Computer Assisted Mechanics and Engineering Sciences, 10: 565–574, 2003.

Adaptive refinement for a local error bound based on duality

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(Received September 11, 2003)

This paper presents the basis of an adaptive mesh refinement technique aimed at reducing a local error, i.e. the error in a local quantity, which is defined as the integral of a stress or a displacement in a given subregion.

Two pairs of dual solutions, one corresponding to the applied load and the other to the virtual action, dual of the local quantity, are used to bound the local error and to provide the element error indicators for the adaptive process.

A test case is used to exemplify the behaviour of the technique.

1. Introduction

The finite element method is widely used and accepted as a numerical tool to "solve" several classes of problems for which an analytical solution is not available. The solutions thus obtained are, as a matter of rule, approximate and it is therefore important to assess the accuracy of the approximation.

Whatever the approach, the lack of an exact solution implies that only an approximation of the error may be obtained. Usually two different behaviours may be required from these approximations: either they must present asymptotic exactness or they must bound the exact error. From a practical point of view it is generally more important to guarantee that, for a given solution, the true error is smaller than the prescribed tolerance. This may be ensured only if an upper bound of the error is computed, from which an upper and a lower bound of the exact solution may be obtained.

The concept of dual analysis, developed by Fraeijs de Veubeke [7], may be used to obtain a bound for the global error of the solution.

In practice, it is generally more relevant to know the error of the solution in specific points or sub-regions, rather than having a global average error. It is therefore important to formulate techniques which provide this type of information.

Greenberg [9] and Washizu [20] proved that, given two pairs of dual solutions, one pair for the real problem and the other for a virtual action (equivalent unit load for displacements and equivalent unit relative displacement for stresses), dual of the desired local effect, bounds of the exact value of a displacement or a stress at a point can be obtained.

In the context of the finite element method, Yang, Kelly and Isles [22] used Washizu's approach to obtain upper bound estimates of the pointwise error in the finite element method and Albanese [1] used Greenberg's approach for electromagnetic problems.

When the goal is to obtain a more accurate approximation of the local value, but not bounds for this value, it is sufficient to have one solution for the real problem and another for the virtual action. This approach was followed by Babuska [4], who also used the associated element error indicators to drive an adaptive refinement process directed at reducing the local error.

In this paper we start by presenting the basic concepts related to static and kinematic admissibility, which are used to compute the local quantities and corresponding error bounds. It is then shown that each bound on the local error may be obtained as the sum of element contributions, which are used as element error indicators to drive an adaptive procedure focused on the required local quantity. A numerical example is used to study the characteristics of this technique.

Preliminary results obtained by the authors were presented in [14].

2. Statically and kinematically admissible solutions

In this paper, a subscript e (e.g. $\boldsymbol{\sigma}_e$) will imply that the corresponding field is derived from a statically admissible solution, whereas a subscript c (e.g. $\mathbf{u}_c$) implies that the field is derived from a kinematically admissible solution. When none of these subscripts is used the solution is exact.

Except for particular cases, it is not possible to obtain the exact solution of a given problem using the finite element method, but it is possible to establish formulations which provide either statically or kinematically admissible solutions.

Kinematically admissible solutions may be directly obtained using both the conventional formulation of the finite element method [19] and a hybrid displacement formulation [3, 11, 17].

Equilibrium finite element models [7] and hybrid stress finite element formulations [2] may be used to directly obtain statically admissible solutions.

It is also possible to indirectly derive a statically admissible solution from a kinematically admissible finite element solution [13] and *vice versa* [15].

3. Calculation of local quantities

A local quantity is a weighted integral of a field in a subregion. If the weighting function is a unit, constant function, the local quantity is equal to the average value of the field, multiplied by the volume/area/length of the subregion, but other functions may be used. If the subregion is a point, the local quantity is the value of the field at that point. In this paper, the local quantity will always be computed on a region that has the dimensions of the boundary of the domain: on an area for solids and on a line for plane problems. To distinguish it from the original boundary this region will be denoted $\overline{\Gamma}$.

Interpreting the weighting function as a virtual action, it is possible to indirectly compute the local quantity using the principle of virtual work, resulting in Eq. (1) for stresses and (2) for displacements. In the following, to distinguish the virtual action an overbar will be used, *e.g.* $\overline{\boldsymbol{\sigma}}$ corresponds to the stresses due to the virtual action.

$$\begin{aligned}\sigma_{\text{local}} &= -\int_{\overline{\Gamma}_u} \overline{\mathbf{u}}_\Gamma^{\mathrm{T}} \mathbf{t}\, \mathrm{d}\Gamma \\ &= -\int_\Omega \overline{\boldsymbol{\varepsilon}}_c^{\mathrm{T}} \boldsymbol{\sigma}\, \mathrm{d}\Omega + \int_\Omega \overline{\mathbf{u}}_c^{\mathrm{T}} \mathbf{b}\, \mathrm{d}\Omega + \int_{\Gamma_t} \overline{\mathbf{u}}_c^{\mathrm{T}} \mathbf{t}_\Gamma\, \mathrm{d}\Gamma\,, \end{aligned} \tag{1}$$

$$\begin{aligned}\delta_{\text{local}} &= \int_{\overline{\Gamma}_t} \mathbf{u}^{\mathrm{T}} \overline{\mathbf{t}}_\Gamma\, \mathrm{d}\Gamma \\ &= \int_\Omega \boldsymbol{\varepsilon}^{\mathrm{T}} \overline{\boldsymbol{\sigma}}_e\, \mathrm{d}\Omega - \int_{\Gamma_u} \mathbf{u}_\Gamma^{\mathrm{T}} \overline{\mathbf{t}}_e\, \mathrm{d}\Gamma\,. \end{aligned} \tag{2}$$

As a consequence of having different solutions available from the dual analyses, more than one approximate value may be computed for each local quantity, *e.g.* $\sigma_{\text{local}}^{\overline{c}c}$ and $\sigma_{\text{local}}^{\overline{c}e}$, the first obtained from a pair of compatible solutions and the second from a compatible solution of the virtual problem and an equilibrated solution of the real problem.

This paper will focus on the computation of local stresses, but the extension of the results presented to local displacements is straightforward.

4. Solution Error

The pointwise error is defined as the difference between the approximate and the exact fields, e.g. $e_c = \sigma_c - \sigma$ and $e_e = \sigma_e - \sigma$.

Similarly, the local error is the difference between the approximate and the exact local quantities.

The most common global measure to assess the error in finite element approximations is the distance, in the energy norm, between the approximate and the exact solutions:

$$\|e_c\| = (e_c, e_c)^{1/2}, \tag{3}$$

$$\|e_e\| = (e_e, e_e)^{1/2}, \tag{4}$$

where $(e_1, e_2) = \int_\Omega e_1^{\mathrm{T}} f e_2 \, \mathrm{d}\Omega$ and $\boldsymbol{\varepsilon} = f\boldsymbol{\sigma}$.

The energy norm of the error of each solution is bounded by the energy norm of the difference between the dual solutions [18]:

$$\|e_e\| = \|\boldsymbol{\sigma}_e - \boldsymbol{\sigma}\| \leq \|\boldsymbol{\sigma}_e - \boldsymbol{\sigma}_c\| = \epsilon, \tag{5}$$

$$\|e_c\| = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_c\| \leq \|\boldsymbol{\sigma}_e - \boldsymbol{\sigma}_c\| = \epsilon. \tag{6}$$

Following Washizu [20], given a pair of dual solutions corresponding to the "real problem", whose error is bounded by ϵ, and a pair of dual solutions corresponding to the "virtual problem", bounded by $\bar{\epsilon}$, an upper bound of the error in the local quantity is given by:

$$\epsilon_{\text{local}} = \epsilon \times \bar{\epsilon}. \tag{7}$$

For a local stress, as shown in [14],

$$|\sigma_{\text{local}} - \sigma_{\text{local}}^{\bar{c}c}| \leq \|\overline{\boldsymbol{\sigma}}_c - \overline{\boldsymbol{\sigma}}_e\| \times \|\boldsymbol{\sigma}_e - \boldsymbol{\sigma}_c\| = \epsilon_{\text{local}}, \tag{8}$$

and, when $\boldsymbol{\sigma}_e$ is a solution obtained from an equilibrium finite element model and $\overline{\sigma}_e$ may be represented using the same model,

$$|\sigma_{\text{local}} - \sigma_{\text{local}}^{\bar{c}e}| \leq \|\overline{\boldsymbol{\sigma}}_c - \overline{\boldsymbol{\sigma}}_e\| \times \|\boldsymbol{\sigma}_c - \boldsymbol{\sigma}_e\| = \epsilon_{\text{local}}. \tag{9}$$

From each approximation of the local quantity and from the corresponding upper bound of the local error, an interval containing the exact value of the local quantity may be defined.

The intersection of each of the two pairs of intervals yields, for a local stress:

$$\max\left(\sigma_{\text{local}}^{\bar{c}c}, \sigma_{\text{local}}^{\bar{c}e}\right) - \epsilon_{\text{local}} \leq \sigma_{\text{local}} \leq \min(\sigma_{\text{local}}^{\bar{c}c}, \sigma_{\text{local}}^{\bar{c}e}) + \epsilon_{\text{local}}. \tag{10}$$

5. Element error indicators for local quantities

The goal of an adaptive refinement is to find, at a minimal cost, a finite element solution that satisfies a required tolerance, which for most of the research on this subject is defined in terms of the relative global error.

The selection of optimal regions for refinement may be based on the contribution of each element or degree of freedom to the estimate of the error. The mesh is then refined by using smaller elements and/or by increasing the degree of the approximations used.

Most of the research in this field has focused on conventional, compatible, finite element models. In the initial work by Babuska [5] and Gago [12], the error was estimated from the equilibrium residuals, but later work by Zienkiewicz and Zhu [23] established a framework based on an improved, recovered, stress field which is now commonly used [24].

Another alternative is to compute an upper bound of the global error using dual principles. Then, as the error bound in (5) and (6) may be expressed as a sum over all the elements,

$$\epsilon = \|\boldsymbol{\sigma}_e - \boldsymbol{\sigma}_c\| = \left(\sum_i \|\boldsymbol{\sigma}_e - \boldsymbol{\sigma}_c\|^2_{(i)} \right)^{1/2} = \left(\sum_i \epsilon^2_{(i)} \right)^{1/2}, \tag{11}$$

naturally yielding element error indicators.

The pair of dual solutions may be obtained from the parallel analysis of an equilibrium and a compatible finite element model, as used by the authors [15]. Alternatively, the primal solution may be used to derive the dual field.

This approach has been used by Ladeveze [13], who devised a procedure to derive equilibrated stress fields from compatible solutions, and by the authors [15], who derived compatible displacement fields from equilibrated solutions.

In adaptive processes focused on the global error, the tolerance is normally expressed relative to the strain energy norm of the solution.

In this paper, the tolerance used in the adaptive process is related to a local quantity, which in general is measured as an absolute quantity. This is justified, from a practical viewpoint, because the designer is generally concerned that the variable provided by the finite element mesh is within given material or design bounds and, from a theoretical viewpoint, because the local quantity may be zero.

The error in the local quantity, ϵ_{local}, which was defined in Eq. (7), may be rewritten:

$$\begin{aligned}
\epsilon_{\text{local}} &= \|\boldsymbol{\sigma}_e - \boldsymbol{\sigma}_c\| \|\overline{\boldsymbol{\sigma}}_e - \overline{\boldsymbol{\sigma}}_c\| \\
&= \left(\left(\sum_i \epsilon^2_{(i)} \right) \left(\sum_i \overline{\epsilon}^2_{(i)} \right) \right)^{1/2} \\
&= \left(\alpha\, \overline{\epsilon}^2 \sum_i \epsilon^2_{(i)} + (1-\alpha)\, \epsilon^2 \sum_i \overline{\epsilon}^2_{(i)} \right)^{1/2} \\
&= \left(\sum_i \left(\alpha\, \overline{\epsilon}^2 \epsilon^2_{(i)} + (1-\alpha)\, \epsilon^2 \overline{\epsilon}^2_{(i)} \right) \right)^{1/2}.
\end{aligned} \tag{12}$$

As the local error may be written as the square root of the sum in (12), the terms therein may be used as element error indicators. Factor α may be used to adjust the relative importance of each solution. Setting α to one corresponds to adapt only for the real solution and setting it to zero will optimise the mesh for the virtual solution.

We will use the mean value, by setting α to 0.5. The following element error indicator is thus obtained:

$$\epsilon^2_{(i)\text{local}} = \frac{1}{2} \left(\overline{\epsilon}^2 \epsilon^2_{(i)} + \epsilon^2 \overline{\epsilon}^2_{(i)} \right). \tag{13}$$

6. Adaptive strategies

Given a finite element mesh and an estimate of the relevant error, the h-adaptive procedure will attempt to provide a new, optimal, mesh for which the error does not exceed a required value.

In the present context, the relevant error is either the bound of the error of the local quantity or the size of the interval containing it. The procedure adopted is a generalisation of the algorithm presented in [8] and [16] in the context of the global error, which provides a size reduction ratio that must be applied to each element in the given mesh.

The strategy presented aims to provide an optimal mesh for only one real and one virtual action. More complex situations, e.g. more than one real action or more than one local quantity, are not considered in this paper.

Admitting that the global convergence rate of a given solution, with respect to the total number of elements, is optimal:

$$\|\sigma_e - \sigma_c\| = C\, NE^{-p/D};$$

where NE is the number of elements in the mesh, p is the lowest degree used in the approximation of σ_e and σ_c, D is the dimensionality of the domain and C is a constant. Then, the error in the local quantity has a similar convergence, with rate $-(p+\overline{p})/D$, where $\overline{p}$ is the equivalent of p for the virtual action.

In general, the same model will be used for all loadings, solving a governing system with multiple right hand sides. In that case, the convergence rate will be $-2p/D$.

In order to obtain the required error, the number of elements in the new mesh, M, must be related to the number of elements in the given mesh by:

$$M = NE \left(\frac{\epsilon_{\text{local}}}{\epsilon_{\text{local}}^{\text{req}}} \right)^{D/(\overline{p}+p)}.$$

Assuming that the convergence rate within each element, in terms of its size, is the same throughout the new mesh, it will be optimal if:

$$\epsilon_{\text{local}(e)}^2 = \frac{\left(\epsilon_{\text{local}}^{\text{req}}\right)^2}{M}.$$

As the increase in the number of elements is $O(h^{-D})$, the corresponding size reduction ratios are:

$$\chi_{(i)} = \left(\frac{\epsilon_{\text{local}}^{\text{req}}}{\epsilon_{\text{local}(i)} \left(NE \left(\frac{\epsilon_{\text{local}}}{\epsilon_{\text{local}}^{\text{req}}} \right)^{D/(\overline{p}+p)} \right)^{1/2}} \right)^{1/(p+\overline{p}+D)}. \tag{14}$$

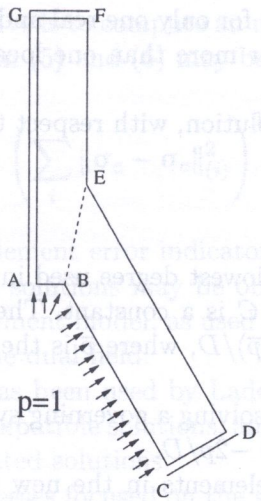
In problems with singularities, the convergence rate within each element will not be uniform throughout the mesh, but the formula presented will be adapted, following a procedure similar to the one presented in [16].

7. Examples

The plane stress problem, presented in Fig. 1, inspired by a structural model used in the design of the Sleipner oil platform [10], where the integral of the tangential stress along line BE, i.e. the shear force, is the local stress, was used to test the proposed error bound and the adaptive technique. Sides AGF and CD are sliding supports and an uniform normal pressure is applied on side ABC; an uniform elasticity modulus and a Poisson's ratio of 0.15 were assumed. The units of the quantities involved are not explicitly stated, but it will be assumed that they are consistent.

Hybrid stress and hybrid displacement formulations of the same degree were used to obtain the statically and kinematically admissible solutions, respectively. An initial mesh, consisting of 44 triangular elements was used.

In Figure 2 the deformed solutions for this mesh, using hybrid displacement elements of degree 5, are presented for the real action, indicated in Fig. 1, and for the virtual action, which is an imposed unit relative tangential displacement along line BE.



Node	x	y
A	0.000	7.001
B	22.680	7.001
C	106.644	−138.429
D	154.275	−110.929
E	40.000	87.001
F	40.000	220.301
G	0.000	220.301

Fig. 1. Adaptation of the structural model used in the Sleipner platform

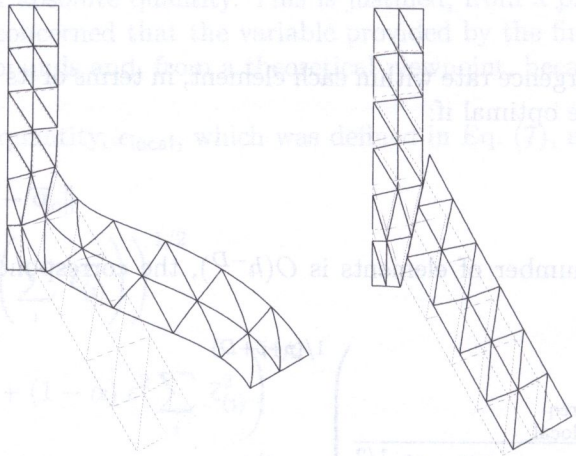


Fig. 2. Deformed shapes of the initial finite element mesh

For elements without singular vertices the adaptive process used the size reduction ratios provided by Eq. (14). This equation was modified for elements with singular vertices to:

$$\chi_{(i)} = \left(\frac{\epsilon_{\text{local}}^{\text{req}}}{\epsilon_{\text{local}(i)} \left(NE \left(\frac{\epsilon_{\text{local}}}{\epsilon_{\text{local}}^{\text{req}}} \right)^{D/(\overline{p}+p)} \right)^{1/2}} \right)^{1/(\lambda+\overline{\lambda})}, \tag{15}$$

where the strengths of the singularities, $\lambda = \overline{\lambda} = 0.61573106$ for point B and $\lambda = \overline{\lambda} = 0.75197455$ for point E, were computed according to [21]. In these example $\overline{p}$ was always equal to p.

The tests presented focus on the influence of the target error reduction in each adaptive step on the efficiency of the process, which can be measured by expressing the error bound for the local quantity as a function of the total number of elements, the total number of degrees of freedom or the total accumulated computational cost.

Normally, a smaller reduction step will provide better individual meshes, but implies a higher accumulated computational cost.

Two alternative types of formulae were considered to select the most appropriate error reduction factor, $\epsilon_{\text{local}}/\epsilon_{\text{local}}^{\text{req}}$, in each iteration:

$$2^{p/2}; 2^p; 2^{2p} \quad \text{and} \quad 2p^2; 4p^2; 8p^2.$$

The first set of formulae corresponds to fixing the multiplying factor which is applied to the number of elements to obtain the optimal convergence rate. The second set is based on an heuristic formula suggested by Beckers [6].

The analysis of all results indicated that the differences between the alternative error reduction factors are more visible when the error bound is plotted versus the number of elements, as presented in Fig. 3.

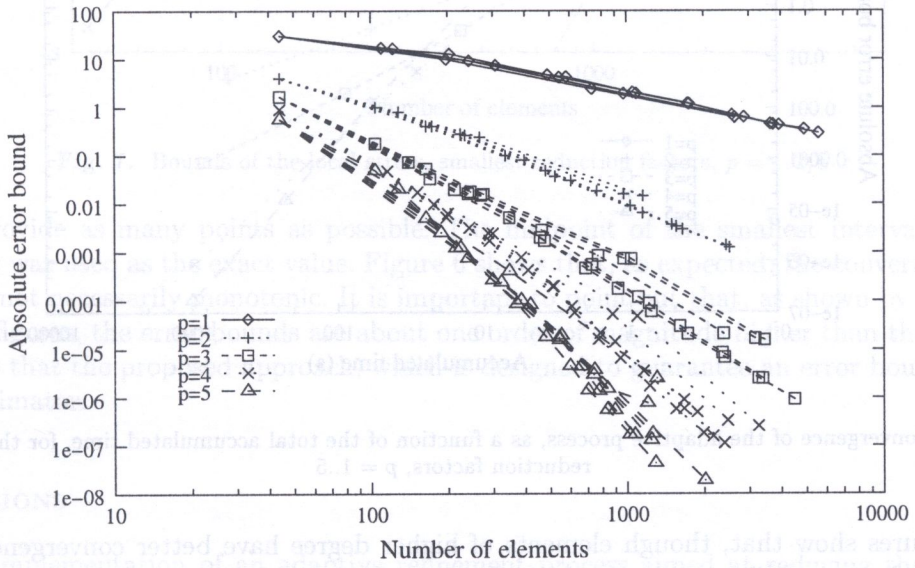


Fig. 3. Convergence of the adaptive process for all error reduction factors and $p = 1..5$

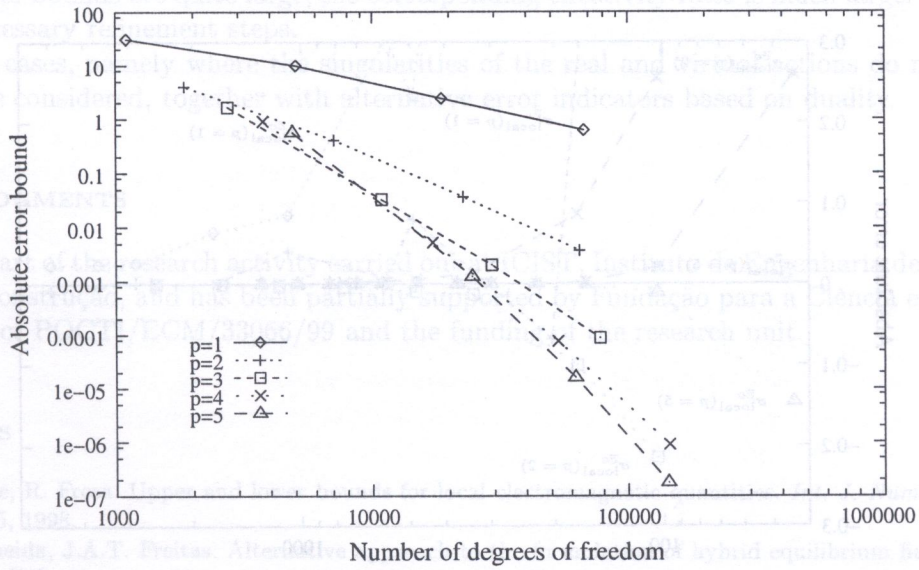


Fig. 4. Convergence of the adaptive process, as a function of the number of degrees of freedom, for the "optimal" reduction factors, $p = 1..5$

When the efficiency is measured in terms of the number of elements in the final mesh it is always better to use the smallest error reduction factor. But when the total accumulated time is considered, this is no longer the case.

Inspection of the results indicates that, for these tests, the growth of the optimal error reduction factors follows the formula proposed by Beckers [6], $4p^2$.

In Figures 4 and 5 the convergence, for these optimal error reduction factors, is presented as a function of the total number of degrees of freedom and of the total accumulated time, respectively.

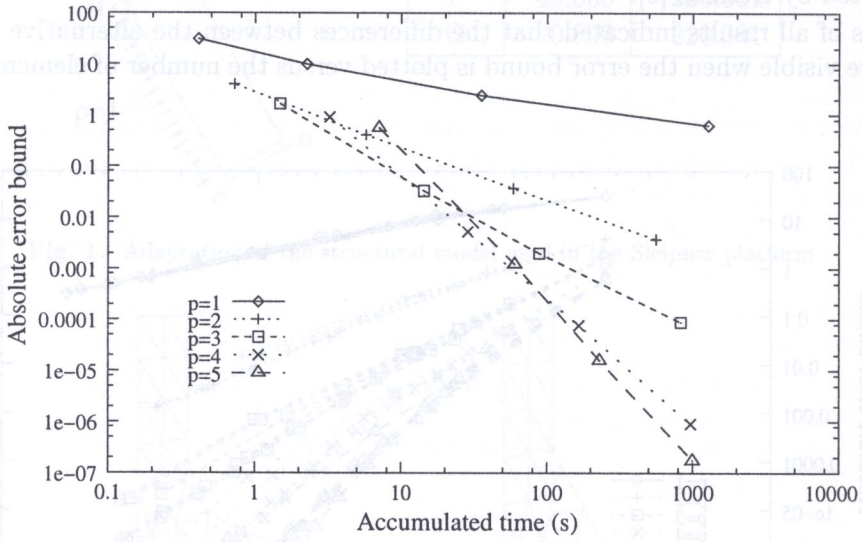


Fig. 5. Convergence of the adaptive process, as a function of the total accumulated time, for the "optimal" reduction factors, $p = 1..5$

These figures show that, though elements of higher degree have better convergence, they only outperform when a small error is required. Furthermore, the comparison in terms of number of degrees of freedom is biased towards high degree elements.

The convergence to the exact solution of the real error and of the error bounds, are presented in Figs. 6 and 7, where the smallest of the six reduction factors that were previously tested was

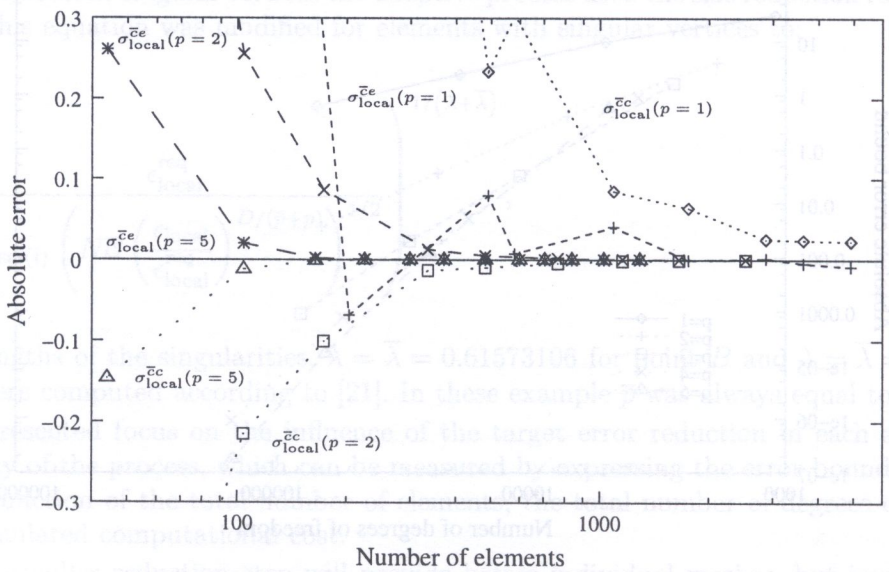


Fig. 6. Error of the approximated local stresses, smallest reduction factors, $p = 1, 2, 5$

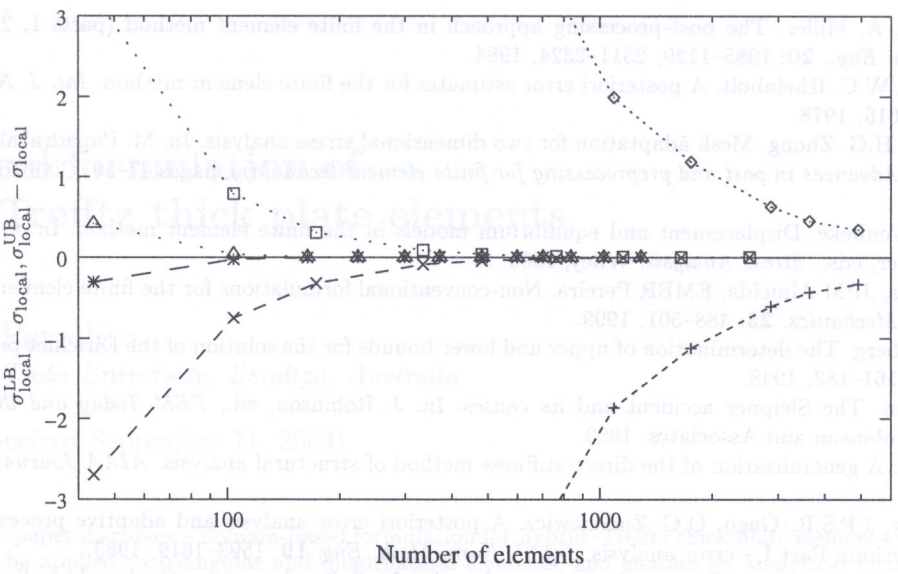


Fig. 7. Bounds of the local stress, smallest reduction factors, $p = 1, 2, 5$

applied, to provide as many points as possible. The midpoint of the smallest interval computed using Eq. (10) was used as the exact value. Figure 6 shows that, as expected, the convergence of the local stress is not necessarily monotonic. It is important to point out that, as shown by the vertical axes in these figures, the error bounds are about one order of magnitude higher than the real error. This indicates that the proposed approach, which is designed to guarantee an error bound, yields a poor error estimator.

8. Conclusions

A successful implementation of an adaptive refinement process aimed at reducing the error in a local stress was achieved.

The use of a formulation based on duality ensured that true upper bounds of the local error were obtained, which is an advantage from the viewpoint of safety.

As the upper bounds are quite large, the corresponding effectivity ratio is much larger than unity, causing unnecessary refinement steps.

Other test cases, namely where the singularities of the real and virtual actions do not overlap, should also be considered, together with alternative error indicators based on duality.

Acknowledgments

This work is part of the research activity carried out at ICIST, Instituto de Engenharia de Estruturas Território e Construção, and has been partially supported by Fundação para a Ciência e Tecnologia through project POCTI/ECM/33066/99 and the funding of the research unit.

References

[1] R. Albanese, R. Fresa. Upper and lower bounds for local electromagnetic quantities. *Int. J. Num. Meth. Eng.*, **42**: 499–515, 1998.

[2] J.P.M. Almeida, J.A.T. Freitas. Alternative approach to the formulation of hybrid equilibrium finite elements. *Computers & Structures*, **40**(4): 1043–1047, 1991.

[3] J.P.M. Almeida, J.A.T. Freitas. Continuity conditions for finite element analysis of solids. *Int. J. Num. Meth. Eng.*, **33**: 845–853, 1992.

[4] I. Babuska, A. Miller. The post-processing approach in the finite element method (parts 1, 2 and 3). *Int. J. Num. Meth. Eng.*, **20**: 1085–1129, 2311–2324, 1984.

[5] I. Babuska, W.C. Rheinbolt. A posteriori error estimates for the finite element method. *Int. J. Num. Meth. Eng*, **12**: 1597–1615, 1978.

[6] P. Beckers, H.G. Zhong. Mesh adaptation for two dimensional stress analysis. In: M. Papadrakakis, B.H.V. Topping, eds., *Advances in post and preprocessing for finite element technology*, pages 47–59. CivilComp, Edinburgh, 1994.

[7] B.M.F. deVeubeke. Displacement and equilibrium models in the finite element method. In: O.C. Zienkiewicz, G.S. Holister, eds., *Stress Analysis*. Wiley, 1965.

[8] JAT Freitas, JPM Almeida, EMBR Pereira. Non-conventional formulations for the finite element method. *Computational Mechanics*, **23**: 488–501, 1999.

[9] H.J. Greenberg. The determination of upper and lower bounds for the solution of the Dirichlet problem. *J. Math. Phys.*, **27**: 161–182, 1948.

[10] B. Jakobsen. The Sleipner accident and its causes. In: J. Robinson, ed., *FEM Today and the Future*, pages 102–108. Robinson and Associates, 1993.

[11] R.E. Jones. A generalization of the direct-stiffness method of structural analysis. *AIAA Journal*, **2**(5): 821–826, 1964.

[12] D.W. Kelly, J.P.S.R. Gago, O.C. Zienkiewicz. A posteriori error analysis and adaptive processes in the finite element method: Part I – error analysis. *Int. J. Num. Meth. Eng*, **19**: 1593–1619, 1983.

[13] P. Ladeveze, D Leguillon. Error estimate procedure in the finite element method and applications. *SIAM J. Num. Anal.*, **20**(3): 483–509, 1983.

[14] O.J.B.A. Pereira, J.P.M. Almeida. Majorantes do erro em grandezas locais obtidas a partir de soluções duais de elementos finitos. In: J.M. Goicolea, C. Mota Soares, M. Pastor, G. Bugeda, eds., *Métodos Numéricos en Ingeniería V.* SEMNI, 2002.

[15] O.J.B.A. Pereira, J.P.M. Almeida, E.A.W. Maunder. Adaptive methods for hybrid equilibrium finite element models. *Comp. Methods in Applied Mechanics and Engineering*, **176**: 19–39, 1999.

[16] O.J.B.A. Pereira, G. Bugeda. Mesh optimality criteria and remeshing strategies for singular point problems. In: M. Papadrakakis, ed., *Innovative Computational Methods for Structural Mechanics*, Edinburgh, 1998. Saxe-Coburg Publications.

[17] T.H.H. Pian, P. Tong. Basis of finite element methods for solid continua. *Int. J. Num. Meth. Eng.*, **1**: 3–28, 1969.

[18] W. Prager, J.L. Synge. Approximations in elasticity based on the concept of function space. *Quart. Appl. Math.*, **5**(3): 241–269, 1947.

[19] M.J. Turner, R.W. Clough, H.C. Martin, L.J. Topp. Stiffness and deflection analysis of complex structures. *Journal of the Aeronautical Sciences*, **23**(9): 805–854, Sep. 1956.

[20] K. Washizu. Bounds for solutions of boundary value problems in elasticity. *J. Math. Phys.*, **32**: 117–128, 1953.

[21] M.L. Williams. Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *Journal of Applied Mechanics*, **19**: 526–528, 1952.

[22] D.J. Yang, W.D. Kelly, J.D. Isles. A posteriori pointwise upper bound estimates in the finite element error. *Int. J. Num. Meth. Eng.*, **36**: 1279–1298, 1993.

[23] O.C. Zienkiewicz, J.Z. Zhu. A simple error estimator and adaptive procedure for practical engineering analysis. *Int. J. Num. Meth. Eng*, **24**: 337–357, 1987.

[24] O.C. Zienkiewicz, J.Z. Zhu. The superconvergent patch recovery (SPR) and adaptive finite element refinement. *Comp. Methods in Applied Mechanics and Engineering*, **101**: 207–224, 1992.