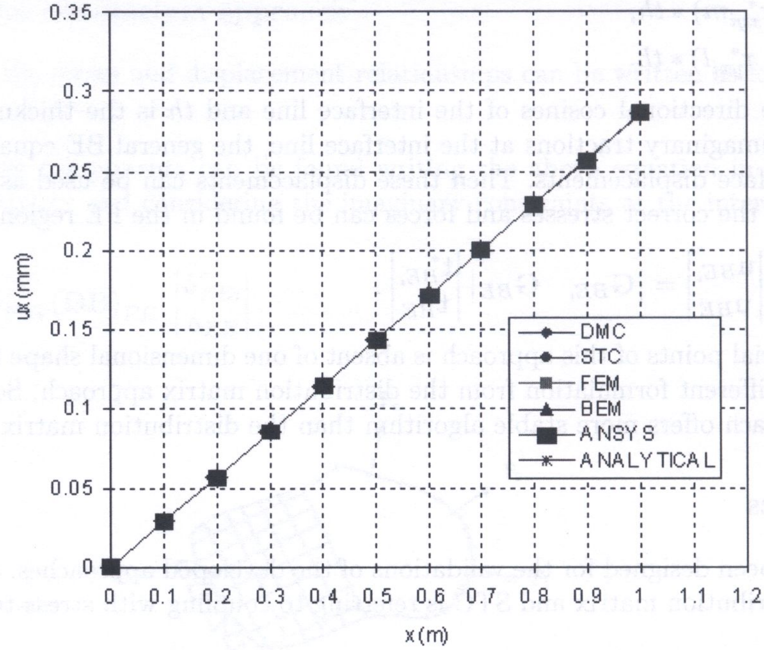

Fig. 4. Axial displacement distribution along upper surface of bar

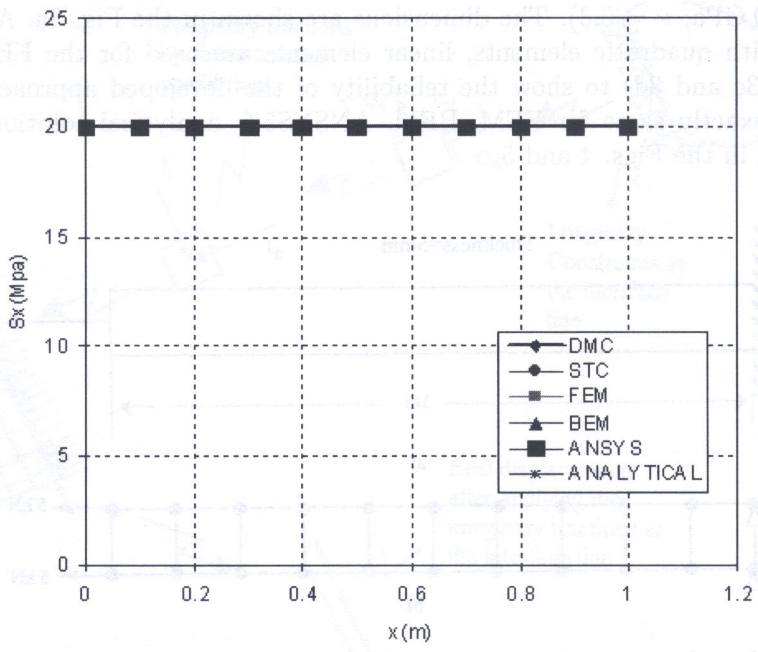

Fig. 5. Axial stress distribution along upper surface of bar

4.2. Cantilever beam

This case represents a cantilever beam under vertical load (Fig. 6a). Quadratic elements are used for the FE, BE and coupling models as shown in Figs. 6b, 6c and 6d. The beam is made of a steel alloy ($E = 210$ GPa; $\nu = 0.3$). Displacements of both STC and DMC are in a good agreement with FE, BE, ANSYS and analytical solution (Fig. 7). Stress results of the stress-traction equilibrium approach are more accurate than the distribution matrix approach (Fig. 8).

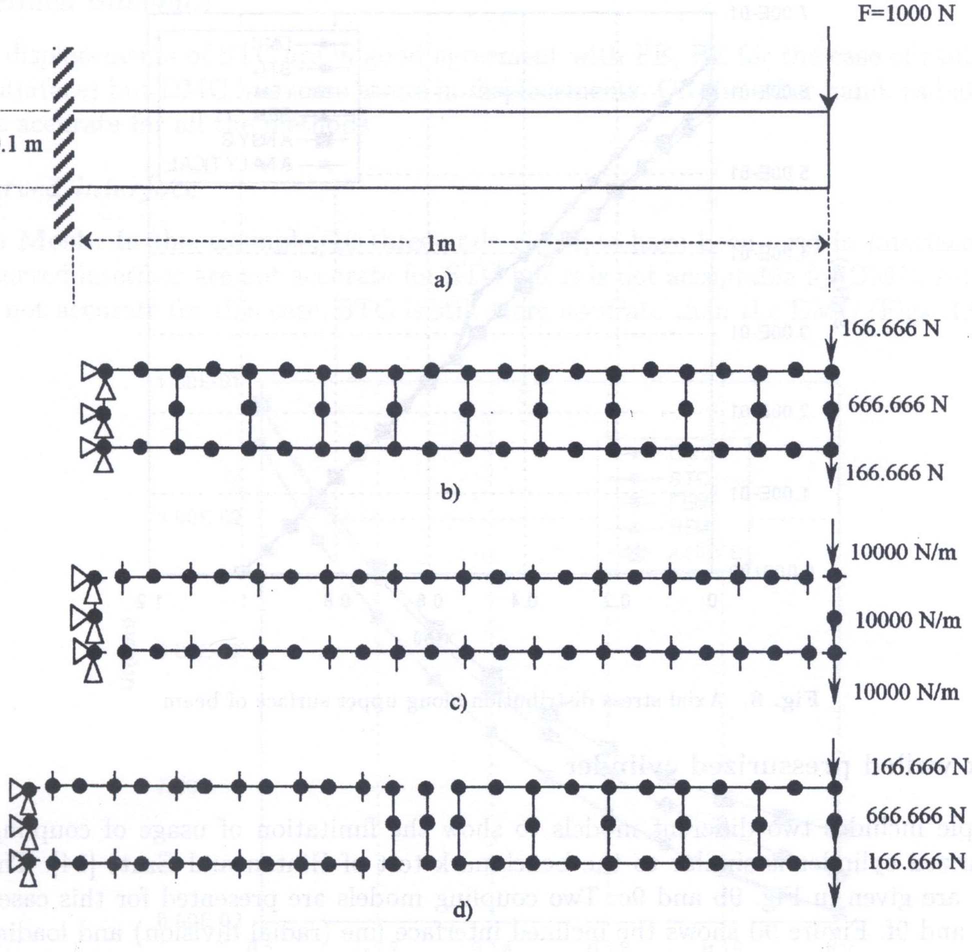

Fig. 6. (a) Cantilever beam, (b) FEM model, (c) BEM model, (d) Coupling model

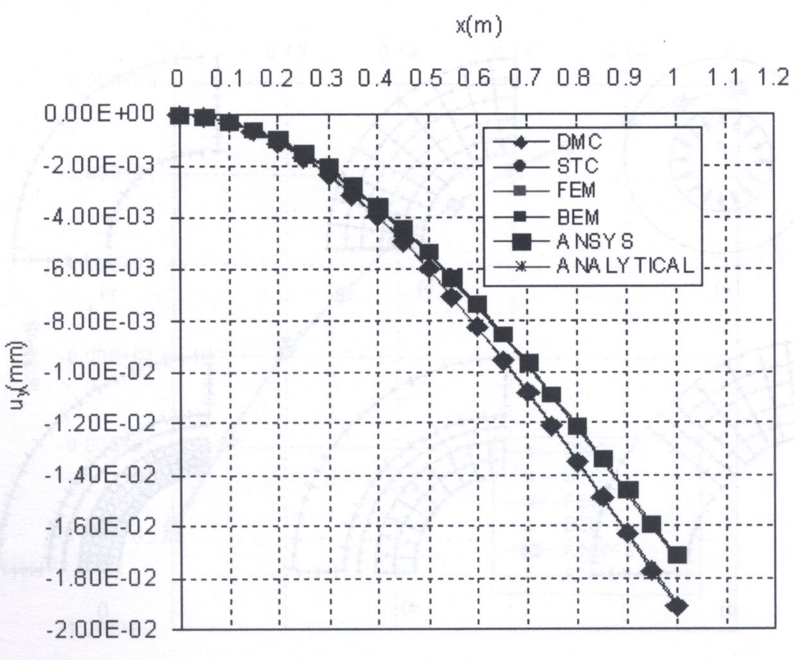

Fig. 7. Vertical displacement along upper surface of beam

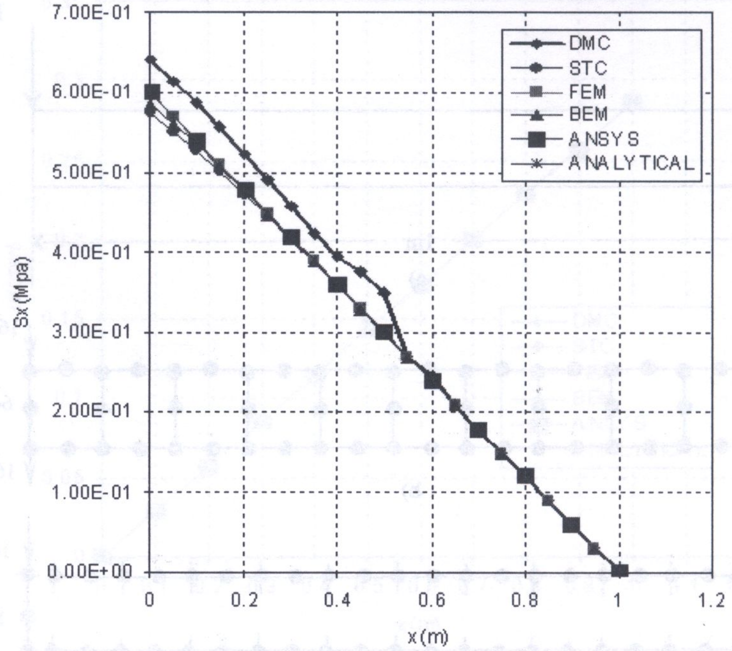

Fig. 8. Axial stress distribution along upper surface of beam

4.3. Thick walled pressurized cylinder

This example includes two different models to show the limitation of usage of coupling process. The pressurized cylinder is similar to the benchmark test of Hinton and Ezatt [14]. The FE and BE models are given in Fig. 9b and 9c. Two coupling models are presented for this case shown in Fig. 9d, 9e and 9f. Figure 9d shows the inclined interface line (radial division) and loading in both regions. Figures 9e and 9f show the curved interface line (longitudinal division) and loading in the BE region. The dimensions, material properties and internal pressure have been given as follows: inner radius $r_i = 0.1$ m; outer radius $r_o = 0.2$ m; $E = 200$ GPa; $\nu = 0.3$; $P_i = 100$ MPa.

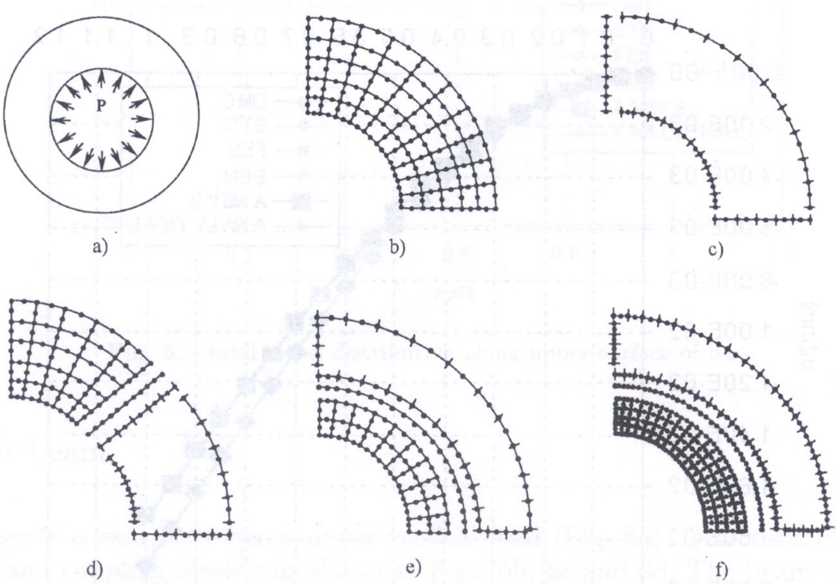

Fig. 9. (**a**) Thick walled cylinder, (**b**) FEM model, (**c**) BEM model (**d**) Coupling model with inclined interface, (**e**) Coarse coupling model with curved interface, (**f**) Fine coupling model with curved interface

4.3.1. Inclined interface

The radial displacements of STC are in good agreement with FE, BE for the case of radial division (inclined interface) but DMC has some errors in displacements. On the other hand, radial and hoop stresses are accurate for all the methods.

4.3.2. Curved interface

a) Coarse Mesh: In this example, 10 three-node elements have been used in interface line. The results of curved interface are not accurate for STC but it is not acceptable for DMC. Although the results are not accurate for this case, STC is still more accurate than the DMC (Figs. 13–15).

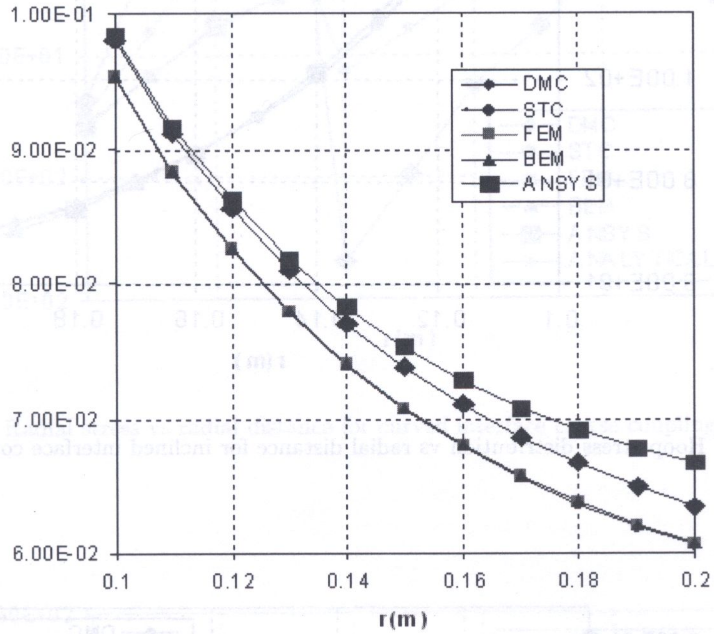

Fig. 10. Radial displacement vs radial distance for inclined interface coupling model

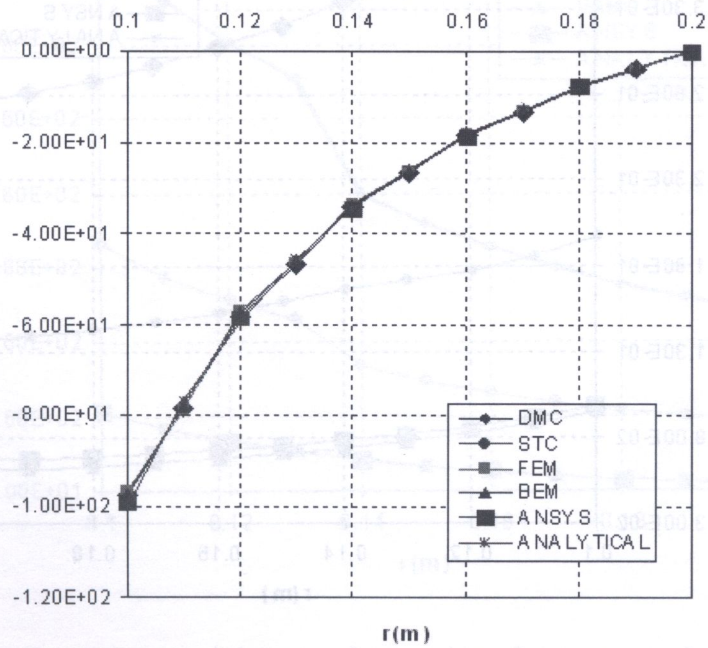

Fig. 11. Radial stress distribution vs radial distance for inclined interface coupling model

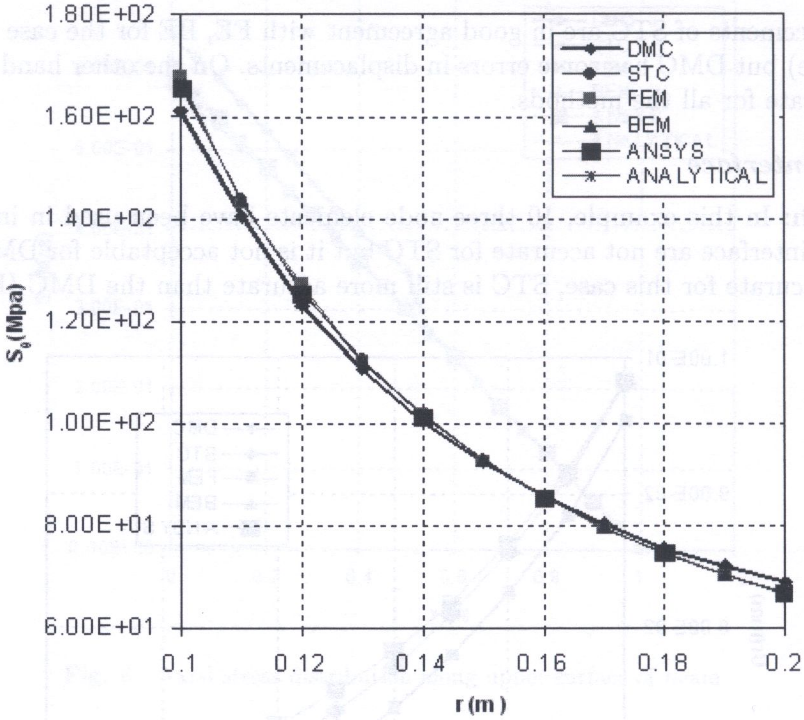

Fig. 12. Hoop stress distribution vs radial distance for inclined interface coupling model

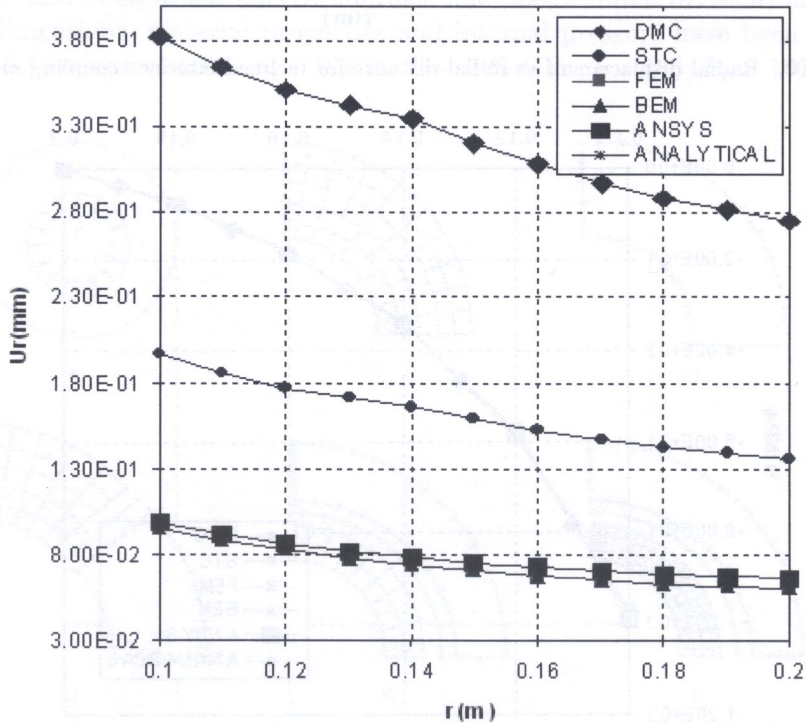

Fig. 13. Radial displacement vs radial distance for curved interface coarse coupling model

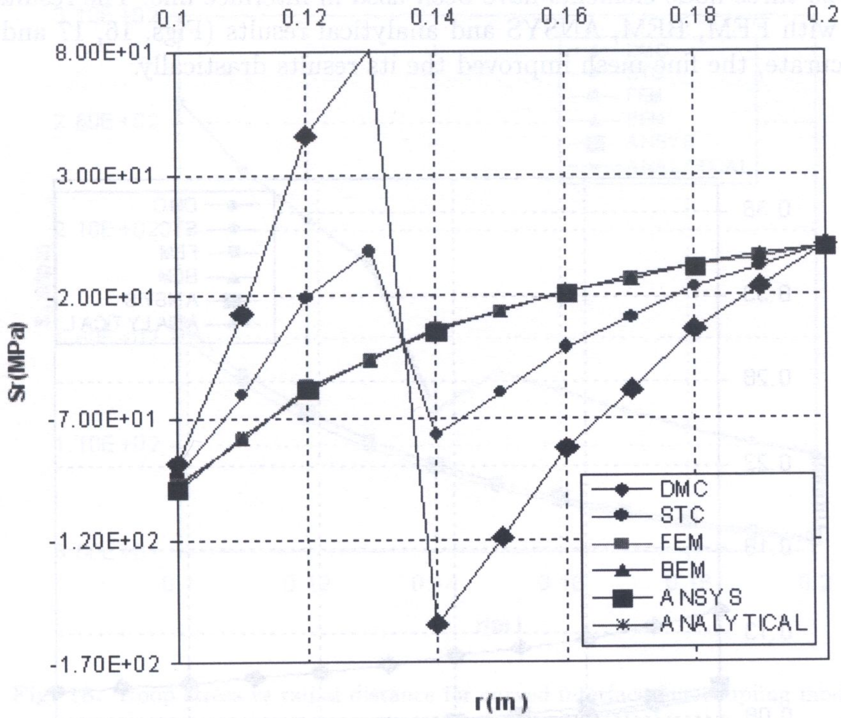

Fig. 14. Radial stress vs radial distance for curved interface coarse coupling model

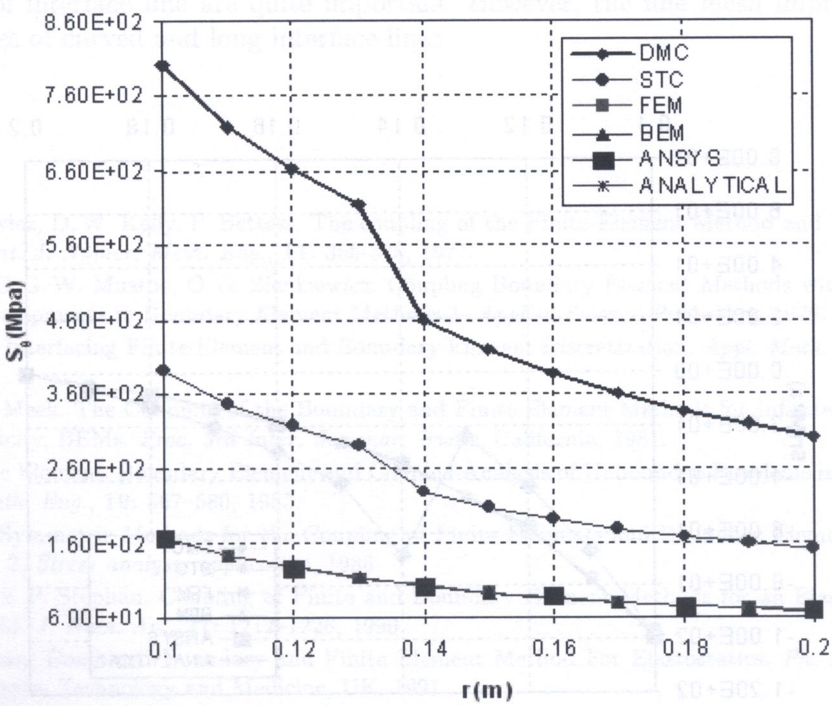

Fig. 15. Hoop stress vs radial distance for curved interface coarse coupling model

b) Fine Mesh: 20 three-node elements have been used in interface line. The results of STC are in good agreement with FEM, BEM, ANSYS and analytical results (Figs. 16, 17 and 18). Although DMC are not accurate, the fine mesh improved the its results drastically.

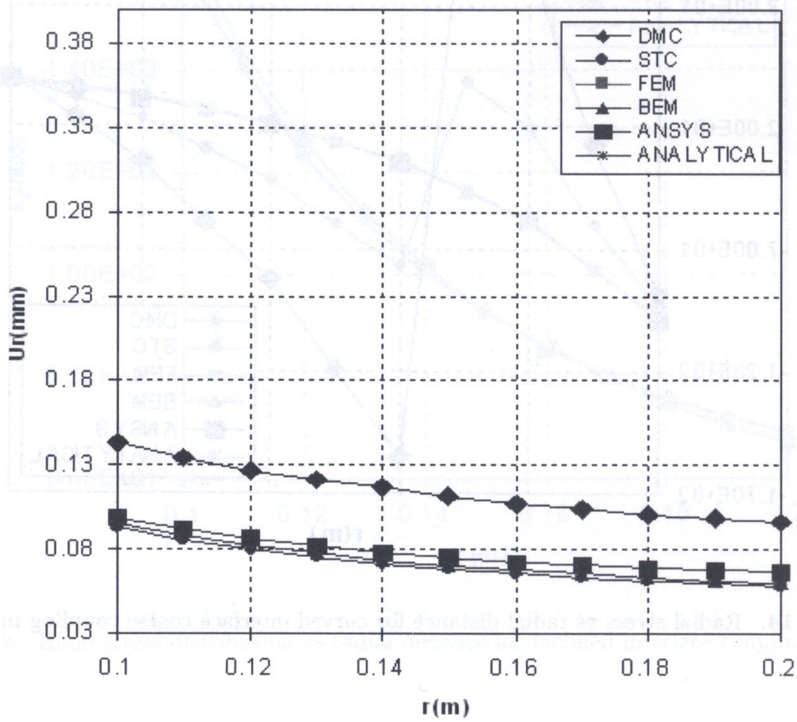

Fig. 16. Radial displacement vs radial distance for curved interface fine coupling model

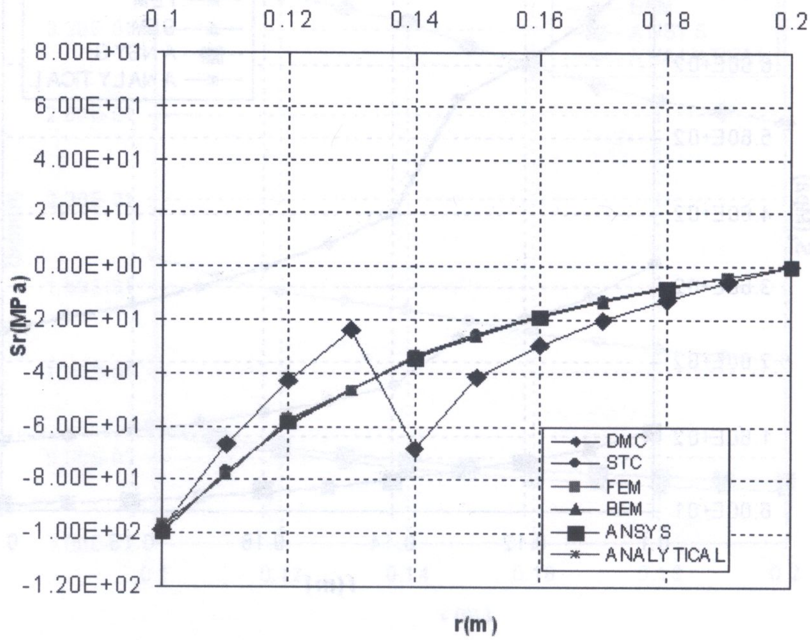

Fig. 17. Radial stress vs radial distance for curved interface fine coupling model

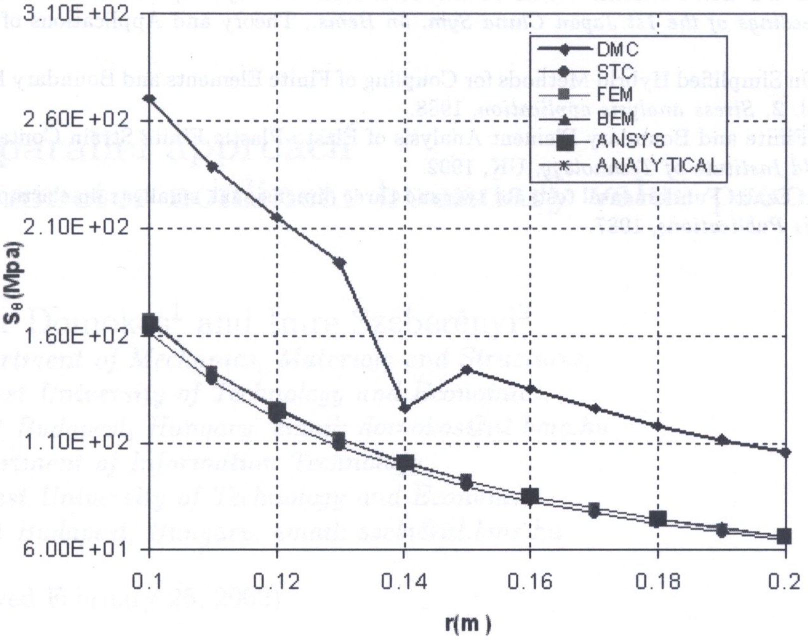

Fig. 18. Hoop stress vs radial distance for curved interface fine coupling model

5. Conclusion

The results prove that the developed strategy using the stress-traction equilibrium approach is more efficient and accurate than the distribution matrix approach even with the difficult cases. This approach makes the coupling process more powerful and efficient. The cases show that the length and the shape of interface line are quite important. However, the fine mesh improves the results even for the cases of curved and long interface lines.

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