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Time integration and the Trefftz Method Part I – First-order and parabolic problems

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The finite element method is applied in the time domain to establish formulations for the integration of first-order and parabolic (transient) problems. The modal decomposition concept is applied using two distinct approaches. The first is based on modal decomposition in the space domain to recover the well-established method for uncoupling the parabolic system of equations. To overcome the limitations of this approach in the implementation of large-scale, non-linear problems, the second approach that is reported consists in inducing uncoupling through modal decomposition in the time domain without using the periodic approximation that characterise analyses in the frequency domain. The methods of modal decomposition are related with the implementation of the Trefftz concept in both time and space.

Keywords: Time integration, first-order problems, parabolic problems, Trefftz method

1. Introduction

The high-performance displayed by Trefftz elements is a direct consequence of the use of approximation bases extracted from the set of formal solutions of the system of differential equations that governs the problem under analysis.

As these bases embody the physics of the problem, Trefftz elements can produce highly accurate solutions using a relatively small number of degrees of freedom. Depending on the discretization technique that is used in their implementation, they may yield different patterns of convergence under static or kinematic admissibility conditions and present different levels of suitability for self-adaptivity and for parallel processing.

This has been well proven and illustrated in recent years through the solution of a wide range of elliptic (static) problems. However, the equivalent work in the solution of parabolic (transient) and hyperbolic (dynamic) problems has been limited to spectral analyses of periodic or periodically extended responses, e.g. [1, 2].

This uneven development has been caused by limitations intrinsic to the time integration techniques that are frequently used in the solution of non-static problems. Most of these techniques destroy the fundamental properties of time-dependent problems, namely parabolicity and hyperbolicity in first- and second-order systems of equations, respectively.

A direct consequence of this limitation is that the bases derived from the equivalent Helmholtz (elliptic) problem emerge grossly truncated. The degree of incompleteness of these bases is so strong that it renders irrelevant the implementation of the Trefftz concept.

The paper addresses the formulation of two basic time integration methods that preserve the properties of time dependent problems, namely parabolicity of transient problems and hyperbolicity of dynamic problems. To lighten the presentation, these problems are analysed separately, in Parts I and II, respectively.

The first method of time integration addressed here is based in modal decomposition in the space domain. The strategy consists in replacing the parabolic problem by an equivalent first-order

problem and subsequently calling a time integration method to replace this system by an equivalent algebraic system of equations. The basis for this approach is well established. It is recalled here to stress that it corresponds, in essence, to the use of the Trefftz concept and to relate this concept with the VIP method suggested by Tamma et al. [3].

It is used to recall also the major obstacles found in the numerical implementation of this approach, namely the difficulty in its extension to the solution of non-linear problems and the cost of solving large-scale eigenvalue problems. Moreover, it hinders the implementation of the Trefftz concept in the discretization of the space component of parabolic problems.

The second method of time integration reported here is based on the alternative option of implementing first modal decomposition in the time domain, and thus replace the parabolic problem by an equivalent elliptic problem, which may be then solved using any of the available methods for numerical solution by discretization in the space domain, namely the finite element method [4]. This is illustrated with the conventional (conform displacement) formulation and with the displacement model of the hybrid-Trefftz formulation.

In order to establish a common background, the presentation of the method of modal decomposition in the time domain is recalled first in the context of the equally well-established method of analysis in the frequency domain. The method is generalised to accept other than Fourier bases, and thus open its implementation to any time basis, namely non-periodic bases, which facilitates, also, its extension to the solution of non-linear problems.

It is shown that besides this advantage over the method of modal decomposition in space, the alterntive method of modal decomposition in time can be implemented solving a single, small-scale eigenvalue in the time domain, the dimension of which is independent of the number of degrees-of-freedom of the subsequent discretization in the space domain of the resulting Helmholtz-type elliptic system of equations.

2. Standard integration procedure of parabolic problems

Linear transient problems are governed by hyperbolic equations of the form,

$$\mathbf{DkD}^*\mathbf{u} + \mathbf{f} = \mathbf{c\dot{u}} \quad \text{in} \quad V \quad \text{and} \quad 0 \le t \le T, \tag{1}$$

where t is the time parameter and V the structural domain referred to a Cartesian system $\mathbf{x}$, $\mathbf{u}(\mathbf{x}, t)$ and $\dot{\mathbf{u}}$ are the displacement and velocity fields, respectively, $\mathbf{D}$ is a linear differential operator in the space domain and $\mathbf{D}^*$ denotes its conjugate, $\mathbf{k}$ and $\mathbf{c}$ are (local) stiffness and damping matrices, which are assumed to be symmetric and with constant coefficients for simplicity of the presentation, and vector $\mathbf{f}$ defines the forcing load and the effect of eventual residual strains and stresses.

The solution of the governing system (1) depends on consistent boundary conditions, which are simplified here to the usual Neumann and Dirichlet forms,

$$\mathbf{NkD}^*\mathbf{u} = \mathbf{t}_\Gamma \quad \text{on } \Gamma_\sigma \quad \text{and} \quad 0 \le t \le T, \tag{2}$$

$$\mathbf{u} = \mathbf{u}_\Gamma \quad \text{on} \quad \Gamma_u \quad \text{and} \quad 0 \le t \le T \tag{3}$$

and on the initial condition set on the displacement field:

$$\mathbf{u}_0(\mathbf{x}) = \mathbf{u}(\mathbf{x}, 0) \quad \text{in} \quad V \quad \text{and} \quad t = 0. \tag{4}$$

The standard integration procedure consists in approximating the displacement field,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x})\mathbf{d}(t) \quad \text{in} \quad V \quad \text{and} \quad 0 \le t \le T, \tag{5}$$

by separation of variables in time (the nodal displacement vector $\mathbf{q}$) and in space (the approximation modes collected in matrix $\mathbf{U}$, and thus reduce system (1) to a first-order differential equation in the time domain,

$$\mathbf{Cv} + \mathbf{Kd} = \mathbf{F}(t), \tag{6}$$

where $\mathbf{v} = \dot{\mathbf{d}}$ is velocity vector, $\mathbf{C}$ and $\mathbf{K}$ are the (symmetric) damping and stiffness matrices,

$$\mathbf{C} = \int \mathbf{U}^{\mathrm{t}} \mathbf{c} \mathbf{U} \mathrm{d}V, \tag{7}$$

$$\mathbf{K} = \int (\mathbf{D}^{*}\mathbf{U})^{\mathrm{t}}\, \mathbf{k}\, (\mathbf{D}^{*}\mathbf{U})\, \mathrm{d}V, \tag{8}$$

and $\mathbf{F}$ is the consistent forcing load vector:

$$\mathbf{F} = \int \mathbf{U}^{\mathrm{t}} \mathbf{f} \mathrm{d}V + \int \mathbf{U}^{\mathrm{t}} \mathbf{t}_{\Gamma} \mathrm{d}\Gamma_{\sigma}. \tag{9}$$

The equations above show that the domain and boundary equilibrium conditions (1) and (2) are enforced on average, in the sense of Galerkin. Conformity is enforced by requiring C^{n-1} continuity, where n is the order of the space differential operator $\mathbf{D}$, and by ensuring locally (in space and in time) the Dirichlet condition (3). It is assumed that conformity applies also to the initial condition (4),

$$\mathbf{u}_0(\mathbf{x}) = \mathbf{U}(\mathbf{x})\mathbf{d}(0) \quad \text{in} \quad V \quad \text{and} \quad t = 0, \tag{10}$$

although these conditions may also, and in general, be enforced on average.

3. Loss of parabolicity and the Trefftz method

A wide variety of procedures is available to integrate Eq. (6) under the initial condition (10). They are usually classified as explicit and implicit integration schemes, the latter being the most commonly used. The central objective of these schemes is to establish (approximate) solutions of Eq. (6) at instant $t = T$,

$$\left\{ \begin{array}{c} \mathbf{d}(T) \\ \mathbf{v}(T) \end{array} \right\} = \mathbf{A} \left\{ \begin{array}{c} \mathbf{d}(0) \\ \mathbf{v}(0) \end{array} \right\} + \mathbf{L}, \tag{11}$$

where $\mathbf{A}$ is the amplification matrix and $\mathbf{L}$ the load vector.

The major disadvantage of this approach is that substitution of approximation (11) in the solving Eq. (6) leads to a non-parabolic Helmholtz-type equation. Consequently, the Trefftz method cannot be applied in the solution of the time component of the problem.

The integration rule (11) can be expressed at local variable level,

$$\left\{ \begin{array}{c} \mathbf{u}(\mathbf{x},T) \\ \mathbf{v}(\mathbf{x},T) \end{array} \right\} = \mathbf{A} \left\{ \begin{array}{c} \mathbf{d}(\mathbf{x},0) \\ \mathbf{v}(\mathbf{x},0) \end{array} \right\} + \mathbf{L}(\mathbf{x}) \tag{12}$$

and enforced in the parabolic Eq. (1) to obtain an elliptic problem to be solved by the Trefftz method in the space domain. Parabolicity is lost again in the resulting equation, with a consequence of wasting fully the advantages offered by the Trefftz method in terms of accuracy and boundary integral definition of the finite element matrices and vectors. This is well illustrated in [5].

The first part of the paper addresses the question of establishing an integration rule (11) that solves locally the first-order system (6). This is used to establish the framework for the derivation, in the second part of the paper, of an integration rule of the form (12) that preserves the parabolicity of Eq. (1).

4. Integration in time using modal decomposition in space

Under the initial condition (10), the closed form solution of Eq. (6) is defined by,

$$\mathbf{d}(\tau) = \sum_m \mathbf{X}_m Y_m(\tau), \tag{13}$$

where $\tau = t/T$ is the non-dimensional time parameter ($0 \leq \tau \leq 1$), $\mathbf{X}_n$ represents the m-th eigenvector of the eigenvalue problem,

$$\mathbf{K}\mathbf{X}_n = \left(\frac{\lambda_n}{T}\right) \mathbf{C}\mathbf{X}_n \tag{14}$$

and Y_n is the solution of equation:

$$Y'_m + \lambda_m Y_m = T\mathbf{X}_m^{\mathrm{t}}\mathbf{F}(\tau). \tag{15}$$

The eigenvectors are assumed to be $\mathbf{C}$-orthonormal, as stated below, where δ_{mn} is the Kronecker symbol and superscript t denotes transposition:

$$\mathbf{X}_m^{\mathrm{t}}\mathbf{C}\mathbf{X}_n = \delta_{mn}. \tag{16}$$

Numerical solutions of the form (11) are used in detriment of the closed-form solution (13) due to their relatively high computational cost and the difficulty of extending this solution to non-linear problems.

However, the modal decomposition in the space domain (13) is useful to inspire the development of numerical methods for the integration of both linear and non-linear first-order problems (6). This is shown below using a finite element approach applied in the time domain. It is implemented on the displacement vector in form (13),

$$\mathbf{d}(\tau) = \sum_n \mathbf{X}_n T_n(\tau) d_n, \tag{17}$$

where now functions $T_n(\tau)$ define the (complete) time approximation basis and the weighting parameter d_n represents a generalised displacement.

4.1. Finite element approximation in the time domain

As in the Galerkin procedure for the integration of elliptic equations by the conventional (conforming) finite element method, the approximation functions are used to enforce on average the equilibrium Eq. (6), to yield,

$$\mathbf{X}_n^{\mathrm{t}} \int_0^T \widehat{T}_n \left[\mathbf{C}\mathbf{v} + \mathbf{K}\mathbf{d} - \mathbf{F}(t)\right] \mathrm{d}t = 0, \tag{18}$$

where $\widehat{T}_n$ denotes the conjugate of (the eventually complex) approximation function T_n. It is recalled that, due to symmetry, the eigenvectors $\mathbf{X}_n$ are real.

Equation (18) is integrated by parts to force the emergence of the boundary terms and thus enforce explicitly the initial condition. The following result is obtained after implementing the displacement approximation (17) in domain $0 \leq \tau \leq 1$ and at instant $\tau = 1$:

$$D_m d_m = f_m, \tag{19}$$

$$D_m = -\int_0^1 \left[\widehat{T}'_m - \lambda_m \widehat{T}_m\right] T_m \mathrm{d}\tau + \left[\widehat{T}_m T_m\right]_{\tau=1}, \tag{20}$$

$$f_m = \widehat{T}_m(0) d_m(0) + T F_m, \tag{21}$$

$$d_m(0) = \mathbf{X}_m^{\mathrm{t}} \mathbf{C} \mathbf{d}(0), \tag{22}$$

$$F_m = \int_0^1 \widehat{T}_m \mathbf{X}_m^{\mathrm{t}} \mathbf{F}(t) \mathrm{d}\tau. \tag{23}$$

Estimate (17) and result (19) can be used to construct the amplification matrix an the load vector present in Eq. (11) written in terms of displacements.

4.2. Trefftz method

To recover the Trefftz method it suffices to constrain the time approximation basis used in the single-field approximation to satisfy locally the homogeneous equation of equilibrium in conjugate form:

$$\widehat{T}'_m - \lambda_m \widehat{T}_m = 0. \tag{24}$$

After implementing the general solution of this equation,

$$T_m = \exp(\hat{\lambda}_m \tau), \tag{25}$$

the domain integral present in the solving Eq. (20) vanishes, as it is typical of the Trefftz method, to yield the following expressions:

$$D_m = \exp(2\hat{\lambda}_m), \tag{26}$$

$$f_m = d_m(0) + T F_m. \tag{27}$$

5. Standard integration procedure in the frequency domain

It can be verified that the Trefftz solution presented above corresponds to the closed form solution of system (6), as defined by Eqs. (13) and (15) at instant $\tau = 1$. This becomes particularly obvious when approximation (17) is extended to include a particular solution of Eq. (15), written, for instance, in the form of the Duhamel integral:

$$\mathbf{d}(\tau) = \sum_m \mathbf{X}_m \left(T_m(\tau) d_m + u_{pm}\right). \tag{28}$$

It can be easily confirmed, also, that this Trefftz interpretation recovers the Exact Integral Operator version of the VIP method by Tamma et al. [3]. In other words, both approaches coincide with the well-established procedure of time integration in the frequency domain with modal decomposition in space.

In order to clarify the presentation below of the integration method based on modal decomposition in the time domain, it is convenient to start by establishing the solution of the first-order problem (6) and of the parabolic problem (1)–(4) in the frequency domain without resorting to modal decomposition in space.

5.1. First-order problems

To avoid its implicit dependency on modal decomposition in space, let the displacement approximation (17) be written in the following form, where $\mathbf{d}_n$ is now a weighting vector:

$$\mathbf{d}(\tau) = \sum_m T_m(\tau)\mathbf{d}_m. \tag{29}$$

The method of Galerkin is still used to enforce Eq. (6) on average, thus replacing Eq. (18) by Eq. (30), which is integrated by parts to yield result (31):

$$\int_0^T \widehat{T}_m \left[\mathbf{C}\mathbf{v} + \mathbf{K}\mathbf{d} - \mathbf{F}(t)\right] \mathrm{d}t = \mathbf{0}, \tag{30}$$

$$\int_0^T \left[-\dot{\widehat{T}}_m \mathbf{C} + \widehat{T}_m \mathbf{K}\right] \mathbf{d}\mathrm{d}t + \left[\widehat{T}_m \mathbf{C}\mathbf{d}\right]_0^T = T\mathbf{F}_m, \tag{31}$$

$$\mathbf{F}_m = \int_0^1 \widehat{T}_m \mathbf{F} \mathrm{d}\tau. \tag{32}$$

Substitution of approximation (29) in the integral term of Eq. (31), and of its value at instant $t = T$ (or $\tau = 1$) in the boundary term, leads, in general, to systems of coupled equations. Uncoupling can be gained, without using modal decomposition in space, by using a Fourier time basis,

$$T_m = \exp(i\omega_m \tau), \tag{33}$$

where $\omega_m = 2m\pi$ and i is the imaginary unit. As this basis implies periodic solutions, the boundary term in Eq. (31) is identically null, and the well-known governing equation for Fourier analysis in the frequency domain is recovered:

$$\left[\mathbf{K} + i\left(\frac{\omega_m}{T}\right)\mathbf{C}\right] \mathbf{d}_m = \mathbf{F}_m. \tag{34}$$

5.2. Parabolic problems

The same procedure can be applied to parabolic problems. The displacement approximation (29) is now written as,

$$\mathbf{u}(\mathbf{x}, t) = \sum_n T_n(\tau)\mathbf{u}_n(\mathbf{x}) \tag{35}$$

to reduce the system of Eqs. (1)–(3) to the following Helmholtz form problem:

$$\left[\mathbf{D}\mathbf{k}\mathbf{D}^* - i\left(\frac{\omega_m}{T}\right)\mathbf{c}\right] \mathbf{u}_m(\mathbf{x}) + \mathbf{f}_m(\mathbf{x}) = \mathbf{0} \quad \text{in } V, \tag{36}$$

$$\mathbf{N}\mathbf{k}\mathbf{D}^* \mathbf{u}_m(\mathbf{x}) = \mathbf{t}_{\Gamma m}(\mathbf{x}) \quad \text{on } \Gamma_\sigma, \tag{37}$$

$$\mathbf{u}_m(\mathbf{x}) = \mathbf{u}_{\Gamma m}(\mathbf{x}) \quad \text{on } \Gamma_u, \tag{38}$$

$$\mathbf{f}_m(\mathbf{x}) = \int_0^1 \widehat{T}_m \mathbf{f}(\mathbf{x}, \tau) \mathrm{d}\tau, \tag{39}$$

$$\mathbf{t}_{\Gamma m}(\mathbf{x}) = \int_0^1 \widehat{T}_m \mathbf{t}_\Gamma(\mathbf{x}, \tau) \mathrm{d}\tau, \tag{40}$$

$$\mathbf{u}_{\Gamma m}(\mathbf{x}) = \int_0^1 \widehat{T}_m \mathbf{u}_\Gamma(\mathbf{x}, \tau) \mathrm{d}\tau. \tag{41}$$

System (36)–(38) can now be solved in the space domain using, for instance, one of the variants of the finite element method, namely the alternative displacement and stress models of the hybrid-mixed, hybrid and hybrid-Trefftz formulations presented in [6, 7].

6. Time integration using modal decomposition in time

The procedure described below is designed to preserve the parabolicity present in systems (34) and (36) while avoiding the periodicity implied by the Fourier time basis (33). For simplicity, the second-order and hyperbolic problems are still analysed separately.

The basic distinction with respect to the frequency domain analysis presented in the above section is that the time approximation basis is no longer assumed to be periodic, orthogonal to itself and to its derivative, as it is implied by definition (33).

Any complete and linearly independent time basis may therefore be used to implement the integration procedure in the time domain described below. However, they are so combined as to ensure the uncoupling of the system of equations discretized in the space domain.

6.1. First-order problems

For a general time basis, Eq. (31) takes the following form:

$$\sum_n \left(\Omega_{mn}\mathbf{C} + TH_{mn}\mathbf{K}\right)\mathbf{d}_n = \widehat{T}_m(0)\mathbf{C}\mathbf{d}(0) + T\mathbf{F}_m, \tag{42}$$

where result (32) holds and:

$$H_{mn} = \int_0^1 \widehat{T}_m T_n \mathrm{d}\tau, \tag{43}$$

$$\Omega_{mn} = \widehat{T}_m(1)T_n(1) - \int_0^1 \dot{\widehat{T}}_m T_n \mathrm{d}\tau. \tag{44}$$

The key idea to uncouple Eq. (42) is to define the time approximation basis so as to ensure that the following relation between matrices $\mathbf{H}$ and matrix $\mathbf{\Omega}$, defined by Eqs. (43) and (44), holds for a diagonal matrix $\mathbf{\Omega}_*$:

$$\mathbf{\Omega} = \mathbf{H}\mathbf{\Omega}_*. \tag{45}$$

As it is shown below, the construction of a basis that ensures modal decomposition in the time domain involves the solution of two eigenvalue problems. It is noted, however, that the dimension of these problems is the dimension of the time basis, $T_n(\tau)$, which is, in general, much smaller than the dimension of the first-order problem (6). Moreover, matrices $\mathbf{H}$ and $\mathbf{\Omega}_*$ are formed by pairs of complex conjugate eigenvectors and eigenvalues, respectively, whenever a complete basis is used.

Consequent upon result (45), Eq. (42) uncouples into form,

$$\left[\mathbf{K} + \left(\frac{\Omega_{*m}}{T}\right)\mathbf{C}\right]\mathbf{d}_m = \frac{\omega_{*m}}{T}\mathbf{C}\mathbf{d}(0) + \mathbf{F}_{*m}, \tag{46}$$

in the manner of Eq. (34), and where:

$$\omega_{*m} = \sum_n H^{-1}_{mn}\widehat{T}_n(0), \tag{47}$$

$$\mathbf{F}_{*m} = \sum_n H^{-1}_{mn}\mathbf{F}_n. \tag{48}$$

The displacement estimate at instant $t = T$ is obtained substituting the solutions of system (46) in approximation (29).

6.2. Construction of the time basis for modal decomposition

Let vector $\overline{\mathbf{T}}(\tau)$ collect a complete and linearly independent set of (real) functions. The procedure described below is designed to establish a linear combination $\mathbf{T}(\tau)$ of these functions that satisfies condition (45) under definitions (43) and (44):

1. Compute the auxiliary matrix $\overline{\mathbf{H}}$ using definition (43) on the time approximation functions $\overline{\mathbf{T}}(\tau)$:

$$\overline{\mathbf{H}} = \int_0^1 \overline{\mathbf{T}}(\tau)\overline{\mathbf{T}}^{\mathrm{t}}(\tau)\mathrm{d}\tau. \tag{49}$$

2. Determine the (real, orthonormal) eigenvectors and the (real) eigenvalues of (symmetric) matrix $\overline{\mathbf{H}}$, and collect the results in matrices $\mathbf{E}$ and (diagonal) $\mathbf{S}^{-2}$, respectively:

$$\overline{\mathbf{H}}\mathbf{E} = \mathbf{E}\mathbf{S}^{-2}. \tag{50}$$

3. Compute the auxiliary matrix $\overline{\mathbf{\Omega}}$ using definition (44):

$$\overline{\mathbf{\Omega}} = \overline{\mathbf{T}}(1)\overline{\mathbf{T}}^{\mathrm{t}}(1) - \int_0^1 \overline{\mathbf{T}}'(\tau)\overline{\mathbf{T}}^{\mathrm{t}}(\tau)\mathrm{d}\tau. \tag{51}$$

4. Determine the (complex conjugate) eigenvectors and eigenvalues of matrix $\overline{\mathbf{\Omega}}$, and collect the results in matrices $\mathbf{Z}$ and (diagonal) $\mathbf{\Omega}_*$, respectively:

$$\overline{\mathbf{\Omega}}\mathbf{Z} = \mathbf{Z}\mathbf{\Omega}_*. \tag{52}$$

5. A linear combination of the time functions $\overline{\mathbf{T}}(\tau)$ that satisfies Eqs. (43) to (45) is defined by:

$$\mathbf{T}(\tau) = \left(\mathbf{E}\mathbf{S}\hat{\mathbf{Z}}^{\mathrm{t}}\mathbf{Z}\right)\overline{\mathbf{T}}(\tau). \tag{53}$$

This can be confirmed implementing definitions (43) and (44) for time basis (53) and using results (49) to (51) to obtain the following expressions:

$$\mathbf{H} = \int_0^1 \mathbf{T}(\tau)\mathbf{T}^{\mathrm{t}}(\tau)\mathrm{d}\tau = \hat{\mathbf{Z}}^{\mathrm{t}}\mathbf{Z}, \tag{54}$$

$$\mathbf{\Omega} = \mathbf{T}(1)\mathbf{T}^{\mathrm{t}}(1) - \int_0^1 \mathbf{T}'(\tau)\mathbf{T}^{\mathrm{t}}(\tau)\mathrm{d}\tau = \hat{\mathbf{Z}}^{\mathrm{t}}\overline{\mathbf{\Omega}}\mathbf{Z}. \tag{55}$$

Equation (45) is recovered substituting Eq. (52) and result (54) in Eq. (55).

The first two steps of the procedure described above can be omitted when the real time basis $\overline{\mathbf{T}}(\tau)$ is orthonormal. The implementation of this procedure is illustrated in [4] using both periodic and non-periodic bases $\overline{\mathbf{T}}(\tau)$ built on polynomial and radial functions and on the simplest form of wavelet systems. The stability and performance analyses of this basis are reported in the same reference.

6.3. Parabolic problems

The time integration procedure described above for first-order problems can be readily extended to parabolic problems. Approximation (35) for the displacement field is used now and the average enforcement of Eqs. (1)–(3) is still implemented in the sense of Galerkin using the time functions $T_m(\tau)$ as weighting functions, as described in Section 5.

Results (43) to (45) still hold and the solving systems equivalent to Eqs. (36) to (38) for frequency domain analysis are the following:

$$\left[\mathbf{DkD}^* - \left(\frac{\Omega_{*m}}{T}\right)\mathbf{c}\right]\mathbf{u}_m(\mathbf{x}) + \frac{\omega_{*m}}{T}\mathbf{cu}(\mathbf{x},0) + \mathbf{f}_{*m} = \mathbf{0} \quad \text{in} \quad V, \tag{56}$$

$$\mathbf{NkD}^*\mathbf{u}_m(\mathbf{x}) = \mathbf{t}_{\Gamma *m}(\mathbf{x}) \quad \text{on } \Gamma_\sigma, \tag{57}$$

$$\mathbf{u}_m(\mathbf{x}) = \mathbf{u}_{\Gamma *m}(\mathbf{x}) \quad \text{on } \Gamma_u. \tag{58}$$

Results (39) to (41) yield the following definitions:

$$\mathbf{f}_{*m} = \sum_n H_{mn}^{-1}\mathbf{f}_n, \tag{59}$$

$$\mathbf{t}_{\Gamma *m} = \sum_n H_{mn}^{-1}\mathbf{t}_{\Gamma m}, \tag{60}$$

$$\mathbf{u}_{\Gamma *m} = \sum_n H_{mn}^{-1}\mathbf{u}_{\Gamma m}. \tag{61}$$

Equation (56) under boundary conditions (57) and (58) may now be solved numerically using any of the available methods for the solution of elliptic equations, namely the variants of the finite element method. This is illustrated below for the conventional formulation of the finite element method and for its hybrid-Trefftz variant.

6.4. Finite element solution in the space domain

The conventional (single-field, conforming) finite element formulation develops from approximation (5), written now in form:

$$\mathbf{u}_m(\mathbf{x}) = \mathbf{U}(\mathbf{x})\mathbf{d}_m \quad \text{in} \quad V. \tag{62}$$

The solving system (46) for first-order problems is recovered applying the procedure recalled in Section 2, provided that Eqs. (7) and (8) are used to define the damping and stiffness matrices of the finite element mesh, and that nodal force definition (48) is replaced by the equivalent form of Eq. (9):

$$\mathbf{F}_{*m} = \int \mathbf{U}^{\mathrm{t}}\mathbf{f}_m \mathrm{d}V + \int \mathbf{U}^{\mathrm{t}}\mathbf{t}_{\Gamma m}\mathrm{d}\Gamma_\sigma. \tag{63}$$

To establish the solving equation for the displacement model of the hybrid-Trefftz finite element formulation, it is convenient (but not necessary) to extend approximation (35) to include a particular term, as in Eq. (28):

$$\mathbf{u}_m(\mathbf{x}) = \mathbf{U}_m(\mathbf{x})\mathbf{d}_m + \mathbf{d}_{Pm}(\mathbf{x}) \quad \text{in} \quad V. \tag{64}$$

This approximation is now complex, in general, and frequency dependent as the displacement approximation function $\mathbf{U}_m(\mathbf{x})$ is constrained to solve the homogeneous form of the Helmholtz Eq. (56),

$$\left[\mathbf{DkD}^* - \left(\frac{\Omega_{*m}}{T}\right)\mathbf{c}\right]\mathbf{U}_m(\mathbf{x}) = \mathbf{0} \quad \text{in} \quad V \tag{65}$$

and vector $\mathbf{d}_{Pm}(\mathbf{x})$ is assumed to be a particular solution of the same equation:

$$\left[\mathbf{DkD}^* - \left(\frac{\Omega_{*m}}{T}\right)\mathbf{c}\right]\mathbf{d}_{Pm}(\mathbf{x}) + \frac{\omega_{*m}}{T}\mathbf{cu}(\mathbf{x},0) + \mathbf{f}_{*m} = \mathbf{0} \quad \text{in} \quad V. \tag{66}$$

As for the conventional formulation, the equilibrium Eq. (56) is enforced on average using the (complex conjugate) displacement functions, $\widehat{\mathbf{U}}_m$, as weighting functions.

Consequent upon the Trefftz constraints (65) and (66), the displacement approximation (64) will not, in general, lead to conformity. It becomes necessary to approximate also the surface forces (on the element interfaces and) on the Dirichlet boundary of the mesh (thus the hybrid labelling):

$$\mathbf{t}_m(\mathbf{x}) = \mathbf{P}(\mathbf{x})\mathbf{p}_m \quad \text{on} \quad \Gamma_u. \tag{67}$$

Still in the manner of Galerkin, the (complex conjugate) of the surface force approximation functions are used to enforce conformity on average. The relaxed form of the Dirichlet Eq. (58) is, for the assumed displacements (64):

$$\int \widehat{\mathbf{P}}^{\mathrm{t}} \left(\mathbf{u}_m - \mathbf{u}_{\Gamma * m}\right) \mathrm{d}\Gamma_u = \mathbf{0}. \tag{68}$$

The following description is found for the solving system,

$$\begin{bmatrix} \mathbf{D}_m & -\mathbf{P}_m \\ -\widehat{\mathbf{P}}_m^{\mathrm{t}} & \mathbf{O}, \end{bmatrix} \begin{Bmatrix} \mathbf{d}_m \\ \mathbf{p}_m \end{Bmatrix} = \begin{Bmatrix} \overline{\mathbf{p}}_m \\ \overline{\mathbf{d}}_m \end{Bmatrix}, \tag{69}$$

where all intervening arrays present boundary integral definitions, as it is typical of the Trefftz method:

$$\mathbf{D}_m = \int \widehat{\mathbf{U}}_m^{\mathrm{t}} \mathbf{N}\mathbf{k}\mathbf{D}^* \mathbf{U}_m \mathrm{d}\Gamma, \tag{70}$$

$$\mathbf{P}_m = \int \widehat{\mathbf{U}}_m^{\mathrm{t}} \mathbf{P} \mathrm{d}\Gamma_u, \tag{71}$$

$$\overline{\mathbf{p}}_m = \int \widehat{\mathbf{U}}_m^{\mathrm{t}} \mathbf{N}\mathbf{k}\mathbf{D}^* \mathbf{u}_{Pm} \mathrm{d}\Gamma + \int \widehat{\mathbf{U}}_m^{\mathrm{t}} \mathbf{t}_{\Gamma m} \mathrm{d}\Gamma_\sigma, \tag{72}$$

$$\overline{\mathbf{d}}_m = \int \widehat{\mathbf{P}}^{\mathrm{t}} \left(\mathbf{u}_{Pm} - \mathbf{u}_{\Gamma * m}\right) \mathrm{d}\Gamma_u. \tag{73}$$

A detailed derivation of the results presented above can be found in [4, 6, 7], where the alternative stress model of the hybrid-Trefftz finite element formulation is also considered, as well as the derivation of a Pian-type hybrid stress element.

7. Closure

Analytical solutions of parabolic equations can be derived either by searching solutions that depend on time and space or by basing this search on the assumption that the solution is separable in time and space, e.g. [8]. The same approach can be followed in the construction of approximate solution methods.

The paper illustrates the implementation of numerical solution procedures based on separation of variables, with the objective of assessing how these procedures relate with the Trefftz method.

As it is well known, when the basis stated a priori is written in the space domain, the parabolic equation collapses into a first-order problem. The Trefftz concept is being used, in fact, when the local solution of this time dependent problem is used to construct the approximation basis, as in the approach suggested by Tamma et al. [3].

The major limitation of this approach is that the resulting algebraic system of equations can be solved efficiently only when modal decomposition in space is assumed. Besides the obstacles found in the extension of the method to the solution of non-linear problems, the resulting solving system is derived at the expense of solving large-scale eigenvalue problems. This cost is frequently minimised

by neglecting the higher frequency modes, which may, however, influence strongly the quality of the estimates for the associated gradient fields, the strain and stress fields in structural analyses.

The second disadvantage of following the path of replacing parabolic systems of equations by equivalent first-order differential systems and algebraic solving systems, in this sequence, is that it pre-empts the use of the Trefftz concept at the space discretization level.

It is to avoid this major limitation that the method reported here is based on the option of starting the approximation procedure on discretization in time. The standard procedure in this approach consists in enforcing a Fourier (or Trefftz) modal decomposition in time. This is recalled here to establish that the resulting analysis in the frequency domain can be emulated using non-periodic time approximation bases. The resulting time integration method reported here [4] is general in the sense that it can be implemented on any complete and linearly independent basis. Moreover, it may be easily extended to the solution of non-linear problems.

Its efficiency hinges on the introduction of the idea of using modal decomposition in the time domain and not (or not necessarily) in the space domain. The dimension of the eigenvalue problem is strongly reduced, as it depends now on the dimension of the time basis and not on the number of degrees-of-freedom of the equation of motion. In addition, orthogonality is not called upon, which releases the approach from direct dependency on the structural material properties.

Finally, and equally relevant, the time integration method is independent of the method used for discretization of the structural problem in the space domain. Therefore, the solution of the resulting hyperbolic problem, in the sense of Helmholtz, can be solved using any of the available methods for the solution of elliptic equations. This is illustrated here using the displacement model of the hybrid-Trefftz formulation. The implementation of the alternative stress model of the same formulation is reported in [4].

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