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Trefftz functions using the fundamental solution with the singularity outside the domain

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Two types of Trefftz (T-) functions are often used – fundamental solutions with their singularities outside the given region and general solutions of homogenous differential equations. For elasticity problems the general solution of the homogeneous differential equation (equilibrium equation in displacements known as Lame–Navier equations) can be found in the polynomial form.

In this paper we present the first type of T-functions. The paper deals with the investigation of accuracy and stability of the resulting system of discretized equations in relation to the position of the source (singularity) point. In this way non-singular reciprocity based boundary integral equations relate the boundary tractions and the boundary displacements of the searched solution to corresponding quantities of the known solutions.

It was found that there exist an optimal relation of the distance of the singularity to the distance of the collocation points where both the integration accuracy and numerical stability are good.

Keywords: point and line Hertzian contact, infinitesimal displacements, large element/sub-domain concept, FEM/BEM technique.

1. Introduction

If spherical bodies in contact have much different curvatures at least in one direction, like it is in roller bearings, the contact area is very small and under normal loading the contact pressure may occur in order of several thousands N/mm^2. The contact area can be small in both contact surface directions (called point contact) or in one direction only, when the difference of the curvatures in the other direction is small (called line contact). For detail description of the problems, see the references [1–14].

The detail distribution of the load over the individual rolling elements and the pressure distribution in the contact area of each element requires a very fine element mesh, high performance computers and effective software for the numerical modelling of the problem.

The solution of such problem can be split into two solutions, the first one, which contains the large gradients, is solved very precisely using the known Boussinesq's solution, and the second, which is smooth enough, is solved by BEM. Such procedure reduces the numerical model considerably.

The present paper deals with an effective modelling of the second problem. The direct reciprocity based BEM formulation is used in the solution. The unknown field, which is looked for, is related to the known solutions, which are defined by fundamental solutions having the source (singular) points outside the domain. In this way, effective procedures can be used for the numerical integration. The fundamental solutions introduce the Trefftz (T-) functions satisfying the solution of homogeneous governing equations in the whole domain.

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The aim of our contribution is to show the influence of position of singular points on the accuracy and stability of numerical procedure in the solution of problems with smooth boundaries and smooth boundary tractions.

2. Local field solution

Very efficient formulation for modelling of displacement and stress fields in the contact regions can be obtained using Trefftz type reciprocity based boundary elements. In all cases the sub-domain solution is presented by the reciprocity based BEM [15–18]

$$\int_{\Gamma} t_i^* u_i \mathrm{d}\Gamma = \int_{\Gamma} t_i u_i^* \mathrm{d}\Gamma + \int_{\Omega} b_i u_i^* \mathrm{d}\Omega , \tag{1}$$

where u_i and b_i are displacements and body forces, respectively and t_i are tractions acting on the domain boundaries Γ of the domain Ω. This equation expresses the reciprocity of works done by two systems of forces, the one without stars which is sought, or given on the domain boundaries, or inside of the domain, and another, reference state (for which all, displacements, stresses and tractions are known inside and on the domain boundaries), denoted by letters with stars.

The reciprocity equations are well known from the BEM formulations, where the Kelvin fundamental solutions are used for reference state and lead to the singular integral equations. Using T-polynomials, or Kelvin solutions with the singular points located outside the domain, all integrals are regular and can be computed more effectively using the numerical quadrature formulas.

Three types of problems leading to large local gradients can be considered [19]:

- Local loads and local contact problems,
- Holes, cracks, elastic inclusions and other types of geometric concentrators,
- Rigid and quasi-rigid inclusions.

As we need to compute all displacements and stresses on and near the boundaries, the problem is defined as follows:

A. Solve the problem (2) for unknown displacements from prescribed tractions on the boundary nodes (located on points on the radius R as shown in Fig. 1 for 2D problem) using the Kelvin's functions with source points y located outside the domain (on radius R_0). In Kelvin's type solution the reciprocal state is defined by the unit force acting in the source point y (outside the domain) in direction y_i. The equation (1) is written then in the form

$$\int_{\Gamma} T_{ij}(y,x)\, u_j(x)\, \mathrm{d}\Gamma(x) = \int U_{ij}(y,x)\, t_j(x)\, \mathrm{d}\Gamma(x) , \tag{2}$$

where for 2D problems Trefftz functions for displacements are

$$U_{ij}(y,x) = -\frac{1}{8\pi(1-\nu)\mu}\left\{(3-4\nu)\,\delta_{ij} ln(r) - r_{,i} r_{,j}\right\} \tag{3}$$

and for tractions they are

$$T_{ij}(y,x) = -\frac{1}{4\pi(1-\nu)r}\left\{\left[(1-2\nu)\,\delta_{ij} + 2r_{,i}r_{,j}\right]\frac{\partial r}{\partial n} + (1-2\nu)\left(r_{,i}n_{,j} - r_{,j}n_{,i}\right)\right\} . \tag{4}$$

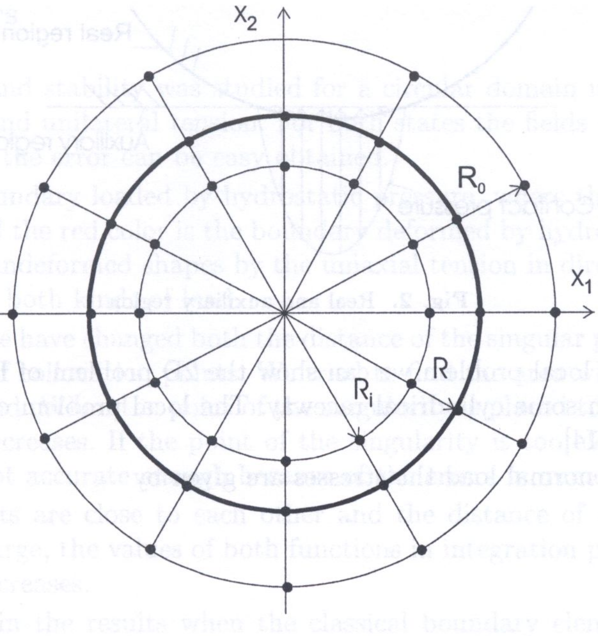

Fig. 1. 2D rolling element

B. The displacements in points y (radius R_i) inside the domain are computed from the boundary displacements and tractions using Eq. (5)

$$u_i(y) = \int_\Gamma U_{ij}(y,x)\, t_j(x)\, \mathrm{d}\Gamma(x) - \int_\Gamma T_{ij}(y,x)\, u_j(x)\, \mathrm{d}\Gamma(x) + \int_\Omega U_{ij}(y,x)\, b_j(x)\, \mathrm{d}\Omega(x). \tag{5}$$

C. For computation of stress components in a point of interest (POI) it is necessary to know the derivatives of displacements. They are computed using Trefftz polynomial interpolation of displacements from the displacements at the closest discrete points. For this purpose even low, second order interpolation gives very good results. Taking into account the traction boundary conditions improves the quality of results for computation of stresses near or on the domain boundaries [19].

The complex problem is decomposed into the local solution (an auxiliary domain, if the local problem is more precisely solved in such domain) and another smooth solution defined on large sub-domains/elements (Fig. 2), giving

$$f^R = f^L + f^S \qquad (f = u_i, \sigma_{ij}, t_i), \tag{6}$$

where the indices R, L and S denote the resulting (total), local (particular solution) and smooth part of the corresponding field, respectively.

The local field is described in an auxiliary region, which can only partially coincide with the real region (Fig. 2). The auxiliary region contains the local fields solution, whereas the real region is solved numerically with prescribed smooth boundary conditions and thus, it does not require very fine discretization.

When the intensity of the local field is known, then the problem can be split into local and smooth part according to (6) and solved separately for each of them, whereas in the case, when its intensity is unknown (e.g. cracks, or other geometrical concentrators), then both the local displacements and traction boundary conditions have to be included into the formulation. As the local solution satisfies the governing equations in a strong sense, the accuracy of the solution will not be destroyed by the large gradients and the fineness of discretization is dictated by the smooth field part of the solution.

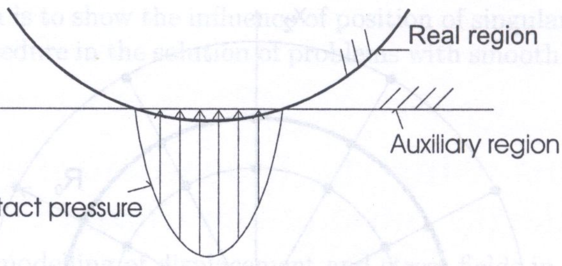

Fig. 2. Real and auxiliary region

As an example of the local problem we can show the 2D problem of Hertz contact of a cylinder with a half plane or with some cylindrical raceway. The local problem can be described using the Boussinesq solution [12–14].

For 2D problem with normal load the stresses are given by

$$\sigma_{11} = -\frac{2P}{\pi}\frac{x_1^2 x_2}{r^4}, \quad \sigma_{22} = -\frac{2P}{\pi}\frac{x_2^3}{r^4}, \quad \sigma_{12} = -\frac{2P}{\pi}\frac{x_1 x_2^2}{r^4}, \tag{7}$$

with

$$r = \sqrt{x_1^2 + x_2^2}, \tag{8}$$

and the displacements

$$u_\alpha = \frac{P_3}{4\pi\mu}\left[\frac{x_\alpha x_3}{r^3} - (1-2\nu)\frac{x_\alpha}{r(r+x_3)}\right], \quad u_3 = \frac{P_3}{4\pi\mu}\left[\frac{x_3^2}{r^3} + \frac{2(1-\nu)}{r}\right], \tag{9}$$

with ($\alpha = 1, 2$), where x_3 is the normal direction of the surface of the half space.

The resulting local stresses and displacements are obtained by integration over the loaded surface and replacing the force by traction in corresponding quadrature point. Singular integrals have to be evaluated and so, some technique known from BEM can be used [17, 18]. Another possibility for evaluation of integrals is to use potential functions, which reduce the order of singularity by one. However, in that case the stresses are obtained from second derivatives of the potential functions, which require some special treatment, usually defined by a discrete numerical procedure.

Further it is assumed that the local problem is described by accurate enough functions and a domain with smooth boundaries is solved numerically. Also the tractions are considered to be smooth enough. The boundary integral Eqs. (1) or (2) are solved by collocation method with equispaced integration points. Such a rule gives the highest numerical accuracy, if the integrands are harmonic functions. However, the kernel functions are the singular Kelvin's functions and sufficient accuracy is obtained only if the singularity is far enough from the integration surface. On the other side, the kernel functions serve also as weighting functions for satisfaction of the equilibrium in integral form and thus, unlike the previous functions, need to give different values in order to result in a stable numerical system of equations. The stability decreases with increasing distance of the source points.

The next section describes a numerical experiment to find the optimal distance of the source points from the domain.

3. Numerical results

The numerical accuracy and stability was studied for a circular domain under two loading states: the hydrostatic pressure and unilateral tension. For both states the fields of displacement, traction and stress are simple and the error can be easy obtained.

Figure 3 shows the boundary loaded by hydrostatic pressure, where the blue color corresponds to unloaded boundary and the red color is the boundary deformed by hydrostatic pressure. Figure 4 shows the deformed and undeformed shapes by the uniaxial tension in direction y. The accuracy is studied in Figs. 5 to 8 for both kinds of load.

In both loading cases we have changed both the distance of the singular points from the boundary domain and the number of collocation points. We tried to find the area where the solution of good numerical accuracy is found. When the point of the singularity is placed too far from the boundary, the numerical stability decreases. If the point of the singularity is too close to the boundary, the result of the solution is not accurate enough because of the errors in numerical integration.

If the collocation points are close to each other and the distance of singular points from the boundary domain is too large, the values of both functions in integration points differ too little and the numerical stability decreases.

Figures 6 and 7 contain the results when the classical boundary elements were used and the boundary displacement and tractions were interpolated by quadratic shape functions.

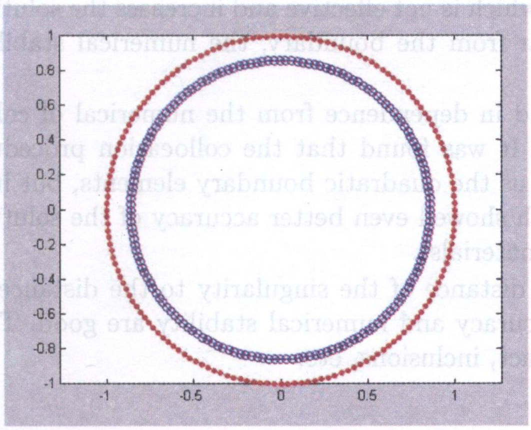

Fig. 3. Hydrostatic pressure

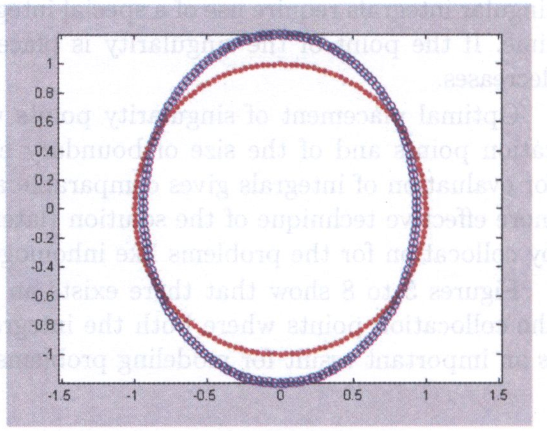

Fig. 4. Uniaxial tension in direction y

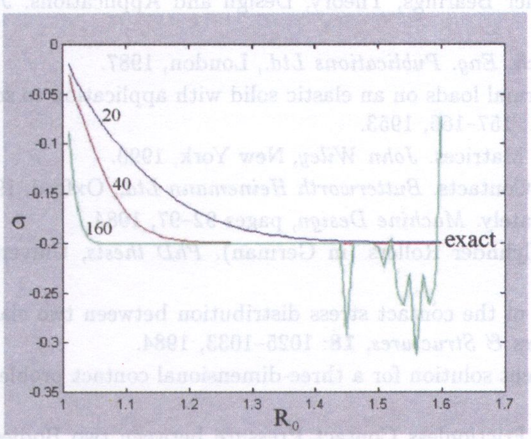

Fig. 5. Uniaxial tension in direction y, collocation method

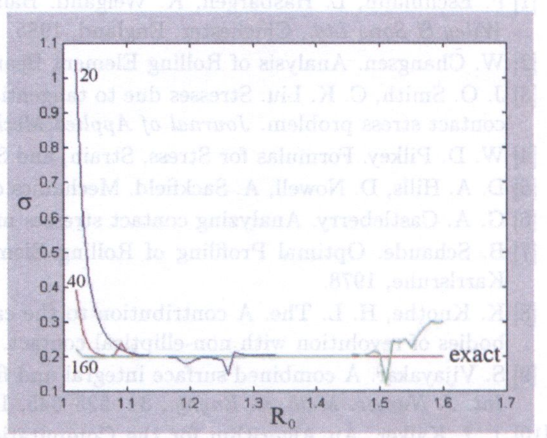

Fig. 6. Uniaxial tension in direction y, quadratic BE

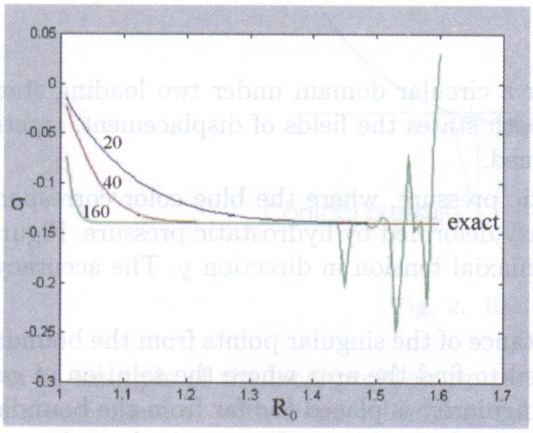

Fig. 7. Hydrostatic pressure, collocation solution

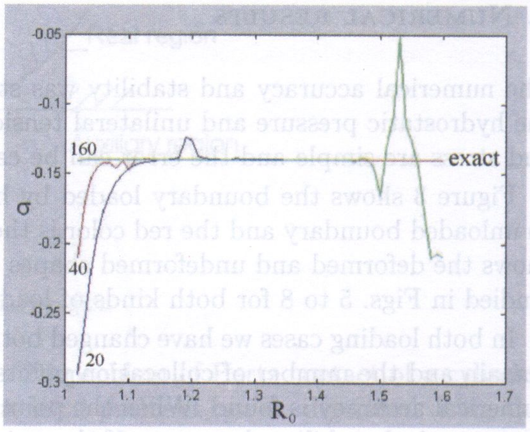

Fig. 8. Hydrostatic pressure, quadratic BE

4. Conclusions

We studied influence of position of singular points on the accuracy and stability of numerical procedure in the solution of problems with smooth boundaries and smooth boundary tractions.

When the singularity of corresponding Trefftz fields is too close to the boundary, the quasi-singular integrals require use of a special integration, which is not effective and increases the solution time. If the point of the singularity is placed too far from the boundary, the numerical stability decreases.

Optimal placement of singularity points was found in dependence from the numerical of collocation points and of the size of boundary elements. It was found that the collocation procedure for evaluation of integrals gives comparable accuracy as the quadratic boundary elements, but it is more effective technique of the solution (later research showed even better accuracy of the solution by collocation for the problems like inhomogeneous materials).

Figures 5 to 8 show that there exists an optimal distance of the singularity to the distance of the collocation points where both the integration accuracy and numerical stability are good. This is an important result for modeling problems of contact, inclusions, etc.

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