Computer Assisted Mechanics and Engineering Sciences, 10: 465–477, 2003.

Time integration and the Trefftz Method Part II – Second-order and hyperbolic problems

João A. Teixeira de Freitas
Departamento de Engenharia Civil e Arquitectura, Instituto Superior Técnico
Technical University of Lisbon, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

(Received September 11, 2003)

The finite element method is applied in the time domain to establish formulations for the integration of second-order and hyperbolic (dynamic) problems. Modal decomposition in the space domain is used to recover the well-established method for uncoupling the equations of motion, which is extended to include general time approximation bases. The limitations of this approach in the implementation of large-scale, non-linear problems while preserving the uncoupling of the equations of motion are overcome by using the alternative concept of modal decomposition in the time domain. Both single- and double-field formulations are presented and the associated Trefftz formulations are established.

Keywords: Time integration, second-order problems, hyperbolic problems, Trefftz method

1. Introduction

The paper addresses the formulation of two basic time integration methods that preserve the hyperbolicity of structural dynamics problems [1]. The first is based on modal decomposition in the space domain and the second in modal decomposition in the time domain.

The basis of the first method is well established. The first step consists in discretising in the space domain the local hyperbolic problem using the conventional formulation of the finite element method where the nodal displacement vector is assumed to be time dependent. The resulting second-order system of equations is subsequently uncoupled through modal decomposition in the space domain and solved using one of the available (implicit or explicit) time integration methods.

This method of numerical integration is formulated here in a more general way, through the implementation in the time domain of two standard finite element formulations, namely a single-field and a double-field (or mixed) formulation. In the first formulation the velocity vector field is taken as the derivative of the displacement approximation in time, while in the second the velocity field is approximated independently. The time approximation bases that can be used are general, in the sense that they are not constrained to solve the governing equations of motion.

The single-field formulation is used to recall that the standard modal decomposition in the space domain corresponds to the application of the Trefftz concept in the time domain. It is used to show also that the generalised VIP method of Tamma *et al.* [2] is fundamentally a Trefftz method. The double-field approximation is used to show how to circumvent the numerical implementation difficulties inherent to the double-field formulation suggested by the same authors. It serves also the purpose of showing that the original version of the VIP method is an incomplete Trefftz approach.

The main advantage offered by the method of modal decomposition in the space domain is the derivation of uncoupled equations of motion with known closed form solution. However, this formal simplicity is gained at the generally very high cost of solving a large-scale eigenvalue problem, as its dimension is dictated by the number of degrees-of-freedom of the finite element discretization in the space domain of a given application.

Besides this limitation and the difficulties found in its extension into non-linear applications, modal decomposition in the space domain inhibits the implementation a priori of the Trefftz discretization of the space component of the governing hyperbolic equations. To open the possibility of using the Trefftz method in space discretization is the main target of the second method of time integration reported in the paper.

The standard method of modal decomposition in the time domain is recalled first to equate Fourier analysis with the Trefftz method. This approach leads to the solution of dynamic problems in the frequency domain, where the Trefftz method has been used with remarkable success, e.g. [3, 4]. This method is extended here to the time domain, to remove thus the strongly limiting assumption of periodicity (periodic extension) of the solution.

As in the case of modal decomposition in space, this method of modal decomposition in the time domain develops from the basic simplification of separating the space and time fields of the solution. A mixed finite element formulation is then established to model the time response in the time domain independently of the of the formulation eventually used to model the space component of the response.

Still as in the case of the modal decomposition in space, the numerical efficacy of the resulting time integration procedure depends directly on the possibility of uncoupling the equations of motion. The distinction now is that this uncoupling targets only the time component of the model.

This is achieved by solving an eigenvalue problem in the time domain that depends uniquely on the time approximation basis being used. Consequently, the dimension of this problem is in general very small (of order n+1 if a polynomial time basis of degree n is used). Moreover, this eigenvalue problem is solved only once, for a given basis, and used subsequently in the implementation of different classes of structural applications. Although it is not particularly relevant in terms of numerical implementation, it is possible to establish closed analytical definitions for the amplification matrix and for the load vector low-degree time approximation bases [1].

Modal decomposition in the time domain is applied first to second-order systems of equations to establish its relation with the alternative technique of modal decomposition in space discussed in the first part of the paper. It is subsequently applied to the hyperbolic equations governing structural dynamics problems to illustrate its implementation with alternative space discretization formulations, namely the standard finite element method displacement formulation and the alternative stress and displacement models of the hybrid-Trefftz finite element formulation.

2. Standard integration procedure in the time domain

Linear structural dynamics problems are governed by hyperbolic equations of the form,

$$\mathbf{DkD^*u} + \mathbf{f} = \mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} \quad \text{in } V \quad \text{and } 0 \leq t \leq T, \tag{1}$$

where t is the time parameter and V the structural domain referred to a Cartesian system $\mathbf{x}$, $\mathbf{u}(\mathbf{x},t)$, $\dot{\mathbf{u}}$, and $\ddot{\mathbf{u}}$ are the displacement, velocity and acceleration fields, respectively, $\mathbf{D}$ is a linear differential operator in the space domain and $\mathbf{D}^*$ denotes its conjugate, $\mathbf{m}$, $\mathbf{c}$ and $\mathbf{k}$ are (local) mass, damping and stiffness matrices, which are assumed to be symmetric and with constant coefficients for simplicity of the presentation, and vector $\mathbf{f}$ defines the forcing load and the effect of eventual residual strains and stresses.

Besides the initial conditions of the problem, which are written as follows,

$$\mathbf{u}_0(\mathbf{x}) = \mathbf{u}(\mathbf{x},0) \quad \text{in } V \quad \text{and } t = 0, \tag{2}$$

$$\mathbf{v}_0(\mathbf{x}) = \dot{\mathbf{u}}(\mathbf{x},0) \quad \text{in } V \quad \text{and } t = 0, \tag{3}$$

the solution of the governing system (1) depends also on consistent boundary conditions, which are simplified here to the usual Neumann and Dirichlet forms:

$$\mathbf{NkD^*u} = \mathbf{t}_\Gamma \quad \text{on } \Gamma_\sigma \quad \text{and } 0 \leq t \leq T, \tag{4}$$

$$\mathbf{u} = \mathbf{u}_\Gamma \quad \text{on } \Gamma_u \quad \text{and } 0 \leq t \leq T. \tag{5}$$

The standard integration procedure consists in approximating the displacement field by separation of variables in time and space,

$$\mathbf{u}(\mathbf{x},t)=\mathbf{U}(\mathbf{x})\mathbf{d}(t) \quad \text{in } V \quad \text{and } 0\le t\le T, \tag{6}$$

to reduce system (1) to a second-order differential equation in the time domain,

$$\mathbf{Ma}+\mathbf{Cv}+\mathbf{Kd}=\mathbf{F}(t), \tag{7}$$

where $\mathbf{a}=\ddot{\mathbf{d}}$ and $\mathbf{v}=\dot{\mathbf{d}}$ are the acceleration and velocity vectors, respectively. The definitions of the (symmetric) mass, damping and stiffness matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$, respectively, and of the consistent forcing load vector, $\mathbf{F}$, are recalled in Appendix A.

These definitions show that the domain and boundary equilibrium conditions (1) and (4) are enforced on average, in the sense of Galerkin. C^{n-1} continuity is then enforced for nth-order problems (n being the order of the space differential operator $\mathbf{D}$) and by ensuring locally (in space and in time) the Dirichlet condition (5). It is assumed that conformity applies also to the initial conditions (2) and (3),

$$\mathbf{u}_0(\mathbf{x})=\mathbf{U}(\mathbf{x})\mathbf{d}(0) \quad \text{in } V \quad \text{and } t=0, \tag{8}$$

$$\mathbf{v}_0(\mathbf{x})=\mathbf{U}(\mathbf{x})\mathbf{v}(0) \quad \text{in } V \quad \text{and } t=0, \tag{9}$$

although these conditions may also, and in general, be enforced on average.

System (7–9) can be integrated using a wide variety of methods. They are usually classified as explicit and implicit integration schemes, the latter being the most commonly used. The central objective of these schemes is to establish (approximate) solutions of equation (7) at instant $t=T$, in form (10), where $\mathbf{A}$ is the amplification matrix and $\mathbf{L}$ the load vector:

$$\left\{\begin{array}{c}\mathbf{d}(T)\\ T\mathbf{v}(T)\end{array}\right\}=\mathbf{A}\left\{\begin{array}{c}\mathbf{d}(0)\\ T\mathbf{v}(0)\end{array}\right\}+\mathbf{L}. \tag{10}$$

3. Time integration using modal decomposition in space

Under conditions (12) and (13), the closed form solution of Eq. (7) is defined by,

$$\mathbf{d}(t)=\sum_{n=1}^{\beta}\mathbf{X}_n Y_n(\tau), \tag{11}$$

where $\tau=t/T$ is the non-dimensional time parameter ($0\le\tau\le 1$), $\mathbf{X}_n$ represents the n-th eigenvector of the (undamped, free vibration with normalised frequency ω_n) eigenvalue problem,

$$\mathbf{K}\mathbf{X}_n=\left(\frac{\omega_n}{T}\right)^2\mathbf{M}\mathbf{X}_n \tag{12}$$

and Y_n is the solution of the scalar second-order equation:

$$Y_n''+2\xi_n\omega_n Y_n'+\omega_n^2 Y_n=T^2\mathbf{X}_n^{\mathrm{t}}\mathbf{F}(\tau). \tag{13}$$

A (Rayleigh) damping orthogonality condition is assumed to hold for $\mathbf{M}$-orthonormal eigenvectors:

$$\mathbf{C}\mathbf{X}_n=2\xi_n\left(\frac{\omega_n}{T}\right)\mathbf{M}\mathbf{X}_n, \tag{14}$$

$$\mathbf{X}_m^{\mathrm{t}}\mathbf{M}\mathbf{X}_n=\delta_{mn}. \tag{15}$$

The closed-form solution (11) is relatively costly in terms of numerical implementation, as it involves the solution of the eigenvalue problem (12) for each application, as its dimension is dictated by the number of degrees-of-freedom, β, of approximation (11). In addition, its extension to non-linear problems is not trivial.

However, the modal decomposition in the space domain (11) is useful to inspire the development of numerical methods for the integration of both linear and non-linear second-order problems encoded in form (7). This is shown below using a finite element approach applied now to the time domain, using two formulations, namely a single-field and a double-field formulation.

The single-field formulation is implemented on the displacement approximation (11) in form (16), where functions $T_n(\tau)$ define a (complete, linearly independent) time approximation basis and the weighting parameter d_n represents a generalised displacement:

$$\mathbf{d}(t) = \sum_{n=1}^{\beta} \mathbf{X}_n T_n(\tau) d_n. \tag{16}$$

In this formulation the velocity field is determined by direct differentiation of the displacement approximation (20):

$$\mathbf{v}(t) = \frac{1}{T}\sum_{n=1}^{\beta} \mathbf{X}_n T_n'(\tau) d_n. \tag{17}$$

What distinguishes the second formulation is that the velocity is approximated independently of the displacement approximation (16):

$$\mathbf{v}(t) = \frac{1}{T}\sum_{n=1}^{\beta} \mathbf{X}_n T_n(\tau) v_n. \tag{18}$$

The resulting formulation qualifies as mixed, as it involves the simultaneous approximation of two fields in the (time) domain of the analysis.

3.1. Single-field approximation in the time domain

As in the standard (Galerkin) procedure for the integration of elliptic equations by the conventional (conforming) finite element method, the approximation functions are used to enforce on average the equilibrium Eq. (7), to yield,

$$\mathbf{X}_n^{\mathrm{t}} \int_0^T \widehat{T}_n \left[\mathbf{Ma} + \mathbf{Cv} + \mathbf{Kd} - \mathbf{F}(t)\right] dt = 0, \tag{19}$$

where $\widehat{T}_n$ denotes the conjugate of (the eventually complex) approximation function T_n. It is recalled that, due to symmetry, the eigenvectors $\mathbf{X}_n$ are real.

Equation (19) is integrated by parts, once on the velocity term and twice on the acceleration term, in order to force the emergence of the boundary terms and thus enforce explicitly the initial conditions. The following result (see Appendix B) is obtained after implementing the displacement approximation (16) in domain $0 \leq \tau \leq 1$ and at instant $\tau = 1$, and the velocity approximation (17) at the same instant:

$$D_n d_n = A_n d_n(0) + \widehat{T}_n(0) T v_n(0) + T^2 F_n. \tag{20}$$

After solving Eq. (20) for a given time step, T, definitions (16) and (17), computed at instant $\tau = 1$, can be used to construct the amplification matrix and of the load vector present in Eq. (10).

3.2. Mixed approximation in the time domain

The mixed formulation develops from the independent approximations (16) and (18) for the displacement and velocity fields. The motion Eq. (7) is still enforced on average, in form (19). Consequent upon the independent approximation of the velocity field, this equation is now integrated by parts only once. The initial conditions are enforced and approximations (16) and (18) are implemented in domain $0 \leq \tau \leq 1$ and at instant $\tau = 1$, to yield the following result, where definitions (B3)–(B5) in Appendix B are used:

$$\Omega_n v_n + \left(2\xi_n\omega_n\Omega_n + \omega_n^2 H_n\right) d_n = 2\xi_n\omega_n\widehat{T}_n(0)d_n(0) + \widehat{T}_n(0)Tv_n(0) + T^2F_n, \tag{21}$$

$$H_n = \int_0^1 \widehat{T}_n T_n d\tau, \tag{22}$$

$$\Omega_n = \widehat{T}_n(1)T_n(1) - \int_0^1 \widehat{T}'_n T_n d\tau. \tag{23}$$

Consistency of the independent displacement and velocity estimates (16) and (18) is enforced also on average in the following form, to exploit the orthogonality condition (15):

$$\mathbf{X}_n^{\mathrm{t}} \int_0^T \widehat{T}_n \mathbf{M}\left(\mathbf{v} - \dot{\mathbf{d}}\right) \mathrm{d}t = 0. \tag{24}$$

Similarly to the procedure used to obtain result (21), this equation is integrated by parts to enforce the initial condition on the displacement field. After implementing approximation (16) at instant $\tau = 1$ and in domain $0 \leq \tau \leq 1$ and the velocity approximation (18) in the same period, the following result is obtained:

$$v_n = \Omega_{*n} d_n - \omega_{*n} d_n(0), \tag{25}$$

$$\Omega_{*n} = H_n^{-1}\Omega_n, \tag{26}$$

$$\omega_{*n} = H_n^{-1}\widehat{T}_n(0). \tag{27}$$

The solving equation is obtained substituting result (25) in Eq. (21), see Appendix B:

$$D_{*n} d_n = A_{*n}\omega_{*n} d_n(0) + \omega_{*n} T v_n(0) + T^2 F_{*n}, \tag{28}$$

$$F_{*n} = H_n^{-1} F_n. \tag{29}$$

Results (28) and (25) are enforced in approximations (16) and (18) at instant $t = T(\tau = 1)$ to establish the definitions of the amplification matrix and of the load vector in Eq. (10).

3.3. Trefftz method

To recover the Trefftz method it suffices to constrain the time basis used in the single-field approximation to satisfy locally the homogeneous equation of motion in conjugate form, see Eq. (13):

$$\widehat{T}''_n - 2\xi_n\omega_n\widehat{T}'_n + \omega_n^2\widehat{T}_n = 0. \tag{30}$$

After implementing the general solution (31), where i is the imaginary unit and ω_{dn} is the (under-) damped free frequency,

$$T_n = \exp\left[(\xi_n\omega_n + i\omega_{dn})\tau\right], \tag{31}$$

$$\omega_{dn} = \omega_n\sqrt{1 - \xi_n^2}, \tag{32}$$

the domain integral present in the solving Eq. (28) vanishes, see Eq. (B1), as it is typical of the Trefftz method. The resulting Trefftz expressions for parameters D_n and A_n in Eq. (28) are given in Appendix B.

It can be verified that this solution corresponds to the closed form solution of system (7). This becomes particularly obvious when a real description for the time basis (31) is adopted and approximation (16) is extended to include a particular solution of Eq. (13), written, for instance, in the form of the Duhamel integral:

$$\mathbf{d}(t) = \sum_{n=1}^{\beta} \mathbf{X}_n \left(T_n(\tau) d_n + u_{pn} \right). \tag{33}$$

3.4. VIP method

It can be easily verified that the Trefftz result presented above corresponds to the generalised VIP method of Tamma et al. [2]. The original version of this method is recovered if the integration by parts of Eq. (19) is not implemented on the damping term. This implies the approximation of the velocity field, meaning that the original VIP method is, indeed, a double-field approximation method.

To avoid the integration on the load term, see Eq. (B5), K. K. Tamma and his co-authors assume a linear approximation of the forcing load over the time interval $0 \leq \tau \leq 1$. The same approximation is assumed for the velocity field in the original VIP method.

The mixed formulation presented above is conceptually distinct from the explicitly double-field approximation suggested in [2], which develops from the equivalent first-order expression of the equation of motion (7), at the cost of doubling the dimension of the problem under analysis, see Appendix B:

$$\dot{\mathbf{y}}_* + \mathbf{A}_* \mathbf{y}_* = \mathbf{F}_*. \tag{34}$$

Similarly to the procedure followed here, this equation is enforced on average using weighting time functions $\mathbf{W}(t)$, and integrated by parts to enforce the initial conditions:

$$\mathbf{W}^{\mathrm{t}}(T)\mathbf{y}_*(T) - \int_0^T \left(\dot{\mathbf{W}}^{\mathrm{t}} - \mathbf{W}^{\mathrm{t}}\mathbf{A}_* \right) \mathbf{y}_* \mathrm{d}t = \mathbf{W}^{\mathrm{t}}(0)\mathbf{y}_*(0) + \int_0^T \mathbf{W}^{\mathrm{t}}\mathbf{F}_* \mathrm{d}t. \tag{35}$$

This is formally equivalent to the coupling of Eqs. (19) and (24) used in the mixed formulation described above. However, two fundamental distinctions remain. The first is that the β velocity degrees-of-freedom are not treated explicitly as dependent variables, as stated by Eq. (21). The second distinction is that Tamma et al. [3], still in the manner of Trefftz, constrain the basis to solve the conjugate homogeneous equation of motion (33):

$$\dot{\mathbf{W}}^{\mathrm{t}} - \mathbf{W}^{\mathrm{t}}\mathbf{A}_* = \mathbf{0}. \tag{36}$$

This Trefftz condition and its associated solution (37), see Eqs. (30) and (31), are then enforced in system (35), to yield result (38):

$$\mathbf{W} = \exp\left[\mathbf{A}_*^{\mathrm{t}} t\right], \tag{37}$$

$$\exp\left[\mathbf{A}_*^{\mathrm{t}} T\right] \mathbf{y}_*(T) = \mathbf{y}_*(0) + \int_0^T \exp\left[\mathbf{A}_*^{\mathrm{t}} t\right] \mathbf{F}_*(t) \mathrm{d}t. \tag{38}$$

It is obvious that the double-field approximation method proposed in [2] collapses to the Trefftz method with modal decomposition in space, that is, to the generalised VIP method, when the exact solution (37) is implemented in Eq. (35). Instead of following this option, K. K. Tamma and his co-authors generate approximate solutions based on the decomposition of matrix $\mathbf{A}_*$ weighted by convergence tuning parameters. This decomposition is developed under the condition that modal decomposition can still be applied in linear dynamic systems in order to use the Lax equivalence theorem, as stated in [2].

4. Standard integration procedure in the frequency domain

In order to clarify the presentation in Section 5 of the integration method based on modal decomposition in the time domain, it is convenient to recall first the standard procedure for integration in the frequency domain by addressing separately second-order problems of the form (7) and hyperbolic problems governed by Eqs. (1) to (5).

4.1. Second-order problems

To avoid its implicit dependency on modal decomposition in space, let the displacement approximation (16) be written in the following form, where $\mathbf{d}_n$ is now a weighting vector and α is the dimension of the time basis (thus opening the possibility of using $\alpha \ll \beta$, the dimension of vector $\mathbf{d}$):

$$\mathbf{d}(t) = \sum_{n=1}^{\alpha} T_n(\tau)\mathbf{d}_n. \tag{39}$$

The method of Galerkin is still used to enforce Eq. (7) on average, thus replacing Eq. (19) by Eq. (40), which is integrated by parts twice:

$$\int_0^T \widehat{T}_n \left[\mathbf{Ma} + \mathbf{Cv} + \mathbf{Kd} - \mathbf{F}(t)\right] \mathrm{d}t = \mathbf{0}, \tag{40}$$

$$\int_0^T \left[\widehat{\ddot{T}}_n\mathbf{M} - \widehat{\dot{T}}_n\mathbf{C} + \widehat{T}_n\mathbf{K}\right]\mathbf{d}\mathrm{d}t + \left[\widehat{T}_n\mathbf{M}\dot{\mathbf{d}} - \widehat{\dot{T}}_n\mathbf{M}\mathbf{d} + \widehat{T}_n\mathbf{C}\mathbf{d}\right]_0^T = \int_0^T \widehat{T}_n\mathbf{F}\mathrm{d}t. \tag{41}$$

Substitution of approximation (39) in the integral term of Eq. (41) and of its value at instant $t = T$ in the boundary term leads, in general, to coupled equations of motions. Uncoupling can be gained, without calling upon modal decomposition in space, by using a Fourier time basis,

$$T_n = \exp(i\omega_n\tau), \tag{42}$$

which represents the undamped form of definition (31) and where now $\omega_n = 2n\pi$. As this basis implies periodic solutions, the boundary term in Eq. (41) is identically null, and the well-known governing equation for Fourier analysis in the frequency domain is recovered:

$$\left[\mathbf{K} + \left(\frac{\omega_n}{T}\right)\mathbf{C} - \left(\frac{\omega_n}{T}\right)^2\mathbf{M}\right]\mathbf{d}_n = \mathbf{F}_n, \tag{43}$$

$$\mathbf{F}_n = \int_0^1 \widehat{T}_n\mathbf{F}\mathrm{d}\tau. \tag{44}$$

4.2. Hyperbolic problems

The same procedure can be applied to the hyperbolic problems. The displacement approximation (38) is now written as,

$$\mathbf{u}(\mathbf{x}, t) = \sum_{n=1}^{\alpha} T_n(\tau)\mathbf{u}_n(\mathbf{x}) \quad \text{in } V, \tag{45}$$

to reduce the system of Eqs. (1), (4) and (5) to the following Helmholtz problem:

$$\left[\mathbf{DkD}^* - i\left(\frac{\omega_n}{T}\right)\mathbf{c} + \left(\frac{\Omega_n}{T}\right)^2\mathbf{m}\right]\mathbf{u}_n(\mathbf{x}) + \mathbf{f}_n(\mathbf{x}) = \mathbf{0} \quad \text{in } V, \tag{46}$$

$$\mathbf{NkD^*u}_n(\mathbf{x}) = \mathbf{t}_{\Gamma n}(\mathbf{x}) \quad \text{on } \Gamma_\sigma, \tag{47}$$

$$\mathbf{u}_n(\mathbf{x}) = \mathbf{u}_{\Gamma n}(\mathbf{x}) \quad \text{on } \Gamma_u, \tag{48}$$

$$\mathbf{f}_n(\mathbf{x}) = \int_0^1 \widehat{T}_n \mathbf{f}(\mathbf{x}, \tau)\mathrm{d}\tau, \tag{49}$$

$$\mathbf{t}_{\Gamma n}(\mathbf{x}) = \int_0^1 \widehat{T}_n \mathbf{t}_\Gamma(\mathbf{x}, \tau)\mathrm{d}\tau, \tag{50}$$

$$\mathbf{u}_{\Gamma n}(\mathbf{x}) = \int_0^1 \widehat{T}_n \mathbf{u}_\Gamma(\mathbf{x}, \tau)\mathrm{d}\tau. \tag{51}$$

System (46)–(48) can now be solved in the space domain using, for instance, one of the variants of the finite element method, namely the alternative displacement and stress models of the hybrid-mixed, hybrid and hybrid-Trefftz formulations presented in [5, 6].

5. Integration using modal decomposition in time

The procedure described below is designed to preserve the hyperbolicity present in systems (43) and (46) while avoiding the periodicity implied by the Fourier time basis (42). For simplicity, the second-order and hyperbolic problems are still analysed separately.

The basic distinction with respect to the frequency domain analysis presented in the previous section is that a mixed formulation is now used, meaning that the displacement approximations (39) and (45) for second-order and hyperbolic problems are now complemented with independent approximations for the velocity field:

$$\mathbf{v}(t) = \frac{1}{T}\sum_{n=1}^{\alpha} T_n(\tau)\mathbf{v}_n, \tag{52}$$

$$\mathbf{v}(\mathbf{x}, t) = \frac{1}{T}\sum_{n=1}^{\alpha} T_n(\tau)\mathbf{v}_n(\mathbf{x}). \tag{53}$$

Moreover, the time approximation basis is no longer assumed to be periodic, orthogonal to itself and to its derivative, as it is implied by definition (42). Any complete and linearly independent time basis may therefore be used to implement the integration procedure in the time domain described below.

5.1. Second-order problems

The time approximation functions present in definitions (39) and (52) are used as weighting functions in the average (Galerkin) enforcement of the equations of motion, in form (40), and of the velocity definition, thus replacing Eq. (24) by the following:

$$\int_0^T \widehat{T}_n \left(\mathbf{v} - \dot{\mathbf{d}}\right) \mathrm{d}t = 0. \tag{54}$$

It can be readily verified that integration by parts of Eqs. (40) and (54) yields the following generalisation for Eqs. (21) and (25),

$$\sum_{n=1}^{\alpha}[\Omega_{mn}\mathbf{M}\mathbf{v}_n + T(\Omega_{mn}\mathbf{C} + TH_{mn}\mathbf{K})\,\mathbf{d}_n] = T\widehat{T}_m(0)\,[\mathbf{M}\mathbf{v}(0) + \mathbf{C}\mathbf{d}(0)] + T^2\mathbf{F}_m, \tag{55}$$

$$\sum_{n=1}^{\alpha} H_{mn}\mathbf{v}_n = \sum_{n=1}^{\alpha}\Omega_{mn}\mathbf{d}_n - \widehat{T}_m(0)\mathbf{d}(0), \tag{56}$$

where result (44) holds and where the following expressions replace Eqs. (22) and (23):

$$H_{mn} = \int_0^1 \widehat{T}_m T_n \mathrm{d}\tau, \tag{57}$$

$$\Omega_{mn} = \widehat{T}_m(1)T_n(1) - \int_0^1 \dot{\widehat{T}}_m T_n \mathrm{d}\tau. \tag{58}$$

As it is stated in Part I, the key idea to uncouple systems (55) and (56) is to define the time approximation basis so as to ensure that the following relation between matrices $\mathbf{H}$ and matrix $\boldsymbol{\Omega}$, defined by Eqs. (57) and (58), holds for a diagonal matrix $\boldsymbol{\Omega}_*$, as in Eq. (26):

$$\boldsymbol{\Omega} = \mathbf{H}\boldsymbol{\Omega}_*. \tag{59}$$

Matrices $\mathbf{H}$ and $\boldsymbol{\Omega}_*$ are formed by pairs of complex conjugate eigenvectors vectors and eigenvalues, respectively, whenever a complete basis is used. Consequent upon result (59), Eqs. (56) and (55) uncouple into form,

$$\mathbf{v}_n = \Omega_{*n}\mathbf{d}_n - \omega_{*n}\mathbf{d}(0), \tag{60}$$

$$\left[\mathbf{K} + \left(\frac{\Omega_{*n}}{T}\right)\mathbf{C} + \left(\frac{\Omega_{*n}}{T}\right)^2\mathbf{M}\right]\mathbf{d}_n = \frac{\omega_{*n}}{T}\left[\left(\mathbf{C} + \frac{\Omega_{*n}}{T}\mathbf{M}\right)\mathbf{d}(0) + \mathbf{M}\mathbf{v}(0)\right] + \mathbf{F}_{*n}, \tag{61}$$

in the manner of Eqs. (25) and (28), respectively, and where the following definitions replace results (27) and (29), with H_{mn}^{-1} representing the coefficient of the inverse of matrix $\mathbf{H}$:

$$\omega_{*m} = \sum_{n=1}^{\alpha} H_{mn}^{-1}\widehat{T}_n(0), \tag{62}$$

$$\mathbf{F}_{*m} = \sum_{n=1}^{\alpha} H_{mn}^{-1}\mathbf{F}_n. \tag{63}$$

The displacement and velocity estimates at instant $t = T$ are obtained substituting the solutions of systems (61) and (60) in approximations (39) and (52), respectively.

5.2. Hyperbolic problems

The time integration procedure described above for second-order problems can be readily extended to hyperbolic problems. Approximations (45) and (53) for the displacement and velocity fields are used now and the average enforcement of Eqs. (1), (4) and (5) is still implemented in the sense of Galerkin using the time functions as weighting functions to obtain the solving systems equivalent to the second-order problem Eqs. (60) and (61):

$$\mathbf{v}_n(\mathbf{x}) = \Omega_{*n}\mathbf{u}_n(\mathbf{x}) - \omega_{*n}\mathbf{u}(\mathbf{x},0), \tag{64}$$

$$\left[\mathbf{D}\mathbf{k}\mathbf{D}^* - \left(\frac{\Omega_{*n}}{T}\right)\mathbf{c} - \left(\frac{\Omega_{*n}}{T}\right)^2\mathbf{m}\right]\mathbf{u}_n(\mathbf{x}) + \frac{\omega_{*n}}{T}\left[\left(\mathbf{c} + \frac{\Omega_{*n}}{T}\mathbf{m}\right)\mathbf{u}(\mathbf{x},0) + \mathbf{m}\mathbf{v}(\mathbf{x},0)\right] + \mathbf{f}_{*n} = \mathbf{0}. \tag{65}$$

This equation preserves the hyperbolicity of Eq. (1), in the sense of Helmholtz, and replaces the spectral analysis equation of motion (46), with:

$$\mathbf{f}_{*m} = \sum_{n=1}^{\alpha} H_{mn}^{-1} \mathbf{f}_n. \tag{66}$$

Results (49) to (51) still hold, and the boundary conditions that replace Eqs. (47) and (48) are:

$$\mathbf{NkD}^* \mathbf{u}_n(\mathbf{x}) = \mathbf{t}_{\Gamma*n}(\mathbf{x}) \quad \text{on } \Gamma_\sigma, \tag{67}$$

$$\mathbf{u}_n(\mathbf{x}) = \mathbf{u}_{\Gamma*n}(\mathbf{x}) \quad \text{on } \Gamma_u, \tag{68}$$

$$\mathbf{t}_{\Gamma*m} = \sum_{n=1}^{\alpha} H_{mn}^{-1} \mathbf{t}_{\Gamma n}, \tag{69}$$

$$\mathbf{u}_{\Gamma*m} = \sum_{n=1}^{\alpha} H_{mn} \mathbf{u}_{\Gamma n}. \tag{70}$$

Equation (65) under boundary conditions (67) and (68) may now be solved numerically using any of the available methods for the solution of elliptic equations, namely the variants of the finite element method. This is illustrated below for the conventional formulation of the finite element method and for its hybrid-Trefftz variant.

5.3. Finite element solution in the space domain

The conventional (single-field, conforming) finite element formulation develops from approximation (6), written now in form:

$$\mathbf{u}_n(\mathbf{x}) = \mathbf{U}(\mathbf{x}) \mathbf{d}_n \quad \text{in } V. \tag{71}$$

The solving system (61) for second-order problems is recovered applying the procedure recalled in Section 2, provided that Eqs. (8) to (10) are used to define the mass, damping and stiffness matrices of the finite element mesh, and that nodal force definition (63) is replaced by the equivalent form of Eq. (11):

$$\mathbf{F}_{*n} = \int \mathbf{U}^t \mathbf{f}_{*n} \mathrm{d}V + \int \mathbf{U}^t \mathbf{t}_{\Gamma*n} \mathrm{d}\Gamma_\sigma. \tag{72}$$

To establish the solving equation for the displacement model of the hybrid-Trefftz finite element formulation, it is convenient (but not necessary) to extend approximation (71) to include a particular term as in Eq. (33):

$$\mathbf{u}_n(\mathbf{x}) = \mathbf{U}_n(\mathbf{x}) \mathbf{d}_n + \mathbf{d}_{Pn}(\mathbf{x}) \quad \text{in } V. \tag{73}$$

Moreover, this approximation is now complex, in general, and frequency dependent as the displacement approximation function $\mathbf{U}_n(\mathbf{x})$ is constrained to solve the homogeneous form of the Helmholtz Eq. (65),

$$\left[\mathbf{DkD}^* - \left(\frac{\Omega_{*n}}{T} \right) \mathbf{c} - \left(\frac{\Omega_{*n}}{T} \right)^2 \mathbf{m} \right] \mathbf{U}_n(\mathbf{x}) = \mathbf{0} \tag{74}$$

and vector $\mathbf{d}_{Pn}(\mathbf{x})$ is assumed to be a particular solution of the same equation:

$$\left[\mathbf{DkD}^* - \left(\frac{\Omega_{*n}}{T} \right) \mathbf{c} - \left(\frac{\Omega_{*n}}{T} \right)^2 \mathbf{m} \right] \mathbf{d}_{Pn}(\mathbf{x}) + \frac{\omega_{*n}}{T} \left[\left(\mathbf{c} + \frac{\Omega_{*n}}{T} \mathbf{m} \right) \mathbf{u}(\mathbf{x}, 0) + \mathbf{m} \mathbf{v}(\mathbf{x}, 0) \right] + \mathbf{f}_{*n} = \mathbf{0}. \tag{75}$$

As for the conventional formulation, the equilibrium Eq. (65) is enforced on average using the (complex conjugate) displacement functions, $\hat{\mathbf{U}}_n$, as weighting functions.

Consequent upon the Trefftz constraints (74) and (75), the displacement approximation (73) will not, in general, lead to conformity. It becomes necessary to approximate also the surface forces (on the element interfaces and) on the Dirichlet boundary of the mesh (thus the hybrid labelling):

$$\mathbf{t}_n(\mathbf{x}) = \mathbf{P}(\mathbf{x})\mathbf{p}_n \quad \text{on } \Gamma_u. \tag{76}$$

The (complex conjugate) of the functions used to approximate the surface forces are used to enforce conformity on average. The Dirichlet Eq. (68) is thus relaxed to form (77), for the assumed displacements (73):

$$\int \hat{\mathbf{P}}^{\mathrm{t}}\left(\mathbf{u}_n - \mathbf{u}_{\Gamma * n}\right) \mathrm{d}\Gamma_u = \mathbf{0}. \tag{77}$$

The following description is found for the solving system,

$$\begin{bmatrix} \mathbf{D}_n & -\mathbf{P}_n \\ -\hat{\mathbf{P}}_n^{\mathrm{t}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{d}_n \\ \mathbf{p}_n \end{Bmatrix} = \begin{Bmatrix} \overline{\mathbf{p}}_n \\ \overline{\mathbf{d}}_n \end{Bmatrix} \tag{78}$$

where all intervening arrays are defined by boundary integral expressions, as it is typical of the Trefftz method:

$$\mathbf{D}_n = \int \hat{\mathbf{U}}_n^{\mathrm{t}} \mathbf{NkD}^* \mathbf{U}_n \mathrm{d}\Gamma, \tag{79}$$

$$\mathbf{P}_n = \int \hat{\mathbf{U}}_n^{\mathrm{t}} \mathbf{P} \mathrm{d}\Gamma_u, \tag{80}$$

$$\overline{\mathbf{p}}_n = \int \hat{\mathbf{U}}_n^{\mathrm{t}} \mathbf{NkD}^* \mathbf{u}_{Pn} \mathrm{d}\Gamma + \int \hat{\mathbf{U}}_n^{\mathrm{t}} \mathbf{t}_{\Gamma * n} \mathrm{d}\Gamma_\sigma, \tag{81}$$

$$\overline{\mathbf{d}}_n = \int \hat{\mathbf{P}}^{\mathrm{t}}\left(\mathbf{u}_{Pn} - \mathbf{u}_{\Gamma * n}\right) \mathrm{d}\Gamma_u. \tag{82}$$

A detailed derivation of the results presented above can be found in [5, 6], where the alternative stress model of the hybrid-Trefftz finite element formulation is also considered.

6. Closure

The results presented in Section 3 generalise the well-established method of modal decomposition in space to establish uncoupled equations of motion.

It is shown that this method can be understood as the Trefftz variant of the finite element method. The generalised VIP method proposed in [2] is recovered using also the Trefftz concept. However, it is shown here how to implement the single-field formulation using general bases, that is, bases that are not included in the set of the solutions of Eq. (30).

The double-field or mixed approximation used here is assessed in [1], where the results obtained using non-Trefftz bases are compared with the solutions reported in Tamma et al. [2] using the quasi-Trefftz method recalled in Section 3.

The major limitation of these methods for integration of the equations of motion is their direct dependency on the solution of the eigenvalue problem (12) and on the orthogonality conditions (14) and (15), as they raise major difficulties in generalisation, particularly to non-linear applications.

The computational costs that the modal decomposition methods imply suggest the frequent use of incomplete (low-frequency) modal bases. However, it is well established that this simplification may induce unacceptable estimates for stress and force fields, as they are not weakly dependent on the high-frequency modes.

As it is shown in Section 5, these limitations can be overcome introducing a fundamental change in the modal decomposition concept, that is, using modal decomposition in the time domain and not (or not necessarily) in the space domain.

This decomposition is constructed so as to preserve hyperbolicity without relying on periodic time bases, thus encoding the solving Eqs. (61) and (65) in the format that typifies the Eqs. (43) and (46) of spectral analysis. The analysis of the stability of this basis obtained by modal decomposition in time and the assessment of the performance of the resulting time integration method are reported in [1].

The dimension of the eigenvalue problem implied by modal decomposition in the time domain is strongly reduced, as it depends now on the dimension of the time basis and not on the number of degrees-of-freedom of the equation of motion.

Moreover, orthogonality is not called upon, which releases the approach from direct dependency on the structural material properties. Finally, and equally relevant, the time integration method is independent of the method used for discretization of the structural problem in the space domain. Therefore, the solution of the resulting Helmholtz elliptic problem can be solved using any of the available methods for the solution of elliptic equations.

Acknowledgment

This work has been developed at ICIST, Instituto Superior Técnico, Technical University of Lisbon, and has been partially supported by Fundação para a Ciência e Tecnologia through contract POCTI/1999/ECM/33066.

Appendix

A. Standard integration procedure in the time domain

The definitions of the (symmetric) mass, damping and stiffness matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$, respectively, and of the consistent forcing load vector, $\mathbf{F}$, present in equation (7) are:

$$\mathbf{M} = \int \mathbf{U}^{\mathrm{t}}\mathbf{m}\mathbf{U}\,\mathrm{d}V, \tag{A1}$$

$$\mathbf{C} = \int \mathbf{U}^{\mathrm{t}}\mathbf{c}\mathbf{U}\,\mathrm{d}V, \tag{A2}$$

$$\mathbf{K} = \int (\mathbf{D}^{*}\mathbf{U})^{\mathrm{t}}\,\mathbf{k}\,(\mathbf{D}^{*}\mathbf{U})\,\mathrm{d}V, \tag{A3}$$

$$\mathbf{F} = \int \mathbf{U}^{\mathrm{t}}\mathbf{f}\,\mathrm{d}V + \int \mathbf{U}^{\mathrm{t}}\mathbf{t}_{\Gamma}\,\mathrm{d}\Gamma_{\sigma}. \tag{A4}$$

B. Time integration using modal decomposition in space

The terms present in the solving Eq. (20) for the single-field formulation are defined by:

$$D_n = \int_0^1 \left[\widehat{T}''_n - 2\xi_n\omega_n\widehat{T}'_n + \omega_n^2\widehat{T}_n\right] T_n\mathrm{d}\tau + \left(\widehat{T}_nT'_n - \widehat{T}'_nT_n + 2\xi_n\omega_n\widehat{T}_nT_n\right)_{\tau=1}, \tag{B1}$$

$$A_n = 2\xi_n\omega_n\widehat{T}_n(0) - \widehat{T}'_n(0), \tag{B2}$$

$$d_n(0) = \mathbf{X}_n^{\mathrm{t}}\mathbf{M}\mathbf{d}(0), \tag{B3}$$

$$v_n(0) = \mathbf{X}_n^{\mathrm{t}}\mathbf{M}\mathbf{v}(0), \tag{B4}$$

$$F_n = \int_0^1 \widehat{T}_n\mathbf{X}_n^{\mathrm{t}}\mathbf{F}(t)\mathrm{d}\tau. \tag{B5}$$

The terms present in the solving Eq. (28) for the double-field formulation are defined by:

$$D_{*n} = \Omega_{*n}^2 + 2\xi_{*n}\omega_n\Omega_{*n} + \omega_n^2, \tag{B6}$$

$$A_{*n} = \Omega_{*n} + 2\xi_{*n}\omega_n. \tag{B7}$$

According to definitions (B1) and (B2), the terms present in the solving Eq. (20) for the single-field formulation under the Trefftz constraint (30) are defined by:

$$D_n = 2A_n\exp(2\xi_n\omega_n), \tag{B8}$$

$$A_n = \xi_n\omega_n + i\omega_{dn}. \tag{B9}$$

The following identifications hold in the equivalent first-order Eq. (34) that supports the development of the double-field VIP formulation

$$\mathbf{y}_* = \begin{Bmatrix} \mathbf{d} \\ \mathbf{v} \end{Bmatrix}, \tag{B10}$$

$$\mathbf{A}_* = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{M}^{-1}\mathbf{K} & \mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \tag{B11}$$

$$\mathbf{F}_* = \begin{Bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{F} \end{Bmatrix}. \tag{B12}$$

It is noted that it is not strictly necessary to call upon the explicit inversion of the mass matrix.

References

[1] J. A. T. Freitas. Integration of second-order and hyperbolic problems by modal decomposition in the time domain. Int. report., icist, Instituto Superior Técnico, Lisbon, 2002.

[2] K. K. Tamma, X. Zhou, and D. Sha. The time dimension: A theory towards the evolution, classification, characterization and design of computational algorithms for transient/dynamic applications. *Archives of Computational Methods in Engineering*, **7**(2): 67–290, 2000.

[3] A.P. Zielinski. Trefftz Method. *CAMES*, **4**(3/4), 1997.

[4] J. A. T. Freitas and J. P. M. Almeida. Trefftz method. *CAMES*, **8**(2/3), 2001.

[5] J. A. T. Freitas. Hybrid-Trefftz displacement and stress elements for elastodynamic analysis in the frequency domain. *CAMES*, **4**: 345–368, 1997.

[6] J. A. T. Freitas. Hybrid finite element formulations for elastodynamic analysis in the frequency domain. *International Journal of Solids and Structures*, **36**: 1883–1923, 1999.