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Equations of motion of serial chains in spatial motion using a recursive algorithm

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In the present study, a recursive algorithm for generating the equations of motion of serial chains that undergo spatial motion is presented. The method is based on treating each rigid body as a collection of constrained particles. Then, the force and moment equations are used to generate the rigid body equations of motion in terms of the Cartesian coordinates of the dynamically equivalent constrained system of particles, without introducing any rotational coordinates and the corresponding rotation matrices. For the open loop case, the equations of motion are generated recursively along the serial chains. Closed loop systems are transformed to open loop systems by cutting suitable kinematic joints and introducing cut-joint constraints. The method is simple and suitable for computer implementation. An example is chosen to demonstrate the generality and simplicity of the developed formulation.

Keywords: multibody system dynamics, equations of motion, system of rigid bodies, mechanisms, machine theory

Notations

- $d_{i,j}$ distance between points i and j,
- $\mathbf{G}_1$ vector sum of the moments of the external forces and force couples acting on the body with respect to particle 1,
- $I_{\xi\xi}$, $I_{\eta\eta}$, $I_{\zeta\zeta}$ moments of inertia of the body with respect to the body attached coordinate frame,
- $I_{\xi\eta}$, $I_{\xi\zeta}$, $I_{\zeta\eta}$ products of inertia of the body with respect to the body attached coordinate frame,
- m mass of the body,
- m_i mass of particle i,
- $m_{i,j}$ mass of the secondary particle that is located between the primary particles i and j ($m_{1,2} = m_{2,1} = m_5, \ldots$ etc.),
- $\mathbf{r}_G$ the position vector of the centre of mass of the body with respect to the body attached coordinate frame,
- $\mathbf{r}_i$, $\dot{\mathbf{r}}_i$, $\ddot{\mathbf{r}}_i$ position, velocity, and acceleration vectors of particle i with respect to the body attached coordinate frame respectively,
- $\mathbf{r}_{i,j}$, $\dot{\mathbf{r}}_{i,j}$, $\ddot{\mathbf{r}}_{i,j}$ relative position, velocity, and acceleration vectors between particles i and j,
- $\mathbf{r}_i^T$, $\mathbf{r}_j$ algebraic notation denotes the dot product operation $(\mathbf{r}_i, \mathbf{r}_j)$,
- $\tilde{\mathbf{r}}_i$, $\mathbf{r}_j$ algebraic notation denotes the cross product operation $(\mathbf{r}_i \times \mathbf{r}_j)$,
- $\mathbf{R}$ vector sum of the external forces acting on the rigid body,
- ξ_i, η_i, ζ_i coordinates of particle i with respect to the body attached coordinate frame.

1. Introduction

There are different formulations for the dynamic analysis of spatial mechanisms which vary in the system of coordinates used and in the way they introduce kinematic constraint equations [1–4]. Each formulation has its own advantages and disadvantages depending on the application and the needs. Some formulations are developed using a two-step transformation in which one system of coordinates is used in the first step and, consequently, a transformation to another system of coordinates is carried out in the second step. Such a transformation process is expected to help gaining advantages of both systems of coordinates. One method [5, 6] uses initially the absolute coordinate formulation where the location of each rigid body in the system is described in terms of a set of translational and rotational coordinates. However, this formulation has the disadvantage of the large number of coordinates defined. Then, the equations of motion are expressed in terms of the relative joint variables which determine the location of each body with respect to the adjacent body and they depend on the type of the kinematic joint connecting the two bodies. Another method uses initially the point coordinate formulation which is originated from the natural coordinates formulation developed by Garcia de Jalon *et al.* [7–9]. In the point coordinate formulation a dynamically equivalent constrained system of particles replaces the rigid bodies [10–12] and the global motion of the constrained system of particles together with the constraints imposed upon them represent both the translational and rotational motions of the rigid body. The external forces and couples acting on the body are distributed over the system of particles. Then, the equations of motion that are expressed in terms of the Cartesian coordinates of the particles are transformed in terms of relative joint variables [10–12]. The main disadvantage of these two-step transformation is the necessity to transform at every time step from the joint variables to the original system which is computationally inefficient.

A recursive dynamical formulation for the dynamic analysis of planar mechanisms with only revolute joints is presented [13]. The method rests upon the idea of replacing the rigid body with a dynamically equivalent constrained system of particles and then uses the concepts of linear and angular momentum to write the rigid body dynamical equations. The method has many advantages such as the absence of the rotational coordinates, the elimination of the necessity to redistribute the external forces and couples over the particles, and the reduced system of differential-algebraic equations. The method can be applied to recursively generate the equations of motion for open and/or closed loop systems.

In this paper, a recursive method for the dynamic analysis of mechanical systems that undergo spatial motion and constitute of open and/or closed loop systems is presented. The method is based upon the idea of treating each rigid body as a collection of constrained particles discussed in [10–12] with essential modifications and improvements. The force and moment equations are used to formulate the rigid body dynamical equations. However, they are expressed in terms of the rectangular Cartesian coordinates of the equivalent constrained system of particles. Some useful geometrical relationships are used to obtain a reduced dynamically equivalent constrained system of particles. This groups the advantages of the automatic elimination of the unknown internal forces as in Newton–Euler formulation which results in a reduced system of differential-algebraic equations. In addition to that, it expresses the general motion of the rigid body in terms of a set of Cartesian coordinates without introducing any rotational coordinates and the corresponding rotation matrices. Also, it eliminates the necessity of distributing the external forces and couples over the particles. Geometric constraints that fix the distances between the particles are introduced while the kinematic constraints and the associated constraint forces are automatically eliminated by properly selecting the locations of the particles.

For the open loop case, the equations of motion are generated recursively along the open chains instead of the matrix formulation derived in [10–11]. Most of the kinematic constraints due to the kinematic joints are automatically eliminated by properly locating the equivalent particles. For the closed loop case, the system is transformed to open loops by cutting suitable kinematic joints and introducing the cut-joint kinematic constraints. An example is chosen to demonstrate the generality and simplicity of the proposed method.

2. THE DYNAMIC FORMULATION

2.1. Construction of the equivalent system of particles

The rigid body and its dynamically equivalent constrained system of particles should have the same mass, the same position of the centre of mass and the same inertia tensor with respect to a body attached coordinate frame which results in ten conditions. A system of ten particles is chosen to replace the rigid body with spatial mass distribution, as shown in Fig. 1. It should be pointed out that only four particles 1, ..., 4, which are denoted as primary particles, can dynamically replace the spatial rigid body [10]. However, additional six particles 5, ..., 10, which are denoted as the secondary particles, each is located at the midpoint between a pair of primary particles.

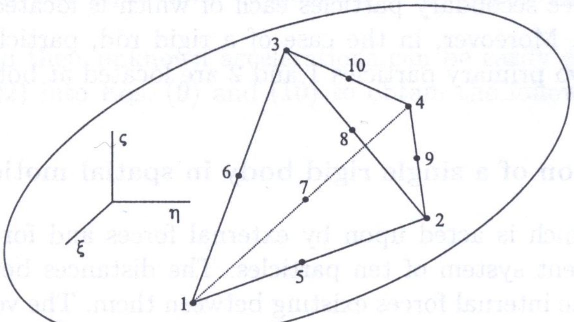

Fig. 1. The rigid body with the equivalent system of 10 particles

The reason for introducing the secondary particles is to allow the solution of ten linear algebraic equations in ten unknown masses for the particles and avoid the solution of nonlinear equations due to the quadratic form of the second moments in the coordinates of the particles. Also, it gives the freedom of possitioning the primary particles on the bodies in accordance with the joints that connect the bodies in order to reduce the number of the primary particles and eliminate some geometric and kinematic constriants. The mass distributions to points must satisfy the following conditions

$$m = \sum_{i=1}^{10} m_i, \tag{1}$$

$$m\mathbf{r}_G = \sum_{i=1}^{10} m_i \mathbf{r}_i, \tag{2}$$

$$I_{\xi\xi} = \sum_{i=1}^{10} m_i(\zeta_i^2 + \eta_i^2), \tag{3}$$

$$I_{\eta\eta} = \sum_{i=1}^{10} m_i(\xi_i^2 + \zeta_i^2), \tag{4}$$

$$I_{\zeta\zeta} = \sum_{i=1}^{10} m_i(\xi_i^2 + \eta_i^2), \tag{5}$$

$$I_{\xi\eta} = \sum_{i=1}^{10} m_i \xi_i \eta_i, \tag{6}$$

$$I_{\xi\zeta} = \sum_{i=1}^{10} m_i \xi_i \zeta_i, \tag{7}$$

$$I_{\zeta\eta} = \sum_{i=1}^{10} m_i \zeta_i \eta_i. \tag{8}$$

Equations (1)–(8) represent 10 linear algebraic equations in 10 unknown masses of the primary and secondary particles. At the same time, the coordinates of the particles can be chosen arbitrary which gives the advantage of the automatic elimination of the kinematic constraints due to some mechanical joints and allows for the two adjacent rigid bodies to contribute to the mass concentrated at the joint connecting them which reduces the total number of particles replacing the whole system.

It should be pointed out that, in the case of spatial motion of a body with planar mass distribution, particle 4 can be conveniently chosen to coincide with particle 3 and consequently the associated secondary particles 8, 9, and 10 can be eliminated. Therefore in this case, we have three primary particles and three secondary particles each of which is located at the mid-point of every pair of primary particles. Moreover, in the case of a rigid rod, particle 3 may be located at the center of the rod while two primary particles 1 and 2 are located at both ends of the rod.

2.2. Equations of motion of a single rigid body in spatial motion

Consider a rigid body which is acted upon by external forces and force couples. The rigid body is replaced by an equivalent system of ten particles. The distances between the ten particles are invariants as a result of the internal forces existing between them. The vector sum of these unknown internal forces or also the vector sum of their moments about any point equals zero by the law of action and reaction [14]. Then, the force equation for the whole system of particles yields,

$$\mathbf{R} = \sum_{i=1}^{10} m_i \ddot{\mathbf{r}}_i. \tag{9}$$

Also, the moment equation for the whole system of particles with respect to particle 1 results in [14]

$$\mathbf{G}_1 = \sum_{i=2}^{10} m_i \mathbf{r}_{i,1} \times \ddot{\mathbf{r}}_i = \sum_{i=2}^{10} m_i \widetilde{\mathbf{r}}_{i,1} \ddot{\mathbf{r}}_i. \tag{10}$$

The distance constraints between the ten particles are given as

$$\mathbf{r}_{2,1}^T \mathbf{r}_{2,1} - d_{2,1}^2 = 0, \tag{11}$$
$$\mathbf{r}_{3,1}^T \mathbf{r}_{3,1} - d_{3,1}^2 = 0, \tag{12}$$
$$\mathbf{r}_{3,2}^T \mathbf{r}_{3,2} - d_{3,2}^2 = 0, \tag{13}$$
$$\mathbf{r}_{4,1}^T \mathbf{r}_{4,1} - d_{4,1}^2 = 0, \tag{14}$$
$$\mathbf{r}_{4,2}^T \mathbf{r}_{4,2} - d_{4,2}^2 = 0, \tag{15}$$
$$\mathbf{r}_{3,4}^T \mathbf{r}_{3,4} - d_{3,4}^2 = 0, \tag{16}$$
$$\mathbf{r}_5 - (\mathbf{r}_1 + \mathbf{r}_2)/2 = 0, \tag{17}$$
$$\mathbf{r}_6 - (\mathbf{r}_1 + \mathbf{r}_3)/2 = 0, \tag{18}$$
$$\mathbf{r}_7 - (\mathbf{r}_2 + \mathbf{r}_3)/2 = 0, \tag{19}$$
$$\mathbf{r}_8 - (\mathbf{r}_1 + \mathbf{r}_4)/2 = 0, \tag{20}$$
$$\mathbf{r}_9 - (\mathbf{r}_2 + \mathbf{r}_4)/2 = 0, \tag{21}$$
$$\mathbf{r}_{10} - (\mathbf{r}_3 + \mathbf{r}_4)/2 = 0. \tag{22}$$

The equations of motion (9), (10) and (11)–(22) represent a system of differential-algebraic equations that can be solved to determine the unknown acceleration vectors of the particles at any instant of time.

It should be noted that, in the case of a spatial rigid body with planar mass distribution, the summation in Eqs. (9) and (10) is up to 6 instead of 10 and the geometric constraint Eqs. (14)–(16) and (20)–(22) are eliminated. While secondary particles 4, 5, and 6 replace 5, 6, and 7 in Eqs. (17)–(19). In the case of a rigid rod, the geometric constraint Eqs. (12)–(16) and (18)–(22) are eliminated and the summation in Eqs. (9) and (10) is up to 3 instead of 10. Also, secondary particle 3 replaces 5 in Eq. (17).

2.3. The reduced form of the equations of motion of a single rigid body

The secondary particles and their unknown accelerations can be easily eliminated by substituting the constraint Eqs. (17)–(22) into Eqs. (9) and (10) to obtain the following reduced form for the equations of motion

$$\mathbf{R} = \sum_{i=1}^{4} \overline{m}_i \ddot{\mathbf{r}}_i, \tag{23}$$

$$\mathbf{G}_1 = \sum_{i=1}^{4} \mathbf{A}_i \ddot{\mathbf{r}}_i, \tag{24}$$

where

$$\overline{m}_i = m_i + \sum_{\substack{j=1\\ j\neq i}}^{4} \frac{1}{2} m_{i,j}, \tag{25}$$

$$\mathbf{A}_i = \overline{\overline{m}}_i \widetilde{\mathbf{r}}_{i,1} + \sum_{\substack{j=2\\ j\neq i}}^{4} \frac{1}{4} m_{i,j} \widetilde{\mathbf{r}}_{j,1}, \tag{26}$$

$$\overline{\overline{m}}_i = m_i + \sum_{\substack{j=1\\ j\neq i}}^{4} \frac{1}{4} m_{i,j}, \tag{27}$$

$m_{i,j}$ – mass of the secondary particle that is located between the primary particles i and j ($m_{1,2} = m_{2,1} = m_5 \ldots$ etc.), $\mathbf{r}_{i,1}, \dot{\mathbf{r}}_{i,1}, \ddot{\mathbf{r}}_{i,1}$ – relative position, velocity, and acceleration vectors between particles i and 1.

Then, Eqs. (23) and (24) in addition to the remaining constraint Eqs. (11)–(16) represent the equations of motion for a single floating rigid body in spatial motion where only the primary particles stay. They can be solved at every time step to determine the unknown acceleration components of the primary particles 1, 2, 3, and 4. The acceleration components of the particles are integrated numerically knowing their Cartesian coordinates and velocities at a certain time to determine the positions and velocities for the next time step. Gear's method [15] for the numerical integration of differential-algebraic equations is used to overcome the instability problem resulting during the modelling process of constraint mechanical systems due to the constraint violation problem. The rectilinear motion of the particles determines completely the translational and rotational motion of the rigid body. If the rigid body is rotating about a fixed point, then particle 1 may be located at the centre of this joint. In this case, Eq. (24) and Eqs. (11)–(16) are used to solve for the unknown Cartesian accelerations of particles 2, 3, and 4. Therefore, Eq. (23) can be solved to determine the unknown reaction forces at the joint N_1 as,

$$N_1 = \sum_{i=1}^{4} \overline{m}_i \ddot{\mathbf{r}}_i - \mathbf{R}. \tag{28}$$

If the rigid body is rotating about a fixed axis, then particles 1 and 2 can be located along the axis of the joint to define its direction. To solve for the unknown acceleration vectors of particles 3 and 4, the constraint Eqs. (12)–(16) can be used in addition to one scalar moment equation that is generated by taking the projection of the vectors in Eq. (24) along the direction of the axis of the revolute joint. Then, Eq. (28) may be used to get the reactions at the axis of the revolute joint.

2.4. Equations of motion of a serial chain of rigid bodies

Figure 2 shows a serial chain of N rigid bodies connected by spherical joints with the equivalent system of $(3N+1)$ particles where connected particles are unified from both bodies.

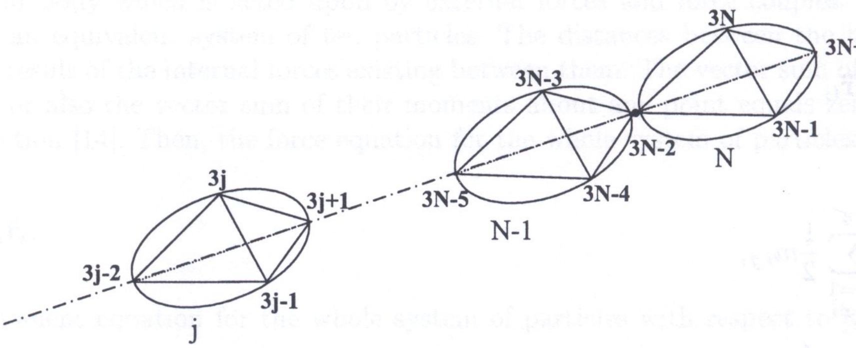

Fig. 2. Serial chain of N rigid bodies with the equivalent system of primary particles

For the last body "N" in the chain, the equations of motion are derived in a similar way as Eq. (24) and Eqs. (11)–(16) of a single rigid body. The moment equation takes the form

$$\mathbf{G}_{3N-2} = \sum_{i=3N-2}^{3N+1} \mathbf{A}_i \ddot{\mathbf{r}}_i, \tag{29}$$

where

$$\mathbf{A}_{3N} = \overline{m}_{3N} \mathbf{r}_{3N,3N-2} + \sum_{\substack{i=3N-1 \\ i \neq 3N}}^{3N+1} \frac{1}{4} m_{3N,i} \mathbf{r}_{i,3N-2},$$

$$\overline{m}_{3N} = m_{3N} + \sum_{\substack{i=3N-2 \\ i \neq 3N}}^{3N+1} \frac{1}{4} m_{3N,i},$$

where $\mathbf{G}_{3N-2}$ is the sum of the moments of the external forces and force couples acting on body N with respect to the location of particle $3N-2$. The acceleration equations of the distance constraints between primary particles belonging to body N are given as

$$\mathbf{r}^T_{3N-2,3N-1}\ddot{\mathbf{r}}_{3N-2} + \mathbf{r}^T_{3N-1,3N-2}\ddot{\mathbf{r}}_{3N-1} = -\dot{\mathbf{r}}^T_{3N-1,3N-2}\dot{\mathbf{r}}_{3N-1,3N-2}, \tag{30}$$

$$\mathbf{r}^T_{3N-2,3N}\ddot{\mathbf{r}}_{3N-2} + \mathbf{r}^T_{3N,3N-2}\ddot{\mathbf{r}}_{3N} = -\dot{\mathbf{r}}^T_{3N,3N-2}\dot{\mathbf{r}}_{3N,3N-2}, \tag{31}$$

$$\mathbf{r}^T_{3N-2,3N+1}\ddot{\mathbf{r}}_{3N-2} + \mathbf{r}^T_{3N+1,3N-2}\ddot{\mathbf{r}}_{3N+1} = -\dot{\mathbf{r}}^T_{3N+1,3N-2}\dot{\mathbf{r}}_{3N+1,3N-2}, \tag{32}$$

$$\mathbf{r}^T_{3N-1,3N}\ddot{\mathbf{r}}_{3N-1} + \mathbf{r}^T_{3N,3N-1}\ddot{\mathbf{r}}_{3N} = -\dot{\mathbf{r}}^T_{3N,3N-1}\dot{\mathbf{r}}_{3N,3N-1}, \tag{33}$$

$$\mathbf{r}^T_{3N-1,3N+1}\ddot{\mathbf{r}}_{3N-1} + \mathbf{r}^T_{3N+1,3N-1}\ddot{\mathbf{r}}_{3N+1} = -\dot{\mathbf{r}}^T_{3N+1,3N-1}\dot{\mathbf{r}}_{3N+1,3N-1}, \tag{34}$$

$$\mathbf{r}^T_{3N,3N+1}\ddot{\mathbf{r}}_{3N} + \mathbf{r}^T_{3N+1,3N}\ddot{\mathbf{r}}_{3N+1} = -\dot{\mathbf{r}}^T_{3N+1,3N}\dot{\mathbf{r}}_{3N+1,3N}. \tag{35}$$

Addition of one more body in the chain leads to the inclusion of an angular momentum vector equation that takes into consideration the contributions of all the ascending bodies in the chain together with three distance constraint equations between the particles belonging to this body. These six scalar equations are appended to the equations of motion derived for the leading bodies in the chain. For body j, the appended equations of motion take the form

$$\mathbf{G}_{3j-2} = \sum_{k=j}^{N} \sum_{i=3k-2}^{3k+1} \mathbf{A}_i \ddot{\mathbf{r}}_i, \tag{36}$$

where

$$\mathbf{A}_{3k} = \overline{m}_{3k}\widetilde{\mathbf{r}}_{3k,3j-2} + \sum_{\substack{i=3k-1\\ i\neq 3k}}^{3k+1} \frac{1}{4} m_{3k,i}\widetilde{\mathbf{r}}_{i,3j-2},$$

$$\overline{m}_{3k} = m_{3k} + \sum_{\substack{i=3k-2\\ i\neq 3k}}^{3k+1} \frac{1}{4} m_{3k,i}.$$

$$\mathbf{r}^T_{3j-2,3j-1}\ddot{\mathbf{r}}_{3j-2} + \mathbf{r}^T_{3j-1,3j-2}\ddot{\mathbf{r}}_{3j-1} = -\dot{\mathbf{r}}^T_{3j-1,3j-2}\dot{\mathbf{r}}_{3j-1,3j-2}, \tag{37}$$

$$\mathbf{r}^T_{3j-2,3j}\ddot{\mathbf{r}}_{3j-2} + \mathbf{r}^T_{3j,3j-2}\ddot{\mathbf{r}}_{3j} = -\dot{\mathbf{r}}^T_{3j,3j-2}\dot{\mathbf{r}}_{3j,3j-2}, \tag{38}$$

$$\mathbf{r}^T_{3j-2,3j+1}\ddot{\mathbf{r}}_{3j-2} + \mathbf{r}^T_{3j+1,3j-2}\ddot{\mathbf{r}}_{3j+1} = -\dot{\mathbf{r}}^T_{3j+1,3j-2}\dot{\mathbf{r}}_{3j+1,3j-2}, \tag{39}$$

$$\mathbf{r}^T_{3j-1,3j}\ddot{\mathbf{r}}_{3j-1} + \mathbf{r}^T_{3j,3j-1}\ddot{\mathbf{r}}_{3j} = -\dot{\mathbf{r}}^T_{3j,3j-1}\dot{\mathbf{r}}_{3j,3j-1}, \tag{40}$$

$$\mathbf{r}^T_{3j-1,3j+1}\ddot{\mathbf{r}}_{3j-1} + \mathbf{r}^T_{3j+1,3j-1}\ddot{\mathbf{r}}_{3j+1} = -\dot{\mathbf{r}}^T_{3j+1,3j-1}\dot{\mathbf{r}}_{3j+1,3j-1}, \tag{41}$$

$$\mathbf{r}^T_{3j,3j+1}\ddot{\mathbf{r}}_{3j} + \mathbf{r}^T_{3j+1,3j}\ddot{\mathbf{r}}_{3j+1} = -\dot{\mathbf{r}}^T_{3j+1,3j}\dot{\mathbf{r}}_{3j+1,3j}, \tag{42}$$

and where $\mathbf{G}_{3j-2}$ is the sum of the moments of the external forces and force couples acting on the chain starting from body j up till the last body N with respect to the location of particle $3j-2$.

Equations (36) and (37)–(42) represent 9 linear algebraic equations that can be solved for the accelerations of particles $3j-1, 3j$, and $3j+1$. For the unknown accelerations of particle $3j-2$, if body "j" is the floating base body in the chain then, three force equations, similar to Eq. (23), are required to solve for the unknown acceleration components of particle $3j-2$. These force equations equate the sum of the external forces acting on all the bodies in the chain to the time rate of change of the vectors of linear momentum of all the equivalent particles that replace the chain which take the form

$$\sum_{k=j}^{N} \mathbf{R}_k = \sum_{k=j}^{N} \sum_{i=3k-2}^{3k} \overline{m}_i \ddot{\mathbf{r}}_i + \overline{m}_{3N+1}\ddot{\mathbf{r}}_{3N+1}, \tag{43}$$

where

$$\overline{m}_{3k} = m_{3k} + \sum_{\substack{i=3k-2\\ i\neq 3k}}^{3k+1} \frac{1}{2} m_{3k,i}.$$

If body j is connected to body $j-1$ by a revolute joint, then we take the projection of all the moment vectors in Eq. (36) along the axis of the joint which is defined by two particles from both bodies that are commonly located on it. Five additional distance constraints, that fix the distances between the third and fourth particles and between each of them and the other two particles along the axis of the joint, together with the moment equation can be used to solve for the acceleration vectors of the third and fourth particles on body j.

In general, for a serial chain of N bodies, an equivalent system of $3N+1$ primary particles and $6N$ secondary particles is first constructed. Then, by eliminating all the secondary particles, we are left with $3N+1$ particles and consequently, $9N+3$ unknown acceleration components. To solve for these unknowns, $3N$ moment equations can be generated recursively along the chain together with $6N$ distance constraints between the particles located on each body, in case of all are spherical joints. In the case of a revolute joint, one scalar moment equation and five distance constraints are used. Finally, three force equations can be used to solve for the unknown acceleration components of particle 1 if body 1 is floating or for the unknown reaction forces if there is a fixation at point 1.

If bodies "j" and "$j-1$" in a serial chain are connected by a prismatic joint, then particles $3j-5, 3j-4, 3j-3$, and $3j-2$ are located on body "$j-1$" while particles $3j-1, 3j, 3j+1$, and $3j+2$ are assigned to body "j'. Particles $3j-5$ and $3j-2$ on body "$j-1$" and particles $3j-1$ and $3j+2$ on body "j" are arbitrarily located along the axis of the prismatic joint. To obtain the equations of motion for body "j", one force equation can be written by taking the projection of all the vectors in Eq. (36) along the axis of the prismatic joint together with the distance constraint Eqs. (37)–(42). Moreover, five independent kinematic constraint equations associated with the prismatic joint are included and take the form,

$$(\mathbf{r}_{3j-5} - \mathbf{r}_{3j-2}) \times (\mathbf{r}_{3j-1} - \mathbf{r}_{3j+2}) = 0, \tag{44a}$$

$$(\mathbf{r}_{3j-5} - \mathbf{r}_{3j-2}) \times (\mathbf{r}_{3j-1} - \mathbf{r}_{3j-2}) = 0, \tag{44b}$$

$$\left(\mathbf{r}_{3j-4,3j-5} - \frac{\mathbf{r}^T_{3j-4,3j-5}\,\mathbf{r}_{3j-2,3j-5}}{|\mathbf{r}_{3j-2,3j-5}|}\right)^T \left(\mathbf{r}_{3j,3j-1} - \frac{\mathbf{r}^T_{3j,3j-1}\,\mathbf{r}_{3j+2,3j-1}}{|\mathbf{r}_{3j+2,3j-1}|}\right) = 0. \tag{44c}$$

Therefore, for a preceeding body "h" in the chain the moment equation is generated recursively along the serial chain as addressed above which take the form,

$$\mathbf{G}_{3h-2} = \sum_{k=h}^{j-1} \sum_{i=3k-2}^{3k+1} \mathbf{A}_i \ddot{\mathbf{r}}_i + \sum_{k=j}^{N} \sum_{i=3k-1}^{3k+2} \mathbf{A}_i \ddot{\mathbf{r}}_i, \tag{45}$$

where $\mathbf{G}_{3h-2}$ is the sum of the moments of the external forces and force couples acting on the chain starting from body h up till the last body N with respect to the location of particle $3h-2$.

If body "h" is the floating base body in the chain, then a force equation, similar to Eq. (43), is written to solve for the unknown acceleration of particle 1 in the form,

$$\sum_{k=h}^{N} \mathbf{R}_k = \sum_{k=h}^{j-1} \sum_{i=3k-2}^{3k} \overline{m}_i \ddot{\mathbf{r}}_i + \sum_{k=j}^{N} \sum_{i=3k-1}^{3k+1} \overline{m}_i \ddot{\mathbf{r}}_i + \overline{m}_{3j-2} \ddot{\mathbf{r}}_{3j-2} + \overline{m}_{3N+2} \ddot{\mathbf{r}}_{3N+2}. \tag{46}$$

Similar treatment can be used in dealing with all other kinds of lower or higher-pair kinematic joints.

In the case of a multi-branch open and/or closed loop system, it can be transformed to a system of serial chains by cutting suitable joints and consequently cut-joint constraints are introduced. Equivalent particles are conveniently chosen to locate at the positions of the connection joints and in terms of their Cartesian coordinates the cut-joint constraint equations are easily formulated. These kinematic constraints substitute for the unknown constraint reaction forces that appear explicitly in the force and moment equations.

3. Dynamic analysis of the double wishbone suspension

Figure 3 presents a quarter car with the double wishbone suspension system. The system is an example of a closed-chain with revolute and spherical joints exist. The mechanical system consists of a main chassis, a double wishbone suspension sub-system, a steering rod, and a wheel. A suspension spring and a shock absorber are included in the suspension sub-system. The system constitutes two closed loops, one due to the four-bar linkage, and the other due to the steering rod. The chassis is constrained to move vertically upward or downward, which can be modelled as a translational joint with axis vertical. The wheel is analytically modelled as a linear translational spring with damping characteristics. The system has three degrees of freedom: the chassis has one degree of freedom due to the vertical motion, the double A-arm suspension has only one degree of freedom due to the steering rod constraint, and the wheel has one degree of freedom corresponding to the rolling motion. The inertia characteristics of the rigid bodies are presented in Table 1. The characteristics of the suspension springs and dampers, and the wheel are presented in Tables 2 and 3 respectively.

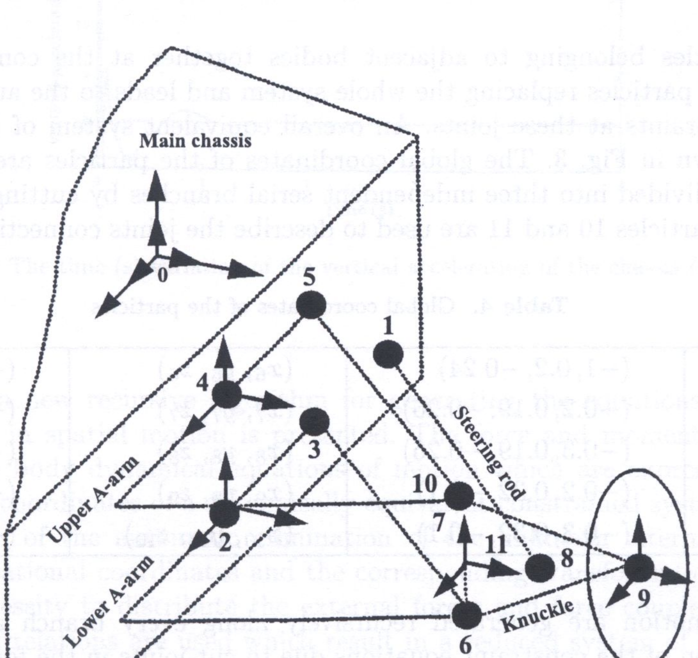

Fig. 3. Schematic diagram of the double A-arm suspension with the primary particles and the body attached coordinate frames

Each rigid body is replaced by an equivalent system of particles. The chassis is replaced by the four primary particles representation. The knuckle and the A-arms each is replaced by the three primary particles representation. The steering rod is modelled with a system of two primary particles.

Table 1. Description of the rigid bodies

Body #	Description	Mass (Kg)	Interia (Kg · m^2) $\xi\xi$, $\eta\eta$, $\zeta\zeta$, $\eta\zeta$, $\xi\zeta$, $\xi\eta$
1	Main chassis	300.0	130, 250, 200, 0, 0, 0
2	Lower *A*-arm	1.0	0.028, 0.002, 0.03, 0, 0, 0
3	Upper *A*-arm	1.0	0.028, 0.002, 0.03, 0, 0, 0
4	Knuckle	8.0	1.6, 1.6, 1.6, 0, 0, 0
5	Wheel	22.0	2.0, 1.35, 2.0, 0, 0, 0

Table 2. The characteristics of the suspension springs and dampers

No.	Connected bodies	K (N/m)	D (N sec/m)	l_0 (m)
1	(1,2)	5.11E+04	1.44E+04	0.345

Table 3. The characteristics of the wheels

Radius	0.35 m
Stiffness	0.3E+06 N/m
Damping Coefficient	0.1E+04 N sec/m

Locating the particles belonging to adjacent bodies together at the connection joints reduces the total number of particles replacing the whole system and leads to the automatic elimination of the kinematic constraints at these joints. An overall equivalent system of 11 primary particles is constructed as shown in Fig. 3. The global coordinates of the particles are presented in Table 4. The mechanism is divided into three independent serial branches by cutting the joints at points 7 and 8. Additional particles 10 and 11 are used to describe the joints connecting separated branches.

Table 4. Global coordinates of the particles

(x_1, y_1, z_1)	$(-1, 0.2, -0.24)$	(x_6, y_6, z_6)	$(-0.2, 0.61, -0.26)$
(x_2, y_2, z_2)	$(-0.2, 0.19, -0.26)$	(x_7, y_7, z_7)	$(-0.2, 0.62, -0.24)$
(x_3, y_3, z_3)	$(-0.3, 0.19, -0.26)$	(x_8, y_8, z_8)	$(-0.2, 0.58, -0.1)$
(x_4, y_4, z_4)	$(-0.2, 0.32, -0.1)$	(x_9, y_9, z_9)	$(-0.2, 0.58, -0.1)$
(x_5, y_5, z_5)	$(-0.3, 0.32, -0.1)$	(x_{10}, y_{10}, z_{10})	$(-0.2, 0.67, -0.24)$

The equations of motion are generated recursively along every branch as discussed in Sec. 4 with the introduction of the constraint equations due to cut-joints in the form,

$$\mathbf{r}_7 - \mathbf{r}_{10} = \mathbf{0}, \qquad \mathbf{r}_8 - \mathbf{r}_{11} = \mathbf{0}.$$

The above equations of motion are used to simulate the free response of the system from the rest position. Figures 4 and 5 present the time variations of the vertical displacement and acceleration of the center of the chassis respectively. Initially, the chassis is subjected to impulsive forces which result in sudden changes in the acceleration of its center. During time progression, due to the motion of the elements of the system, the variable force elements start imposing constraint forces. Then, the chassis undergoes damped oscillations up to the steady state. The comparison with DAP-3D program, which is based on the absolute coordinates [4], shows a complete agreement with the results of the simulation.

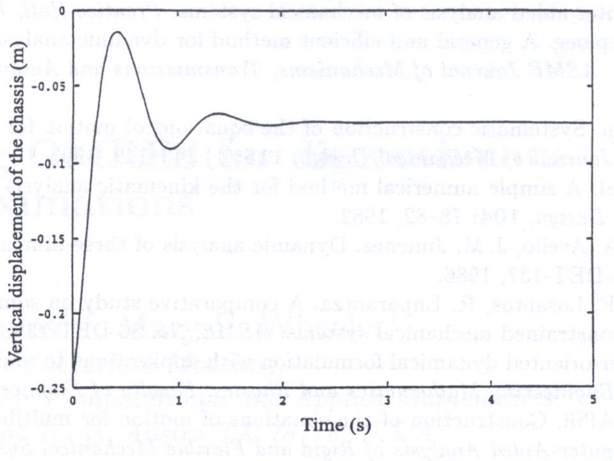

Fig. 4. The time (s) variation of the vertical displacement of the chassis (m)

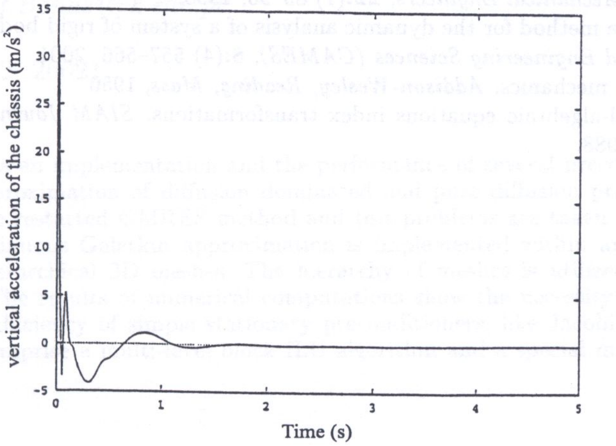

Fig. 5. The time (s) variation of the vertical acceleration of the chassis (m/s^2)

Conclusion

In the present work, a new recursive algorithm for generating the equations of motion of serial chains of rigid bodies in spatial motion is presented. The force and moment equations are used to formulate the rigid body dynamical equations of motion which are expressed in terms of the rectangular Cartesian coordinates of a dynamically equivalent constrained system of particles. This groups the advantages of the automatic elimination of the unknown internal constraint forces, the absence of any rotational coordinates and the corresponding transformation matrices, and the elimination of the necessity to distribute the external forces and force couples over the particles. Some useful geometric relations are used which result in a reduced system of differential-algebraic equations without introducing any rotational coordinates. The formulation can be applied to open and/or closed loop systems with the common types of kinematic joints.

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