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Quadrilateral folded plate structure elements of reduced Trefftz type

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General strategy for developing finite elements of general geometric shape explained on quadrilateral folded plate structure element ensuring invariance properties is presented in this paper. The basic idea of this strategy consists in using the natural coordinate system only for defining the element geometry and performing the element integration in a mapped biunit square. For defining the approximation functions a suitable local Cartesian coordinate system defined from the directions of the covariant base vectors and the perpendicular contravariant base vectors is used. The origin of the local coordinate system is located at the element centroid (centre of gravity).

Hybrid and boundary finite elements of reduced Trefftz type for analysing the folded plate structures are also presented. The folded plate structure element is a combination of a plate bending element and a plane stress element.

1. Introduction

The obstacles in formatting displacement and hybrid finite elements are reported in many publications [1–4]. Different suggestion has been made for selecting the element coordinate systems and constructing the approximation basis for various finite element types. For the hybrid quadrilateral Kirchhoff plate-bending element developed in [4], the element local coordinate system is selected in a skew (but not curvilinear) coordinate system located at the geometric center of the element and defined in the direction of the natural coordinate system. For the hybrid Trefftz elements developed in [3, 5], the origin of the local coordinate system is located at the element centroid and the displacement functions are assumed in those coordinates but it seems that no attention is paid to the selection of their direction. The invariance properties are preserved through constructing the approximation basis in the local coordinates divided by an average distance between element centroid and element corners. In general, the most widely used strategy in formatting finite elements of general geometric form depends on an approximation basis selected directly in a natural coordinate system [6–10].

In this paper a general strategy for developing finite elements of general geometric shape explained on quadrilateral folded plate structure element ensuring invariance properties is presented. The basic idea of this strategy consists in using the natural coordinate system only for defining the element geometry and performing the element integration in a mapped biunit square. For defining the approximation functions a suitable local Cartesian coordinate system defined from the directions of the covariant base vectors and the perpendicular contravariant base vectors is used. The origin of the local coordinate system is located at the element centroid (centre of gravity). Such strategy enhances basically the application of complex formulations like the Trefftz method in the framework of the finite element method and enables preserving invariance properties as well as insensitivity to nodal point numbering for finite elements of general geometric shapes; see for example [11–14].

A slightly modified geometrical interpolation technique for constructing the finite element shape functions is also presented. The displacement approximation basis starts with approximation func-

tions in parametric form involving a homogeneous part and a particular part. This modified technique leads to a constructed displacement basis with shape functions, which involve also in addition to the homogeneous part a particular part. The modified interpolation technique enables considering the effect of the external loading on the displacements at the finite element level [15–17].

The folded plate structure element is a combination of a plate bending element and a plane stress element.

Although the displacement approximation basis is constructed in the defined local Cartesian coordinate system the advantages of using the natural coordinate system are exploited. All quantities needed in the application are expressed in the natural coordinate system by using the isoparametric transformation.

The integration over the interval $[-1, +1]$ is used to derive the element matrices and load vectors. The integration can be performed either numerically or exactly. It is also not necessary to transform the energy expressions to the natural coordinate system in order to evaluate the element matrices and the load vectors. The energy expressions can be evaluated as they are related to the Cartesian coordinate system after replacing the Cartesian variables with their natural equivalents.

A cellular hollow box and shell roof with complex cross section geometry meshed by folded plate structure elements of Trefftz type are subjected to static and eigenvalue analysis. The results are compared with the results produced by hybrid and mixed hybrid elements developed in [18–22]. The results, obtained both by static and eigenvalue analysis, are of the same order.

The deformation of the cross section of the studied structures under external loading signifies the mechanical behavior of the cellular hollow box and the shell roof as surface structures. Such behavior can not be modeled by applying the beam theory. The mode shapes corresponding to the first four eigenvalues obtained by an eigenvalue analysis of the structures studied recognize also the same multi-dimensional behavior.

2. Defining the local coordinate systems

A general strategy for formatting finite elements of general geometric form should be presented by means of the quadrilateral plane stress element and Kirchhoff plate bending element shown in Fig. 1 and Fig. 2, respectively.

Various difficulties may be encountered in developing such element geometry forms especially in association with a Trefftz type approach. The first difficulty, encountered, consists in satisfying the requirement of the approach in constructing the approximation basis, so that the Lagrangian equation is a priori fulfilled. In addition, difficulties encountered in formatting finite elements of general geometric form in association with displacement and hybrid versions of the finite element method concerning convergence, invariance and nodal point numbering insensitivity are to be overcome. Further difficulties appear especially in applying complex formulations like the Trefftz method in the framework of finite element method for solving dynamic problems. In this case, approximation functions, which satisfy the Lagrangian equation in the dynamic case, are required. Such requirement can not be easily achieved in general for Lagrangian equations formulated in a natural coordinate system.

The use of a Cartesian coordinate system enhances basically the application of the Trefftz concept in the framework of the finite element method. For sake of further simplicity, it is also possible to use displacement approximation functions of a static case in applying a dynamic case together with variational concept involving the inertia forces. In such case, it is necessary to evaluate domain integrals, which means that the Trefftz approach is only used in a "static sense".

In order to circumvent the difficulties associated with the application of the Trefftz type approach and the formatting of finite elements of general geometric form for other versions of finite element methods (i.e. displacement and hybrid versions), the following steps of the general application procedure are adopted:

1. Beside global Cartesian coordinate system $(\tilde{x^1}, \tilde{x^2})$, parallel local coordinate system located at the element centroid $(\overset{*}{x^1}, \overset{*}{x^2})$ and the natural coordinate system (θ^1, θ^2), a suitable Cartesian coordinate system (x^1, x^2) located at the element centroid is defined in the following way:

Firstly, the coordinate system $(\overset{*}{x^1}, \overset{*}{x^2})$ is defined by translating the origin of the global coordinate system into the element centroid. The coordinates of an arbitrary point of the element related to system $(\overset{*}{x^1}, \overset{*}{x^2})$ with the unit vectors $(e_{\overset{*}{x^1}}, e_{\overset{*}{x^2}})$can be rewritten as:

$$\overset{*}{x^i} = \tilde{x^i} - \tilde{x^i_{(c)}}\,, \tag{1}$$

$\tilde{x^i}$ are the global Cartesian coordinates of the arbitrary point and $\tilde{x^i_{(c)}}$ are the global coordinates of the element centroid defined as

$$\tilde{x^i_{(c)}} = \int_A \tilde{x^i} \mathrm{d}A \Big/ \int_A \mathrm{d}A\,. \tag{2}$$

Now the following differential geometry properties of the element are defined [21]:

$$\begin{bmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \overset{*}{x^1_{(1)}} & \overset{*}{x^2_{(1)}} \\ \overset{*}{x^1_{(2)}} & \overset{*}{x^2_{(2)}} \\ \overset{*}{x^1_{(3)}} & \overset{*}{x^2_{(3)}} \\ \overset{*}{x^1_{(4)}} & \overset{*}{x^2_{(4)}} \end{bmatrix}. \tag{3}$$

The coordinates of an arbitrary point of the element can be written as follows

$$\overset{*}{x^1} = a_0 + a_1\theta^1 + a_2\theta^2 + a_3\theta^1\theta^2, \tag{3a}$$

$$\overset{*}{x^2} = b_0 + b_1\theta^1 + b_2\theta^2 + b_3\theta^1\theta^2. \tag{3b}$$

The covariant base vectors and the contravariant base vectors are orthogonal and their directions construct naturally a suitable Cartesian coordinate basis for defining the local coordinate system. Thus, the local coordinate system (x^1, x^2) or (x_1, x_2) with the unit vectors (e_{x^1}, e_{x^2}) or (e^{x^1}, e^{x^2}) is, for example, defined from the directions of the covariant base vectors and the perpendicular contravariant base vectors evaluated at the geometric centre of the element, where ($\theta^1 = 0$, $\theta^2 = 0$), as follows

$$\begin{bmatrix} e_{x^1} \\ e_{x^2} \end{bmatrix} = \frac{1}{\sqrt{2(s_1)^2(s_2)^2 + 2j_0 s_1 s_2}} \begin{bmatrix} a_1 s_2 + b_2 s_1 & b_1 s_2 - a_2 s_1 \\ a_2 s_1 - b_1 s_2 & a_1 s_2 + b_2 s_1 \end{bmatrix} \begin{bmatrix} e_{\overset{*}{x^1}} \\ e_{\overset{*}{x^2}} \end{bmatrix}, \tag{4}$$

$$s_1 = \sqrt{(a_1)^2 + (b_1)^2}\,; \qquad s_2 = \sqrt{(a_2)^2 + (b_2)^2}\,; \qquad j_0 = a_1 b_2 - a_2 b_1. \tag{5}$$

Note that a_1, b_1 and a_2, b_2 are the components of the covariant base vectors where b_2/j_0, $-a_2/j_0$ and $-b_1/j_0$, $-a_1/j_0$ are the components of the contravariant base vectors evaluated in the geometric center of the element.

The coordinates in the local system and the unit vectors can be calculated and the transformation relation between both systems can be uniquely defined

$$x_i = a_i^{\overset{*}{j}}\, x_{\overset{*}{j}}\,; \qquad x^i = a^i_{\overset{*}{j}}\, x^{\overset{*}{j}}\,; \qquad e_i = a_i^{\overset{*}{j}}\, e_{\overset{*}{j}}\,; \qquad e^i = a^i_{\overset{*}{j}}\, e^{\overset{*}{j}}\,, \tag{6}$$

2. The displacement approximation basis is constructed in the defined local coordinate system so that the Lagrangian equation is satisfied. In such a way, the difficulties in constructing trial functions, which satisfy the Lagrangian equation for applying the Trefftz method, can be overcome.

3. The advantages of using natural coordinate systems are exploited. All quantities needed in the application are expressed in the natural coordinate system by using the isoparametric transformation.

4. The integration over the interval $[-1,+1]$ is used to derive the element matrices and load vectors. The integration can be performed either numerically or exactly [13]. It is also not necessary to transform the energy expressions to the natural coordinate system in order to evaluate the element matrices and the load vectors. The energy expressions can be evaluated as they are related to the Cartesian coordinate system after replacing the Cartesian variables with their natural equivalents.

3. Reduced hybrid and boundary technique of Trefftz type

The global variational approximation is based on the static version of the extended variational form of the natural boundary conditions (7)

$$\delta I = \int_s \sigma^{ij} n_j \delta u_i \mathrm{d}s - \int_{s_\sigma} \overline{T}^i \delta u_i \mathrm{d}s = 0, \tag{7}$$

$\sigma^{ij} n_j$ denotes the boundary tractions on the part of boundry s_σ where the forces $\overline{T}^i$ are prescribed.

The current approach exploits the basic concept of the Trefftz method in the use of assumed internal displacement field (8) that solves the Lagrangian equation

$$u_i = M_i^{m(m)} c_{m(m)} + \overline{M}_{i(p)} \overline{q}^{(p)}, \tag{8}$$

$M_i^{m(m)}$ are homogeneous trial approximating functions satsify the homogeneous part of Lagrangian equation, $c_{m\,(m)}$ are parameters to be determined depending on the degrees of freedom of the element, $\overline{M}_{i(p)}$ are non-homogeneous approximating functions satisfy the particular part of the Lagrangian equation and $\overline{q}^{(p)}$ are the nodal values of the element loading.

Different techniques are possible in relating the undetermined parameters $c_{m(m)}$ to the nodal degrees of freedom $u_{i(e)}$ [13]. Whenever it is possible to satisfy the continuity requirement using the geometrical interpolation technique, the essential boundary conditions at the finite element level can be used to construct 'interpolation' functions over an element that approximate the internal displacement field. This technique leads to a displacement approximation basis that involves both homogeneous shape functions and particular shape functions.

Two alternative techniques are followed to enforce conformity and inter-element continuity when the interpolation cannot ensure the continuity requirement. The 'frame function' concept is used in both cases. The first technique is hybrid in form of equation (9a) and consists simply in applying the Gauss divergence theorem for the stress variation at the finite element level, after imposing the essential boundary conditions of the element and enforcing the equation of equilibrium, in order to eliminate the undetermined parameters. The second technique is a boundary equivalent. It results from the application of condition (9b) at the finite element level in order to relate the undetermined parameters $c_{m(m)}$ with the nodal degrees of freedom $u_{i(e)}$.

$$\int_V u_{i,j} \delta\sigma^{ij} \mathrm{d}V = \int_{s_\sigma} (\delta\sigma^{ij} n_j) u_i \mathrm{d}s + \int_{s_u} (\delta\sigma^{ij} n_j) \overline{u}_i \mathrm{d}s, \tag{9a}$$

$$\int_s (\delta\sigma^{ij} n_j) u_i \mathrm{d}s = \int_{s_\sigma} (\delta\sigma^{ij} n_j) u_i \mathrm{d}s + \int_{s_u} (\delta\sigma^{ij} n_j) \overline{u}_i \mathrm{d}s. \tag{9b}$$

In order to apply the boundary technique, first we derive from u_i as given in (8), the corresponding displacement functions $u_{i((e)(b))}$ on the element boundary

$$u_{i((e)(b))} = M^{m(m)}_{i((e)(b))} c_{m(m)} + \overline{M}_{i((e)(b))(p)} \overline{q}^{(p)}. \tag{10}$$

At the element boundary the conjugate vector of boundary tractions is determined from (8) and can be written using different index notations as follows

$$T^{i((e)(b))} = R^{i((e)(b))m(m)} c_{m(m)} + \overline{R}^{i((e)(b))}_{(p)} \overline{q}^{(p)}, \tag{11a}$$

$$T^{i((e)(b))} = R^{i((e)(b))n(n)} c_{n(n)} + \overline{R}^{i((e)(b))}_{(p)} \overline{q}^{(p)}, \tag{11b}$$

$$T^{i((e)(b))} = R^{i((e)(b))l(l)} c_{l(l)} + \overline{R}^{i((e)(b))}_{(p)} \overline{q}^{(p)}. \tag{11c}$$

Assuming that the variation of the load terms is zero, the variation of the boundary tractions is then given by (12)

$$\delta T^{i((e)(b))} = R^{i((e)(b))n(n)} \delta c_{n(n)}. \tag{12}$$

The index $^{((e)(b))}$ varies over the four element sides, along which we assume for $\overline{u}_i$ a displacement field that ensures the inter-element continuity in the customary way

$$\overline{u}_{i((e)(b))} = L^{k(k)}_{i((e)(b))} u_{k(k)}\,; \qquad \overline{u}_{i((e)(b))} = L^{k(k)}_{i((e)(b))} u_{k(k)}. \tag{13}$$

Matrix $L^{m(m)}_{i((e)(b))}$ or $L^{k(k)}_{i((e)(b))}$ contains shape functions on the element boundaries.

The undetermined parameters $c_{m(m)}$ (or $c_{l(l)}$) may now be eliminated with the aid of (9a), (9b) using a boundary technique. By substituting the conjugate vector of boundary tractions (11) and the prescribed displacement field from (13) into (9b) the relation (14) can be evaluated and the relationship (19) between the undetermined parameters and the nodal degrees of freedom can be established.

$$\delta c_{m(m)} H^{m(m)n(n)} c_{n(n)} + \delta c_{m(m)} \overline{H}^{m(m)}_{(p)} \overline{q}^{(p)} = \delta c_{m(m)} T^{m(m)n(n)} u_{n(n)}. \tag{14}$$

This leads to the boundary $H^{m(m)n(n)}$ and $\overline{H}^{m(m)}_{(p)}$ matrices defined by equations (15b) and (16b), which are quasi equivalent to the hybrid $H^{m(m)n(n)}$ and $\overline{H}^{m(m)}_{(p)}$ matrices results in by applying the hybrid Trefftz technique (9a) and defined by (15a), (16a).

$$H^{m(m)n(n)} = \int_A M^{m(m)}_{,ij} E^{ijkl} M^{n(n)}_{,kl} \mathrm{d}A, \tag{15a}$$

$$H^{m(m)n(n)} = \int_s M^{m(m)}_{i((e)(b))} R^{i((e)(b))n(n)} \mathrm{d}s, \tag{15b}$$

$$\overline{H}^{m(m)}_{(p)} = \int_A \overline{M}_{(p),ij} E^{ijkl} M^{m(m)}_{,kl} \mathrm{d}A, \tag{16a}$$

$$\overline{H}^{m(m)}_{(p)} = \int_s \overline{M}_{i((e)(b))(p)} R^{i((e)(b))m(m)} \mathrm{d}s. \tag{16b}$$

The matrix $H^{m(m)n(n)}$ of (15b) is quasi symmetric. It is possible to evaluate the integral (17) instead of the integral (15b) to ensure the symmetry (see for example [23])

$$H^{m(m)n(n)} = \frac{1}{2} \left(\int_s M^{m(m)}_{i((e)(b))} R^{i((e)(b))n(n)} \mathrm{d}s + \int_s R^{i((e)(b))n(n)} M^{m(m)}_{i((e)(b))} \mathrm{d}s \right). \tag{17}$$

The remaining matrix in Eq. (14) on the right side is defined as follows

$$T^{m(m)n(n)} = \int_s R^{i((e)(b))n(n)} L^{m(m)}_{i((e)(b))} \mathrm{d}s. \tag{18}$$

Elimination of the undetermined parameters $c_{m(m)}$ (or $c_{l(l)}$) using Eq. (14) recovers relation (19), where $H_{l(l)n(n)}$ is the inverse matrix of (17)

$$c_{l(l)} = H_{l(l)m(m)}(T^{m(m)n(n)} u_{n(n)} - \overline{H}^{m(m)}_{(p)} \overline{q}^{(p)}). \tag{19}$$

Definitions (11a)–(11c) and (13) for the boundary tractions and the boundary displacements, respectively, can be used to derive the following relation by evaluating the variational expression (7)

$$\delta u_{k(k)} \left\{ \int_s (R^{i((e)(b))l(l)} c_{l(l)} + \overline{R}^{i((e)(b))}_{(p)} \overline{q}^{(p)}) L^{k(k)}_{i((e)(b))} \mathrm{d}s - \int_{s_\sigma} L^{k(k)}_{i((e)(b))} \overline{T}^{i((e)(b))} \mathrm{d}s \right\} = 0. \tag{20}$$

The last equation can be recast after observing (18) and introducing the definitions (22) and (23) in the form (21)

$$\delta u_{k(k)}(T^{k(k)l(l)} c_{l(l)} + \overline{T}^{k(k)}_{(p)} \overline{q}^{(p)} - \overline{r}^{0k(k)}) = 0, \tag{21}$$

$$\overline{r}^{0k(k)} = \int_{s_\sigma} \overline{T}^{i((e)(b))} L^{k(k)}_{i((e)(b))} \mathrm{d}s, \tag{22}$$

$$\overline{T}^{k(k)}_{(p)} = \int_{s_\sigma} \overline{R}^{i((e)(b))}_{(p)} L^{k(k)}_{i((e)(b))} \mathrm{d}s. \tag{23}$$

Substituting the free parameters $c_{l(l)}$ defined by Eq. (19) into Eq. (21) leads to the following equation

$$\delta u_{k(k)}(T^{k(k)l(l)} H_{l(l)m(m)} T^{m(m)n(n)}) u_{n(n)} + \delta u_{k(k)}(T^{k(k)l(l)} H_{l(l)m(m)} \overline{H}^{m(m)}_{(p)} \overline{q}^{(p)}) + \delta u_{k(k)} \overline{T}^{k(k)}_{(p)} \overline{q}^{(p)} - \delta u_{k(k)} \overline{r}^{0k(k)}) = 0. \tag{24}$$

This leads by recasting the previous equation to the following "force-displacement" finite element relationship

$$k^{k(k)n(n)} u_{n(n)} - \overline{r}^{0k(k)} = \overline{r}^{k(k)}. \tag{25}$$

In this equation the symmetric finite element stiffness matrix and the equivalent nodal force vectors are defined by the equations below

$$k^{k(k)n(n)} = T^{k(k)l(l)} H_{l(l)m(m)} T^{m(m)n(n)}, \tag{26}$$

$$\overline{r}^{k(k)} = - T^{k(k)l(l)} H_{l(l)m(m)} \overline{H}^{m(m)}_{(p)} \overline{q}^{(p)} + \overline{T}^{k(k)}_{(p)} \overline{q}^{(p)}. \tag{27}$$

4. Extending to dynamic problems

The global approximation basis of the reduced Trefftz-type approach in case of kinetic application operates on the natural boundary conditions enforced in an integral form (Eq. (7)) between two fixed time points $t_1$and t_2 of the vibration process (the virtual variation δu_i of the actual displacement field u_i is to be assumed in such a way that the governing differential equation and the kinematic boundary conditions are exactly satisfied)

$$\int_{t_1}^{t_2} \left\{ \int_{s_\sigma} \left(\sigma^{ij} n_i - \overline{T}^i \right) \delta u_i \mathrm{d}s \right\} \mathrm{d}t = 0. \tag{28}$$

A possible extension to deal with dynamic problems consists in adding the equation of motion to the last variational expression with the aid of Lagrange multiplier

$$\int_{t_1}^{t_2} \left\{ \int_{s_\sigma} \left(\sigma^{ij} n_i - \overline{T}^i \right) \delta u_i \mathrm{d}s + \int_V \left(\sigma^{ij}_{,j} + \overline{f}^i + \rho \ddot{u}_i \right) \delta u_i \right\} \mathrm{d}t = 0. \tag{29}$$

In the last equation, $\overline{f}^i$, $\rho \ddot{u}^i$ are the body and inertia forces, respectively.

Applying a displacement approximation basis constructed for a static case in a dynamic case presented in the potential form of (29) leads after performing the variation to the following standard FEM-relation (in the absence of damping effects)

$$k^{m(m)n(n)} u_{n(n)} + m^{m(m)n(n)} \ddot{u}_{n(n)} = \overline{r}^{0k(k)} + \overline{r}^{k(k)}. \tag{30}$$

The mass matrix $m^{m(m)n(n)}$ is defined by the following equations

$$m^{m(m)n(n)} = \int_A N_i^{m(m)} \rho^{ij} N_j^{n(n)} \mathrm{d}A, \tag{31}$$

$N_i^{m(m)}$ are the shape functions, in which the displacements and rotations are included and ρ^{ij} is the corresponding mass density matrix.

5. Quadrilateral folded plate structure element

5.1. Plane stress element

The four-node quadrilateral plane stress element shown in Fig. 1 has three degrees of freedom per nodal point, these are the displacement components in the directions of the defined local axes x^1, x^2 and the rotation about the third axis x^3 normal to the element plane, that means $\{u^0_{x^1}, u^0_{x^2}, \varphi_{x^3}\}$.

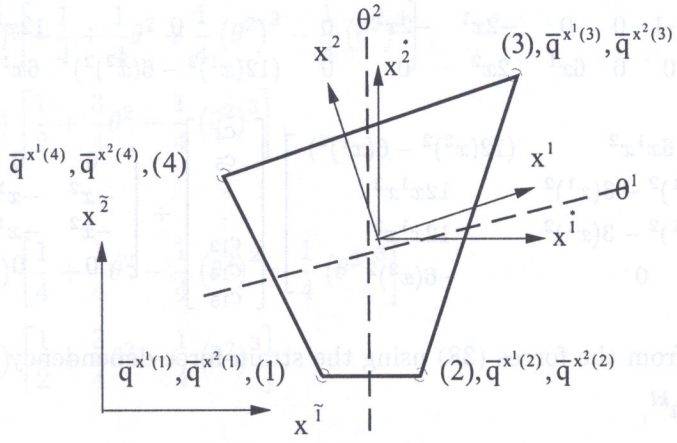

Fig. 1. Quadrilateral finite element for plane stress. Coordinate systems and element loading

The local axes x^1, x^2 are defined using the directions of the element base vectors in an analogous procedure used in section 2. The approximation basis is constructed using a stress function $F(x^1, x^2)$ approximated in parametric form

$$F(x^1, x^2) = M^n c_n, \tag{32a}$$

$$M^n = \left[1 \quad x^1 \quad x^2 \quad (x^1)^2 \quad x^1 x^2 \quad (x^2)^2 \quad (x^1)^3 \quad (x^1)^2 x^2 \quad x^1 (x^2)^2 \quad (x^2)^3 \right.$$
$$\left. (x^1)^4 \quad (x^1)^3 x^2 \quad (x^1)^2 (x^2)^2 \quad x^1 (x^2)^3 \quad (x^2)^4 \right], \tag{32b}$$

$$c_n = \{c_1 \quad c_2 \quad \ldots \quad c_{14} \quad c_{15}\}. \tag{32c}$$

In order to satisfy the differential equations (33) the stress function (32a) may be substituted into the differential equations to yield the relationship (34) between a subset of undetermined parameters

$$\frac{\partial^4 F(x^1,x^2)}{(\partial x^1)^4} + 2\frac{\partial^4 F(x^1,x^2)}{(\partial x^1)^2(\partial x^2)^2} + \frac{\partial^4 F(x^1,x^2)}{(\partial x^2)^4} = 0, \tag{33}$$

$$24c_{11} + 8c_{13} + 24c_{15} = 0. \tag{34}$$

The possible solution (35) of Eq. (34) enables the rearranging of Eq. (32a) in the form (36)

$$c_{13} = 3c_{11} - 3c_{15}, \tag{35}$$

$$\begin{aligned} F(x^1,x^2) = {} & c_1 + c_2x^1 + c_3x^2 + c_4(x^1)^2 + c_5x^1x^2 + c_6(x^2)^2 + c_7(x^1)^3 \\ & + c_8(x^1)^2x^2 + c_9x^1(x^2)^2 + c_{10}(x^2)^3 + c_{11}((x^1)^4 - 3(x^1)^2(x^2)^2) \\ & + c_{12}(x^1)^3x^2 + c_{14}x^1(x^2)^3 + c_{15}((x^2)^4 - 3(x^1)^2(x^2)^2). \end{aligned} \tag{36}$$

The conjugate force vector to the previous stress function can be derived using the following relation

$$\begin{aligned} n_{x^1x^1} &= \frac{\partial^2 F(x^1,x^2)}{(\partial x^2)^2}, \\ n_{x^1x^2} &= \frac{\partial^2 F(x^1,x^2)}{\partial x^1 \partial x^2} - x^2\overline{q}^{x^1} - x^1\overline{q}^{x^2}; \qquad n_{x^2x^1} = n_{x^1x^2}, \\ n_{x^2x^2} &= \frac{\partial^2 F(x^1,x^2)}{(\partial x^1)^2}. \end{aligned} \tag{37}$$

This yields the following approximation basis for the force functions

$$\begin{bmatrix} n_{x^1x^1} \\ n_{x^2x^1} \\ n_{x^1x^2} \\ n_{x^2x^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 2x^1 & 6x^2 & -6(x^1)^2 & 0 \\ 0 & -1 & 0 & 0 & -2x^1 & -2x^2 & 0 & 0 & 12x^1x^2 \\ 0 & -1 & 0 & 0 & -2x^1 & -2x^2 & 0 & 0 & 12x^1x^2 \\ 2 & 0 & 0 & 6x^1 & 2x^2 & 0 & 0 & (12(x^1)^2 - 6(x^2)^2) & 6x^1x^2 \end{bmatrix}$$

$$\begin{bmatrix} 6x^1x^2 & (12(x^2)^2 - 6(x^1)^2) \\ -3(x^1)^2 - 3(x^1)^2 & 12x^1x^2 \\ -3(x^1)^2 - 3(x^1)^2 & 12x^1x^2 \\ 0 & -6(x^2)^2 \end{bmatrix} \begin{bmatrix} c_4 \\ c_5 \\ \cdot \\ \cdot \\ c_{12} \\ c_{14} \\ c_{15} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -x^2 & -x^1 \\ -x^2 & -x^1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{q}^{x^1} \\ \overline{q}^{x^2} \end{bmatrix}. \tag{38}$$

The strains follow from the forces (38) using the strain-force dependency

$$\varepsilon_{ij} = E_{ijkl}n^{kl},$$

$$\begin{bmatrix} \varepsilon_{x^1x^1} \\ \varepsilon_{x^2x^1} \\ \varepsilon_{x^1x^2} \\ \varepsilon_{x^2x^2} \end{bmatrix} = \frac{1}{Et} \begin{bmatrix} 1 & 0 & 0 & -\nu \\ 0 & (1+\nu)/2 & (1+\nu)/2 & 0 \\ 0 & (1+\nu)/2 & (1+\nu)/2 & 0 \\ -\nu & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_{x^1x^1} \\ n_{x^2x^1} \\ n_{x^1x^2} \\ n_{x^2x^2} \end{bmatrix}. \tag{39}$$

E is the elastic modulas, ν is the Poisson's ratio.

For the displacement $\overline{u}_i$ along the four element sides we assume a displacement field that ensure the inter-element continuity in the customary way

$$\overline{u}_i = L^{m(m)}_{i((e)(b))}u_{m(m)}. \tag{40}$$

The matrix $L^{m(m)}_{i((e)(b))}$ contains the shape functions of the element boundaries.

Now the implementation of the element can be carried out in an analogy to the steps described in section 3.

The element matrices can be derived by performing the integration over the biunit interval $[-1,+1]$. The use of 2×2 Gaussian integration formula leads to a singular $H^{m(m)n(n)}$ matrix. The exact integration of $H^{m(m)n(n)}$ matrix shows its regularity. The use of at least 3×3 Gaussian integration formula is necessary.

For considering the inertia forces we define the displacement functions inside the finite element (42) using the shape functions of the element boundaries (41a), (41b):

$$u_1 = \frac{1}{2}(1-\theta^1)\overline{u}_{(4)(1)} + \frac{1}{2}(1+\theta^1)\overline{u}_{(2)(3)}, \tag{41a}$$

$$u_2 = \frac{1}{2}(1-\theta^2)\overline{u}_{(1)(2)} + \frac{1}{2}(1+\theta^2)\overline{u}_{(3)(4)}, \tag{41b}$$

$$u_i = N_i^{m(m)} u_{m(m)}, \tag{42}$$

where the shape functions in the last equation are defined by (43a), (43b):

$$\begin{aligned}
N_1^{1(1)} &= \frac{1}{2}(1-\theta^1)\left[\frac{1}{2} - \frac{3}{4}\theta^2 + \frac{1}{4}(\theta^2)^3\right], \\
N_1^{2(1)} &= 0, \\
N_1^{3(1)} &= \frac{1}{2}(1-\theta^1)\left[-\frac{1}{4} + \frac{1}{4}\theta^2 + \frac{1}{4}(\theta^2)^2 - \frac{1}{4}(\theta^2)^3\right], \\
N_1^{1(2)} &= \frac{1}{2}(1+\theta^1)\left[\frac{1}{2} - \frac{3}{4}\theta^2 + \frac{1}{4}(\theta^2)^3\right], \\
N_1^{2(2)} &= 0, \\
N_1^{3(2)} &= \frac{1}{2}(1+\theta^1)\left[-\frac{1}{4} + \frac{1}{4}\theta^2 + \frac{1}{4}(\theta^2)^2 - \frac{1}{4}(\theta^2)^3\right], \\
N_1^{1(3)} &= \frac{1}{2}(1+\theta^1)\left[\frac{1}{2} + \frac{3}{4}\theta^2 - \frac{1}{4}(\theta^2)^3\right], \\
N_1^{2(3)} &= 0, \\
N_1^{3(3)} &= \frac{1}{2}(1+\theta^1)\left[\frac{1}{4} + \frac{1}{4}\theta^2 - \frac{1}{4}(\theta^2)^2 - \frac{1}{4}(\theta^2)^3\right], \\
N_1^{1(4)} &= \frac{1}{2}(1-\theta^1)\left[\frac{1}{2} + \frac{3}{4}\theta^2 - \frac{1}{4}(\theta^2)^3\right], \\
N_1^{2(4)} &= 0, \\
N_1^{3(4)} &= \frac{1}{2}(1-\theta^1)\left[\frac{1}{4} + \frac{1}{4}\theta^2 - \frac{1}{4}(\theta^2)^2 - \frac{1}{4}(\theta^2)^3\right];
\end{aligned} \tag{43a}$$

$$\begin{aligned}
N_2^{1(1)} &= 0, \\
N_2^{2(1)} &= \frac{1}{2}(1-\theta^2)\left[\frac{1}{2} - \frac{3}{4}\theta^1 + \frac{1}{4}(\theta^1)^3\right], \\
N_2^{3(1)} &= \frac{1}{2}(1-\theta^2)\left[\frac{1}{4} - \frac{1}{4}\theta^1 - \frac{1}{4}(\theta^1)^2 + \frac{1}{4}(\theta^1)^3\right], \\
N_2^{1(2)} &= 0,
\end{aligned} \tag{43b}$$

$$
\begin{aligned}
N_2^{2(2)} &= \frac{1}{2}\left(1-\theta^2\right)\left[\frac{1}{2}+\frac{3}{4}\theta^1-\frac{1}{4}\left(\theta^2\right)^1\right],\\
N_2^{3(2)} &= \frac{1}{2}\left(1-\theta^2\right)\left[-\frac{1}{4}-\frac{1}{4}\theta^1+\frac{1}{4}\left(\theta^1\right)^2+\frac{1}{4}\left(\theta^1\right)^3\right],\\
N_2^{1(3)} &= 0,\\
N_2^{2(3)} &= \frac{1}{2}\left(1+\theta^2\right)\left[\frac{1}{2}+\frac{3}{4}\theta^1-\frac{1}{4}\left(\theta^1\right)^3\right],\\
N_2^{3(3)} &= \frac{1}{2}\left(1+\theta^2\right)\left[-\frac{1}{4}-\frac{1}{4}\theta^1+\frac{1}{4}\left(\theta^1\right)^2+\frac{1}{4}\left(\theta^1\right)^3\right],\\
N_2^{1(4)} &= 0,\\
N_2^{2(4)} &= \frac{1}{2}\left(1+\theta^2\right)\left[\frac{1}{2}-\frac{3}{4}\theta^1+\frac{1}{4}\left(\theta^1\right)^3\right],\\
N_2^{3(4)} &= \frac{1}{2}\left(1+\theta^2\right)\left[\frac{1}{4}-\frac{1}{4}\theta^1-\frac{1}{4}\left(\theta^1\right)^2+\frac{1}{4}\left(\theta^1\right)^3\right].
\end{aligned}
\tag{43b}
$$

[cont.]

5.2. Quadrilateral 12-DOF plate bending element

The four-node element has three degrees of freedom per nodal point, namely the transverse displacement and the two rotations. The application allows the element to be loaded by a distributed load $\overline{q}^{x^3}(x^1, x^2)$ with different nodal values $\overline{q}^{x^3(1)}$, $\overline{q}^{x^3(2)}$, $\overline{q}^{x^3(3)}$, $\overline{q}^{x^3(4)}$ at the element nodes (1), (2), (3), (4), respectively (see Fig. 2). The homogeneous displacement approximation basis is taken from the 12-DOF rectangular plate bending element. The approximation basis is extended to involve a particular part, which satisfies the differential equation of the plate problem. Thus, the application can be classified under the displacement version of the reduced Trefftz type approach.

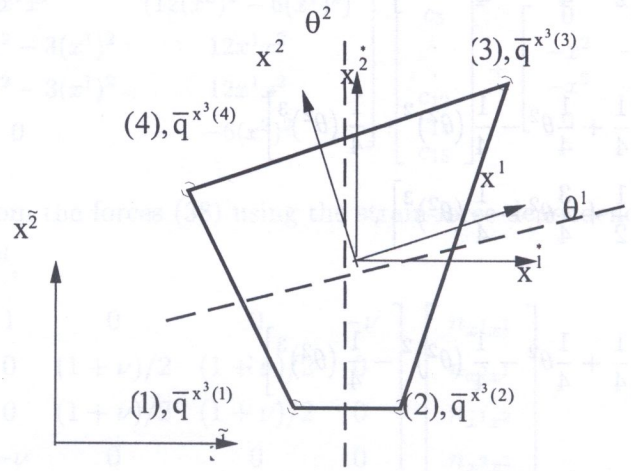

Fig. 2. Quadrilateral finite element for plate bending. Coordinate systems and element loading

The internal displacement field of the finite element is approximated in the defined local coordinate system (x^1, x^2) in parametric form involving a homogeneous part with the same dimension as the degrees of freedom of the element and a particular part dependent on the element loading as follows:

$$u^0_{x^3}(x^1, x^2) = M^{m(m)} c_{m(m)} + \overline{M}_{(p)} \overline{q}^{(p)},$$

$$M^{m(m)} = \left[1 \;\; x^1 \;\; x^2 \;\; (x^1)^2 \;\; x^1 x^2 \;\; (x^2)^2 \;\; (x^1)^3 \;\; (x^1)^2 x^2 \;\; x^1 (x^2)^2 \;\; (x^2)^3 \;\; (x^1)^3 x^2 \;\; x^1 (x^2)^3\right], \tag{44}$$

$$\overline{M}_{(p)} = \frac{1}{D}\left[\frac{(x^1)^2(x^2)^2}{8} \quad \frac{(x^1)^4 x^2}{24} \quad \frac{x^1(x^2)^4}{24} \quad \frac{(x^1)^3(x^2)^3}{72}\right] \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} & \overline{A}_{13} & \overline{A}_{41} \\ \overline{A}_{21} & \overline{A}_{22} & \overline{A}_{23} & \overline{A}_{42} \\ \overline{A}_{31} & \overline{A}_{32} & \overline{A}_{33} & \overline{A}_{43} \\ \overline{A}_{41} & \overline{A}_{42} & \overline{A}_{34} & \overline{A}_{44} \end{bmatrix}.$$

The known values $\overline{A}_{11}, ..., \overline{A}_{44}$ depend only on the coordinates of the nodal points of the element. Now the 12-nodal degrees of freedom $u_{k(e)}$ (i.e. the nodal displacement and slopes at the four element nodes) are used to form the element shape functions.

The folded plate structure element is a combination of the plate element and the plane stress element.

6. Numerical results

6.1. Invariance and insensitivity to nodal point numbering

The invariance study of the element is performed using the single quadrilateral folded plate structure element shown in Fig. 3. The global coordinate system $x^{\widetilde{1}}$, $x^{\widetilde{2}}$, $x^{\widetilde{3}}$ is located in nodal point (1). The structure is rotated about its axes with different rotation angles as can be recovered from Fig. 3a,

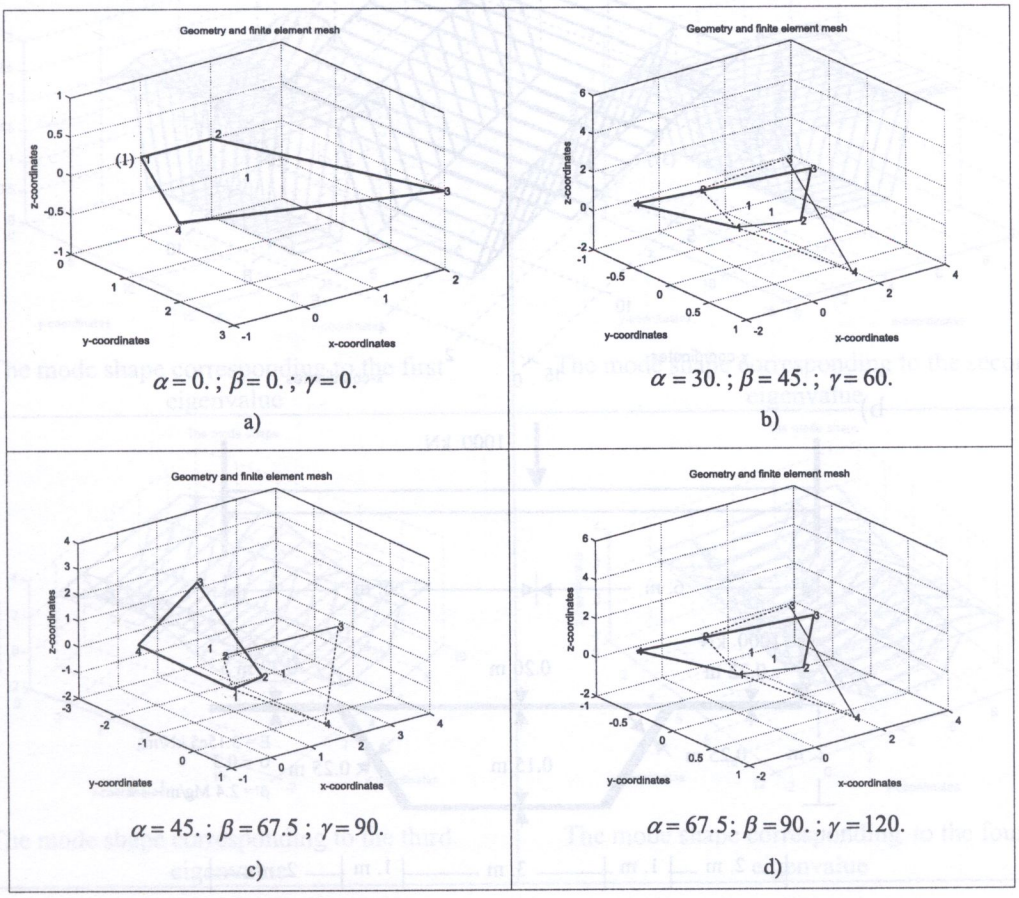

Fig. 3. General single folded plate structure element

3b, 3c, 3d. The nodal point numbering is also changed for the element. The geometry properties are defined by the elasticity modulus $E = 1365$ [kN/m^2], mass density $\rho = 5.$ [Mg/m^3], thickness $h = 0.2$ [m] and Poisson ratio $\nu = 0.3$. The boundary conditions are chosen such that the structure is clamped at nodal point (1).

An eigenvalue analysis is performed on the structure for all the geometry definitions explained. It is found that the eigenvalues remain unchanged in all these cases (see Table 1).

Table 1. The frequencies of the folded plate structure for all cases presented in Fig. 3

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.194123	0.520251	1.18842	1.96525	2.45231	2.9974
ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}
3.96588	7.65959	9.60759	11.1233	14.9202	17.5173

6.2. A cellular hollow box

The cellular hollow box shown in Fig. 4 and meshed by 72 folded plate structure elements (Fig. 4a) is subjected to static and eigenvalue analysis. Details on mechanical system, geometry and loading

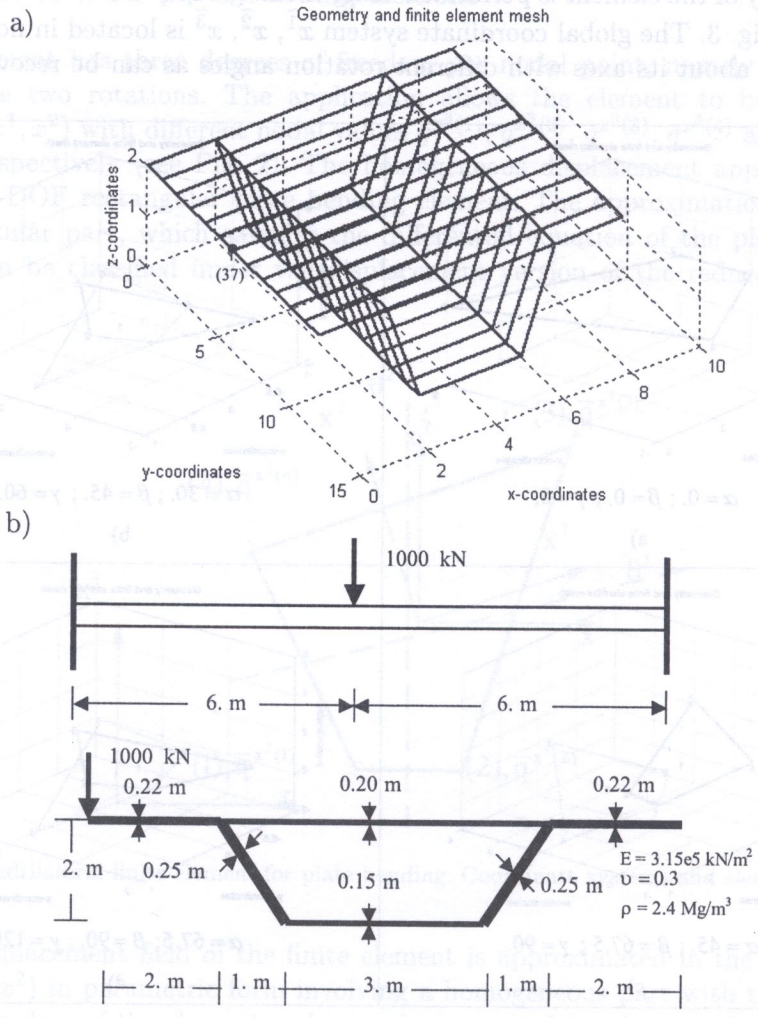

Fig. 4. Hollow box. a) system and finite element mesh b) system, loading, and cross section geometry

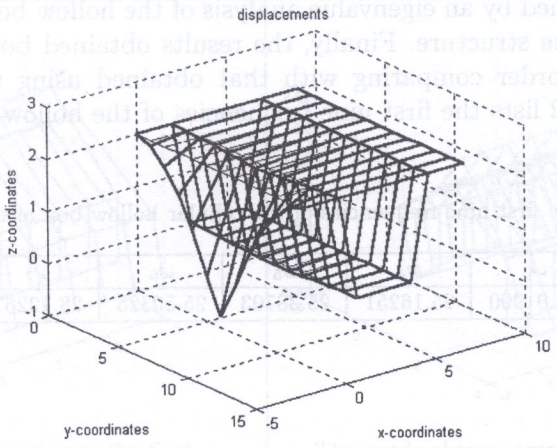

Fig. 5. Hollow box: 3-D representation of the deformation caused by the concentrated load

are given in Fig. 4b. The hollow box is fully clamped on both its end cross sections. A concentrated load of 1000 kN is specified at the position shown in Fig. 4b. Figure 5 shows a 3-D representation of the displacements components caused by the concentrated load. The deformation of the cross section of the hollow box signifies the mechanical behavior of the box as a surface structure. Such behavior can not be modeled by applying the beam theory. The mode shapes corresponding to the

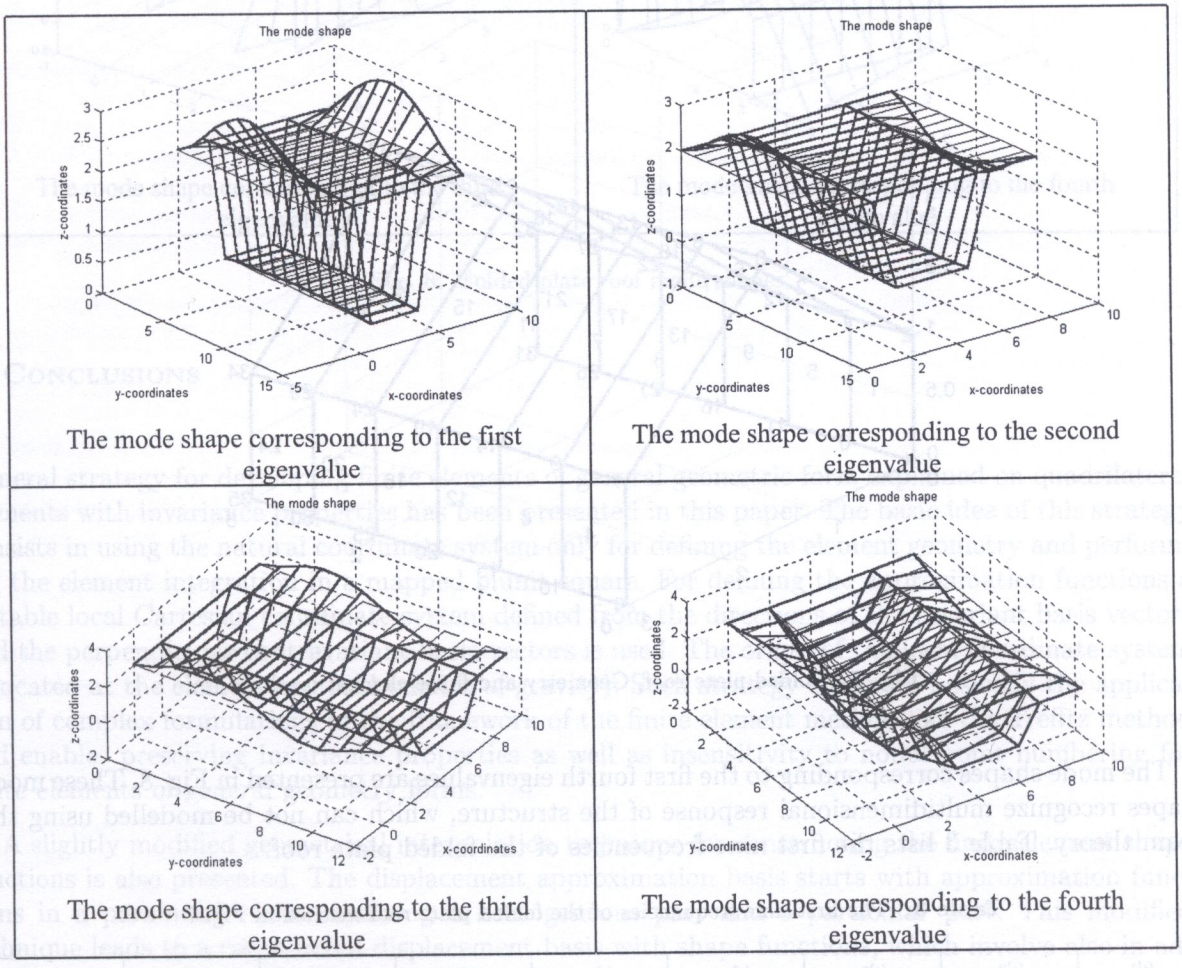

Fig. 6. Hollow box mode shapes

first four eigenvalues obtained by an eigenvalue analysis of the hollow box recognize the same multi-dimensional behavior of the structure. Finally, the results obtained both by static and eigenvalue analysis are of the same order comparing with that obtained using the hybrid mixed elements adopted in [18–22]. Table 2 lists the first nine frequencies of the hollow box and Fig. 6 shows their first four mode shapes.

Table 2. The first nine frequencies of the cellular hollow box meshed in Fig. 3a

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9
16.74340	17.14055	25.01900	25.16251	25.38703	25.50325	28.12267	35.95455	36.17884

6.3. Folded plate roof

The folded plate roof, presented in Fig. 7 and meshed by 6×4 folded plate structure elements is subjected to eigenvalue analysis. The boundary conditions are specified at the supported nodes as follows:

$$u_{x^2(1)} = 0; \quad u_{x^3(1)} = 0; \quad u_{x^3(5)} = 0; \quad u_{x^1(31)} = 0; \quad u_{x^2(31)} = 0; \quad u_{x^3(31)} = 0; \quad u_{x^3(35)} = 0.$$

In Fig. 7 the axes (x^1, x^2, x^3) are designated as (x, y, z).

The geometry properties are defined by the elasticity modulus $E = 34700000$ kN/m^2, mass density $\rho = 2.4$ Mg/m^3, thickness $h = 0.1$ m and Poisson ratio $\nu = 0.2$.

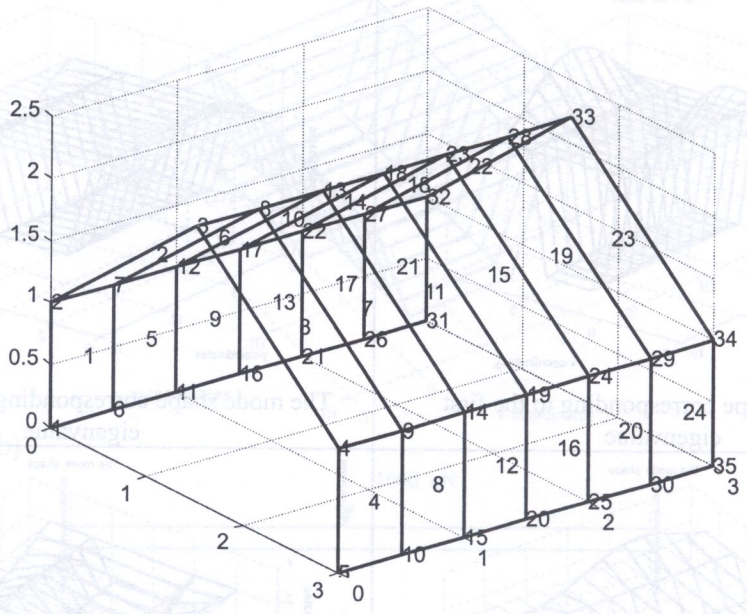

Fig. 7. Folded plate roof, Geometry and finite element mesh

The mode shapes corresponding to the first fourth eigenvalues are presented in Fig. 8. These mode shapes recognize multidimensional response of the structure, which can not be modelled using the beam theory. Table 3 lists the first nine frequencies of the folded plate roof.

Table 3. The first nine frequencies of the folded plate roof meshed in Fig. 7

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9
22.13216	104.44011	221.54477	250.83380	387.22121	411.06980	616.83469	621.32724	684.12496

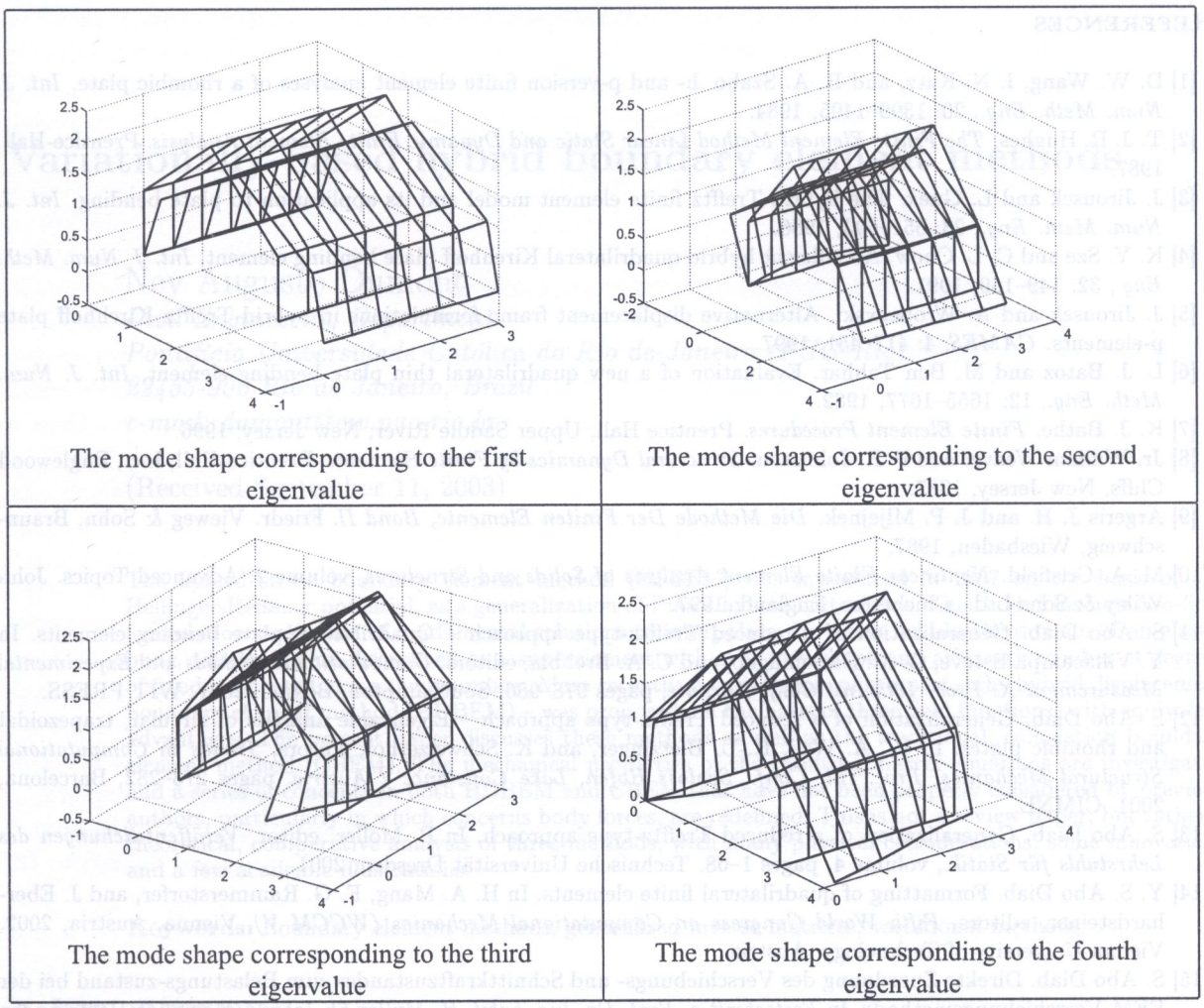

Fig. 8. Folded plate roof mode shapes

7. Conclusions

General strategy for developing finite elements of general geometric form explained on quadrilateral elements with invariance properties has been presented in this paper. The basic idea of this strategy consists in using the natural coordinate system only for defining the element geometry and performing the element integration in a mapped biunit square. For defining the approximation functions a suitable local Cartesian coordinate system defined from the directions of the covariant basis vectors and the perpendicular contravariant basis vectors is used. The origin of the local coordinate system is located at the element centroid (centre of gravity). Such strategy enhances basically the application of complex formulations in the framework of the finite element method like the Trefftz method and enables preserving invariance properties as well as insensitivity to nodal point numbering for finite elements of general geometry forms.

A slightly modified geometrical interpolation technique for constructing the finite element shape functions is also presented. The displacement approximation basis starts with approximation functions in a parametric form involving a homogeneous part and a particular part. This modified technique leads to a constructed displacement basis with shape functions, which involve also in addition to the homogeneous part a particular part. Such interpolation procedure enables considering the effect of the external loading on the displacements at the finite element level.

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