Computer Assisted Mechanics and Engineering Sciences, 10: 609–618, 2003.


# Trefftz type method for 2D problems of electromagnetic scattering from inhomogeneous bodies

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(Received September 11, 2003)

A new numerical method for scattering from inhomogeneous bodies is presented. The cases of E and H-polarizated incident wave scattered by an infinite 2D cylinder are considered. The scattered field is looked for in two different domains. The first one is a bounded region inside the scattering body with an inhomogeneous permittivity $\varepsilon(x,y)$. The second one is an unbounded homogeneous region outside the scatterer. An approximate solution for the scattered field inside the scatterer is looked for by applying the QTSM technique. The method of discrete sources is used to approximate the scattered field in the unbounded region outside the scattering body. A comparison of the numerical and analytic solutions is performed.

## 1. Introduction

Recently various numerical methods have been developed to compute electromagnetic scattering by bodies of irregular form. Boundary methods: the extended boundary condition method (EBCM) [1, 2] or T-matrix method [3], the methods based on the generalized multipole technique (GMT) [4, 5] and the method of discrete sources (MDS) [6, 7] are the fastest and most powerful tools in this field.

In this paper a new numerical method of the Trefftz type is presented for this goal. We consider the 2D problem of scattering of a plane, electromagnetic wave by a penetrable continuously inhomogeneous dielectric cylinder. The axis of the cylinder is along $OZ$. We denote the section of the cylinder in the $XY$-plane as $\Omega$. The region surrounding the cylinder is free space. It is assumed throughout that all fields are taken to be harmonic and that the time dependence, taken to be $e^{i\omega t}$, has been factored out. Suppose also that the magnetic permeability constant $\mu$ is a fixed constant everywhere: $\mu = \mu_0$. Under these assumptions Maxwell's system is:

$$\mathrm{rot}\mathbf{H} = ik_0\varepsilon\mathbf{E}, \qquad \mathrm{rot}\mathbf{E} = -ik_0\mathbf{H}. \tag{1}$$

It is written in dimensionless form. Here $\mathbf{E}$ and $\mathbf{H}$ are vectors of electric and magnetic fields correspondingly, $\varepsilon$ is the relative permittivity, the wave number $k_0 = 2\pi l_s/\lambda$, $\lambda$ is the wave-length of an incident field and $l_s$ is the scaling length ( e.g. a typical size of a scatterer). The relative permittivity is a smooth enough complex-valued function inside $\Omega$. And it is taken to be 1 outside the scatterer (free space).

The both cases of polarization of the incident wave E-polarization: $\mathbf{E}^{(\mathrm{inc})} = \{0, 0, E_z^{(\mathrm{inc})}\}$, $\mathbf{H}^{(\mathrm{inc})} = \{H_x^{(\mathrm{inc})}, H_y^{(\mathrm{inc})}, 0\}$ and H-polarization: $\mathbf{E}^{(\mathrm{inc})} = \{E_x^{(\mathrm{inc})}, E_y^{(\mathrm{inc})}, 0\}$, $\mathbf{H}^{(\mathrm{inc})} = \{0, 0, H_z^{(\mathrm{inc})}\}$ are considered. Under the assumptions listed before the scattered field $\{\mathbf{E}^{(s)}, \mathbf{H}^{(s)}\}$ has the same polarization as the incident one. Let us use the following notations:

$$u^{(i)}(x,y) = \begin{cases} E_z^{(s)}(x,y), & \text{E-polarization} \\ H_z^{(s)}(x,y), & \text{H-polarization} \end{cases} \quad (x,y) \in \Omega.$$

$$u^{(e)}(x,y) = \begin{cases} E_z^{(s)}(x,y), \text{ E-polarization} \\ H_z^{(s)}(x,y), \text{ H-polarization} \end{cases} (x,y) \in \mathcal{R}^2 \setminus \Omega.$$

Using Maxwell's system (1) one can get the systems of the scalar equations:

$$\left[\Delta + k_0^2\right] u^{(e)}(x,y) = 0, \quad (x,y) \in \mathcal{R}^2 \setminus \Omega, \tag{2}$$

$$\left[\Delta + k_0^2 \varepsilon^{(i)}(x,y)\right] u^{(i)}(x,y) = 0, \quad (x,y) \in \Omega, \tag{3}$$

in the E-polarization case and

$$\left[\Delta + k_0^2\right] u^{(e)}(x,y) = 0, \quad (x,y) \in \mathcal{R}^2 \setminus \Omega, \tag{4}$$

$$\left[\frac{\partial}{\partial x}\left(\frac{1}{\varepsilon^{(i)}(x,y)}\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\varepsilon^{(i)}(x,y)}\frac{\partial}{\partial y}\right) + k_0^2\right] u^{(i)}(x,y) = 0, \quad (x,y) \in \Omega, \tag{5}$$

in the case of the H-polarized incident wave.

The boundary conditions are:

$$\text{E-polarization}: \quad u^{(i)} - u^{(e)} = u^{(\text{inc})}, \quad \frac{\partial u^{(i)}}{\partial n} - \frac{\partial u^{(e)}}{\partial n} = \frac{\partial u^{(\text{inc})}}{\partial n} \quad \text{on } \partial\Omega, \tag{6}$$

$$\text{H-polarization}: \quad u^{(i)} - u^{(e)} = u^{(\text{inc})}, \quad \frac{1}{\varepsilon^{(i)}}\frac{\partial u^{(i)}}{\partial n} - \frac{\partial u^{(e)}}{\partial n} = \frac{\partial u^{(\text{inc})}}{\partial n} \quad \text{on } \partial\Omega. \tag{7}$$

Here $\dfrac{\partial}{\partial n}$ denotes the derivative in the direction of the normal vector $\mathbf{n} = (n_x, n_y)$ of $\partial\Omega$ and we denote $u^{(\text{inc})} = E_z^{(\text{inc})}$, or $H_z^{(\text{inc})}$.

We also suppose that the exterior field $u^{(e)}$ is an outgoing cylindrical wave at a large distance from the scatterer:

$$u^{(e)} \sim \frac{1}{\sqrt{r}} \exp(-ik_0 r), \quad r \to \infty, \quad r = \sqrt{x^2 + y^2}. \tag{8}$$

## 2. Numerical algorithm

The method presented belongs to the group of boundary methods when an approximate solution is looked for in the form of a linear combination

$$u(x,y) = \sum_{k=1}^{K} q_k \Psi_k(x,y),$$

where the trial functions $\Psi_k(x,y)$ satisfy exactly the corresponding partial differential equation (PDE) but do not necessary satisfy the boundary conditions which are imposed on the solution. As far as the homogeneous exterior region $\mathcal{R}^2 \setminus \Omega$ is considered one can use a version of the method of discrete sources (MDS) – a very effective technique for scattering problems in homogeneous medium developed in the last few years and which is gaining more and more interest because of its many practical applications. The description of the recent development of MDS and other references can be found in [7]. An approximate solution is looked for in the form of a linear combination of Green's functions which satisfy the radiation condition at infinity:

$$u^{(e)}(x,y) = \sum_{k=1}^{K_e} q_k^{(e)} H_0^{(2)}\left(k_0\sqrt{(x-x_k)^2 + (y-y_k)^2}\right) = \sum_{k=1}^{K_e} q_k^{(e)} \Psi_k^{(e)}(x,y).$$

Here $H_0^{(2)}$ denotes the Hankel function of the second kind and zero order. The source points $(x_k, y_k)$ are placed inside the scatterer $\Omega$ and $q_k^{(e)}$ are the free parameters.

To write the similar approximate solution

$$u^{(i)}(x,y) = \sum_{k=1}^{K_i} q_k^{(i)} \Psi_k^{(i)}(x,y), \tag{9}$$

for inhomogeneous region $\Omega$ a new numerical technique developed in [8–10] is used. First, it assumes applying the embedding procedure to the irregular solution domain. In this work it is assumed that the solution domain $\Omega$ may be embedded in the square $\Omega_0 = [-1,1] \times [-1,1]$. Of course, if this is not the case originally, then appropriate translation and scaling operations may be performed to make it so. Let us consider each polarization separately.

### 2.1. E-polarization

According to the embedding idea the initial differential operator $L^{(i)} = \Delta + k_0^2 \varepsilon^{(i)}(x,y)$ is replaced with $\widetilde{L}^{(i)} = \Delta + k_0^2 \widetilde{\varepsilon}^{(i)}(x,y)$. Here $\widetilde{\varepsilon}^{(i)}(x,y)$ is a truncated series

$$\widetilde{\varepsilon}^{(i)}(x,y) = \sum_{n,m=-N}^{+N} G_{nm} e^{i\pi(nx+my)}, \quad (x,y) \in \Omega_0, \tag{10}$$

which approximates $\varepsilon^{(i)}(x,y)$ for $(x,y) \in \Omega$. A numerical technique called the *C– expansion* procedure proposed by Smelov is used to get this approximation. With more details this technique is described in [8–10].

The trial functions $\Psi_k^{(i)}(x,y)$ are taken as solutions of the PDE:

$$\widetilde{L}^{(i)} \Psi_k^{(i)} = \left[\Delta + k_0^2 \widetilde{\varepsilon}^{(i)}(x,y)\right] \Psi_k^{(i)} = I_{M,l}(\mathbf{x}|\zeta_k), \tag{11}$$

where $I_{M,l}(\mathbf{x}|\zeta)$ is a $\delta$-shaped source function which essentially differs from zero only inside some neighborhood of the source point $\zeta = (\xi, \eta)$, i.e. this is an approximate Dirac's $\delta$-function. However, this is an infinitely differentiable function in the form of truncated series over the trigonometric system $\exp[i\pi(nx+my)]$:

$$I_{M,l}(\mathbf{x}|\zeta) = \sum_{n,m=-M}^{M} C_{n,m}(\zeta) \exp[i\pi(nx+my)], \tag{12}$$

$$C_{n,m}(\zeta) = \frac{1}{4} r_n(M,l)\ r_m(M,l) \exp[-i\pi(n\xi + m\eta)], \tag{13}$$

where the regularization coefficients are:

$$r_0(M,l) = 1, \qquad r_n(M,l) = \sigma_n^l(M), \qquad \sigma_n(M) = \sin\frac{n\pi}{(M+1)} \Big/ \frac{n\pi}{(M+1)}.$$

Here $\sigma_n(M)$ are so-called the Lanczos sigma-factors. Note that the regularization coefficients $r_n(M,l)$ coincide with $\sigma_n^l(M)$ only in the case of the trigonometric expansion. The general method of the regularization procedure is presented in [11].

An approximate solution of (11) is looked for in the same form of a truncated series:

$$\Psi_k^{(i)}(x,y) = \sum_{n,m=-M}^{M} U_{n,m}^{(k)} e^{i\pi(nx+my)}. \tag{14}$$

Integrating over $\Omega_0$ with the weight functions $e^{-i\pi(nx+my)}$, $n,m=0,\pm1,...,\pm M$ one gets:

$$-\pi^2(n^2+m^2)U_{nm}^{(k)}+k_0^2\sum_{|i-n|,|j-m|\le N}G_{n-i,m-j}U_{i,j}^{(k)}=C_{n,m}(\zeta_k)=C_{n,m}^{(k)}. \tag{15}$$

These equations can be rewritten in the form:

$$\sum_{i,j=1}^{M}A_{n,m}^{i,j}U_{i,j}^{(k)}=C_{n,m}^{(k)},$$

with

$$A_{n,m}^{i,j}=-\pi^2(n^2+m^2)\delta_{i,n}\delta_{j,m}+k_0^2G_{n-i,m-j}.$$

Introducing the $(2M+1)^2$-vectors:

$$\mathbf{U}_k=\left\{U_{-M,-M}^{(k)},...,U_{-M,+M}^{(k)},...,U_{+M,-M}^{(k)},...,U_{+M,+M}^{(k)}\right\},$$

$$\mathbf{C}_k=\left\{C_{-M,-M}^{(k)},...,C_{-M,+M}^{(k)},...,C_{+M,-M}^{(k)},...,C_{+M,+M}^{(k)}\right\},$$

they can be presented as a set of $(2M+1)^2\times(2M+1)^2$ – linear systems with the same matrix and different vectors of the right hand side:

$$\widehat{\mathbf{A}}\mathbf{U}_k=\mathbf{C}_k,\qquad k=1,\dots,K_i. \tag{16}$$

It is important to note that the coefficients $A_{n,m}^{i,j}$ of the matrix are obtained in an analytic way without any numerical integration. This is due to the extension of the initial irregular domain $\Omega$ to the square $\Omega_0$ and replacing the initial complex permittivity $\varepsilon^{(i)}(x,y)$ with the truncated series $\widetilde{\varepsilon}^{(i)}(x,y)$.

Solving these systems one gets the trial functions for expansion (9). Note that using these trial functions means that in fact the Helmholtz equation is solved with the nonzero right hand side:

$$\left[\Delta+k_0^2\widetilde{\varepsilon}^{(i)}(x,y)\right]u^{(i)}=\sum_{k=1}^{K_i}q_k^{(i)}I_{M,l}(\zeta_k). \tag{17}$$

When the source points $\zeta_k$ are removed from $\partial\Omega$ inside $\Omega_0$, then the right hand side of (17) is a small value for $(x,y)\in\Omega$. So, we introduce an additional error in an approximate solution when these basis functions are used. But as the numerical experiments have shown, this error can be of the same level as the one due to the approximate satisfaction of the boundary condition.

### 2.2. H-polarization

Here we begin with an approximation of the function $\nu(x,y)=1/\varepsilon^{(i)}(x,y)$. It is assumed that it can be approximated by the expansion similar to (10):

$$\widetilde{\nu}(x,y)=\sum_{n,m=-N}^{+N}G_{nm}e^{i\pi(nx+my)},\quad (x,y)\in\Omega_0. \tag{18}$$

The trial functions $\Psi_k^{(i)}(x,y)$ are taken now as solutions of the PDE (cf. (11)):

$$\left[\frac{\partial}{\partial x}\left(\widetilde{\nu}(x,y)\frac{\partial}{\partial x}\right)+\frac{\partial}{\partial y}\left(\widetilde{\nu}(x,y)\frac{\partial}{\partial y}\right)+k_0^2\right]\Psi_k^{(i)}=I_{M,l}(\mathbf{x}|\zeta_k). \tag{19}$$

An approximate solution of (19) is looked for in the same form as the one in the E-polarization case (see (14)). By integrating over $\Omega_0$ with the weight functions $e^{-i\pi(nx+my)}$, $n, m = 0, \pm 1, ..., \pm M$ one gets:

$$-\pi^2 \sum_{|i-n|,|j-m|\le N} (in + jm) G_{n-i,m-j} U_{i,j}^{(k)} + k_0^2 U_{n,m}^{(k)} = C_{n,m}^{(k)}. \tag{20}$$

It can be rewritten in the form of a linear system like (16) with respect to the unknowns $U_{i,j}^{(k)}$. The coefficients of the matrix now are:

$$A_{n,m}^{i,j} = -\pi^2 (in + jm) G_{n-i,m-j} + k_0^2 \delta_{i,n} \delta_{j,m}.$$

Note that, as above, when the approximation (18) is known, then the coefficients can calculated in an analytic way without any numerical integration.

Having the two linear combinations

$$u^{(e)}(x,y) = \sum_{k=1}^{K_e} q_k^{(e)} \Psi_k^{(e)}(x,y), \qquad u^{(i)}(x,y) = \sum_{k=1}^{K_i} q_k^{(i)} \Psi_k^{(i)}(x,y), \tag{21}$$

we get the free parameters $q_k^{(e)}, q_k^{(i)}$ as a solution of an over-determined system of algebraic equations. The system is obtained from the boundary conditions (6) or (7) using the collocation procedure at the collocation points $(x_j, y_j)$, $j = 1, ..., N_c$ distributed uniformly on $\partial\Omega$. The resulting linear system with $2\,N_c$ equations and $K_e + K_i$ unknowns $q_k^{(e)}, q_k^{(i)}$ is solved by the least squares procedure.

## 3. Numerical examples

To test the algorithm described above the axially symmetric problems are considered. Let $\Omega$ be a disk of radius $a$ centered at the point $(0,0)$ in the $XY$-plane and let $(r, \psi)$ be the polar coordinates:

$$x = r\cos\psi, \qquad y = r\sin\psi. \tag{22}$$

All the calculations were performed with $a = 0.5$. Following [12] the source points are placed as far as possible from the boundary $\partial\Omega$. So, they are uniformly distributed on the circles of the radiuses $r^{(i)} = 0.95$ and $r^{(e)} = 0.1$ for approximation of the inner and outer solution correspondingly.

Without loss of generality, the incident field is taken in the form of a plane wave propagating along the $X$-axis:

$$u^{\text{inc}}(x) = \exp(ik_0 x) = \exp(ik_0 r\cos\psi) = J_0(k_0 r) + 2\sum_{m=1}^{\infty} i^m J_m(k_0 r)\cos m\psi. \tag{23}$$

Here $J_m(\dots)$ denotes the Bessel function of the first kind and $m$-th order.

*1) Constant permittivity.*

The relative permittivity inside $\Omega$ is taken to be constant $\varepsilon^{(i)} = \text{const}$. In this case both systems (2), (3) and (4), (5), written in polar coordinates, are:

$$\frac{\partial^2 u^{(e)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(e)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u^{(e)}}{\partial \psi^2} + k_0^2 u^{(e)} = 0, \qquad r > a, \tag{24}$$

$$\frac{\partial^2 u^{(i)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(i)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u^{(i)}}{\partial \psi^2} + k_0^2 \varepsilon^{(i)} u^{(i)} = 0, \qquad r < a. \tag{25}$$

The boundary conditions, e.g. for H-polarization, are:

$$u^{(i)}(a,\psi) - u^{(e)}(a,\psi) = J_0(k_0 a) + 2\sum_{m=1}^{\infty} i^m J_m(k_0 a)\cos m\psi, \tag{26}$$

$$\frac{1}{\varepsilon^{(i)}}\frac{\partial u^{(i)}}{\partial r}(a,\psi) - \frac{\partial u^{(e)}}{\partial r}(a,\psi) = k_0 J_0'(k_0 a) + 2k_0\sum_{m=1}^{\infty} i^m J_m'(k_0 a)\cos m\psi, \tag{27}$$

for all $0 \le \psi \le 2\pi$. Here we denote $J_m'(z) = dJ_m(z)/dz$. This problem has an analytic solution in the form of expansions over the orthogonal system of the functions $\cos m\psi$:

$$u^{(e)}(r,\psi) = \sum_{m=0}^{\infty} A_m H_m^{(2)}(k_0 r)\cos m\psi, \qquad r > a, \tag{28}$$

$$u^{(i)}(r,\psi) = \sum_{m=0}^{\infty} B_m J_m\left(\sqrt{\varepsilon^{(i)}} k_0 r\right)\cos m\psi, \qquad r < a, \tag{29}$$

where $H_m^{(2)}$ denotes the Hankel function of the second kind and $m$-th order. Using orthogonality of the $\cos m\psi$ basis one gets a $2 \times 2$-linear system for each pair $A_m$, $B_m$. We denote the resulting solution as $u^{(e)}_{\text{anlt}}$ and $u^{(i)}_{\text{anlt}}$ and use it to compare with the numerical solution $u^{(e)}_{\text{num}}$ and $u^{(i)}_{\text{num}}$. Note that in this case the trial functions can be found in the analytical way. Indeed, as it follows from (18)

$$G_{n,m} = \begin{cases} 1/\varepsilon^{(i)}, & n = m = 0; \\ 0, & \text{otherwise.} \end{cases}$$

and so, Eq. (20) can be easily resolved

$$U^{(k)}_{n,m} = \frac{\varepsilon^{(i)} C^{(k)}_{n,m}}{\varepsilon^{(i)} k_0^2 - \pi^2\left(n^2 + m^2\right)}.$$

Suppose

$$\varepsilon^{(i)} k_0^2 - \pi^2\left(n^2 + m^2\right) \neq 0, \qquad n, m = 1, 2, \ldots$$

To estimate the accuracy of the calculations we use the maximal relative error. For the internal solution it is defined as:

$$e_{\text{in}} = \max_{n=1,\ldots,N_t}\left\{\frac{\left|u^{(i)}_{\text{num}}(x_n^{(i)}, y_n^{(i)}) - u^{(i)}_{\text{anlt}}(x_n^{(i)}, y_n^{(i)})\right|}{\left|u^{(i)}_{\text{num}}(x_n^{(i)}, y_n^{(i)})\right| + \left|u^{(i)}_{\text{anlt}}(x_n^{(i)}, y_n^{(i)})\right|}\right\}. \tag{30}$$

The similar value $e_{\text{out}}$ is used to compare $u^{(e)}_{\text{num}}$ and $u^{(e)}_{\text{anlt}}$. We use $N_t = 100$ test points $x_n^{(i)}, y_n^{(i)}$ which are distributed uniformly inside $\Omega$ to compute $e_{\text{in}}$. To compute $e_{\text{out}}$ we choose $N_t = 100$ test points uniformly located on the circles of radius $r_m = a(m+1)$, $m = 1, \ldots, 10$ outside $\Omega$. These errors are placed in Table 1 for different numbers of harmonics $M$ and degrees of freedom (DOF) $K_i = K_e$ (we take them equal). We also compute the errors $e_{\text{bcd}}$ and $e_{\text{bcn}}$ in approximation of the boundary conditions:

$$e_{\text{bcd}} = \sqrt{\frac{1}{N_c}\sum_{j=1}^{N_c}\left[u_j^{(i)} - u_j^{(e)} - u_j^{(\text{inc})}\right]^2}, \qquad e_{\text{bcn}} = \sqrt{\frac{1}{N_c}\sum_{j=1}^{N_c}\left[\frac{1}{\varepsilon^{(i)}}\frac{\partial u_j^{(i)}}{\partial n} - \frac{\partial u_j^{(e)}}{\partial n} - \frac{\partial u_j^{(\text{inc})}}{\partial n}\right]^2}, \tag{31}$$

where summation is performed using all the collocation points.

**Table 1.** H-polarization. The constant permittivity $\varepsilon^{(i)} = 1.2$, the wave number $k_0 = 1$. The maximal relative errors $e_{\text{in}}$, $e_{\text{out}}$ inside and outside the scatterer

| DOF | $M = 10$ | | | | $M = 20$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $e_{\text{in}}$ | $e_{\text{out}}$ | $e_{\text{bcd}}$ | $e_{\text{bcn}}$ | $e_{\text{in}}$ | $e_{\text{out}}$ | $e_{\text{bcd}}$ | $e_{\text{bcn}}$ |
| 10 | $1.5 \cdot 10^{-3}$ | $3.8 \cdot 10^{-2}$ | $7.6 \cdot 10^{-3}$ | $2.6 \cdot 10^{-2}$ | $1.6 \cdot 10^{-3}$ | $3.6 \cdot 10^{-2}$ | $8.0 \cdot 10^{-3}$ | $2.7 \cdot 10^{-2}$ |
| 20 | $2.9 \cdot 10^{-4}$ | $4.4 \cdot 10^{-3}$ | $9.8 \cdot 10^{-5}$ | $3.0 \cdot 10^{-4}$ | $3.6 \cdot 10^{-5}$ | $1.0 \cdot 10^{-4}$ | $8.7 \cdot 10^{-5}$ | $1.5 \cdot 10^{-4}$ |
| 30 | $3.0 \cdot 10^{-4}$ | $4.6 \cdot 10^{-3}$ | $1.7 \cdot 10^{-5}$ | $8.0 \cdot 10^{-5}$ | $2.1 \cdot 10^{-7}$ | $1.1 \cdot 10^{-6}$ | $5.9 \cdot 10^{-7}$ | $1.4 \cdot 10^{-6}$ |

This example demonstrates two different kinds of error in the approximate solution. The error of the first kind is caused by approximate satisfaction of the boundary condition on $\partial\Omega$. It is inherent in all boundary methods and it depends on the DOF which are used to approximate the boundary data. For a small number of harmonics $M$, increasing of the DOF decreases the boundary error $e_{bc}$. However, this does not improve the accuracy of approximation of the scattered field. This can be explained by using of the source functions $I(\mathbf{x}|\xi)$ instead of the Dirac functions $\delta(\mathbf{x}-\xi)$. In this case we do not have enough harmonics to obtain the appropriate condensed ones. When $M$ increases, $I(\mathbf{x}|\xi)$ approaches the delta-function. As a result, the errors in computation of the scattered field have the same level as the ones in the boundary conditions.

*2) Variable permittivity, E-polarization.*

The relative permittivity inside $\Omega$ is taken as

$$\varepsilon^{(i)}(x,y) = 1 + \beta^2(x^2+y^2) \equiv 1 + \beta^2 r^2, \qquad \beta = \text{const}. \tag{32}$$

Here Eqs. (2), (3) take the form:

$$\frac{\partial^2 u^{(i)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(i)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u^{(i)}}{\partial \psi^2} + k_0^2(1+\beta^2 r^2)u^{(i)} = 0, \qquad r < a, \tag{33}$$

$$\frac{\partial^2 u^{(e)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(e)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u^{(e)}}{\partial \psi^2} + k_0^2 u^{(e)} = 0, \qquad r > a. \tag{34}$$

The boundary conditions are:

$$u^{(i)}(a,\psi) - u^{(e)}(a,\psi) = J_0(k_0 a) + 2\sum_{m=1}^{\infty} i^m J_m(k_0 a)\cos m\psi,$$

$$\frac{\partial u^{(i)}}{\partial r}(a,\psi) - \frac{\partial u^{(e)}}{\partial r}(a,\psi) = k_0 J_0'(k_0 a) + 2k_0\sum_{m=1}^{\infty} i^m J_m'(k_0 a)\cos m\psi,$$

for all $0 \le \psi \le 2\pi$. Similar to the example considered above this problem has an analytic solution in the form of expansion over the same system $\cos m\psi$:

$$u^{(e)}(r,\psi) = \sum_{m=0}^{\infty} A_m H_m^{(2)}(k_0 r)\cos m\psi, \qquad r > a, \tag{35}$$

$$u^{(i)}(r,\psi) = \sum_{m=0}^{\infty} B_m V_m(r|k_0,\beta)\cos m\psi, \qquad r < a, \tag{36}$$

where $V_m(r|k_0,\beta)$ is a solution of the equation

$$\left[r^2\frac{d^2}{dr^2} + r\frac{d}{dr} - m^2 + k_0^2(1+\beta^2 r^2)r^2\right] V_m(r|k_0,\beta) = 0,$$

bounded at the point $r = 0$.

It can easily be verified that in this case $V_m(r/k_0|k_0,\beta)$ satisfies the equation

$$\left[r^2\frac{d^2}{dr^2}+r\frac{d}{dr}-m^2+(1+\frac{\beta^2}{k_0^2}r^2)r^2\right]V_m(r/k_0|k_0,\beta)=0.$$

So, with a fixed $m$ it depends on the only parameter $\alpha=\beta/k_0$. We denote a solution of the equation

$$\left[\rho^2\frac{d^2}{d\rho^2}+\rho\frac{d}{d\rho}-m^2+(1+\alpha^2\rho^2)\rho^2\right]Z_m=0, \tag{37}$$

bounded at the point $\rho=0$ as $Z_m(\rho|\alpha)$. So we get

$$V_m(r|k_0,\beta)=Z_m\left(k_0r|\frac{\beta}{k_0}\right).$$

The function $Z_m(\rho|\alpha)$ is looked for in the form of a power series:

$$Z_m(\rho|\alpha)=\rho^m\sum_{n=0}^{\infty}b_n(m,\alpha)\rho^{2n}. \tag{38}$$

Substituting in (37) we get the recursive formula:

$$b_1=-\frac{b_0}{4(m+1)},\qquad b_n=-\frac{b_{n-1}+\alpha^2b_{n-2}}{4n(m+n)}, \tag{39}$$

with the free parameter $b_0$.

Note that the value $\alpha=0$ corresponds to the scattering by a homogeneous circular cylinder. In this case the functions $V_m$ should be replaced by the Bessel's functions of the first kind $J_m$. This can be provided by taking

$$b_0=\frac{1}{m!2^m}, \tag{40}$$

because in this case

$$Z_m(\rho|0)=J_m(\rho). \tag{41}$$

Numerical experiments show the rapid converges of (38) at least for $\rho,\alpha$ satisfying $0\le\rho\le0.5$, $0\le\alpha\le5$ . The summation is truncated as $|b_n(m,\alpha)|$ becomes less than $10^{-18}$.

We use finite sums with the number of terms equal to 20 in evaluation (35), (36). This provides a truncating error less than $10^{-10}$ in all of the calculations performed.

The data in Table 2 were obtained for the wave number $k_0=2$. The coefficient $\beta=1$. In all of the calculations performed we use $N=6$ (see (10)). This provides the maximal absolute error $e_a\le10^{-6}$ in the approximation of $\varepsilon^{(i)}(x,y)$ inside $\Omega$.

**Table 2.** E-polarization. Scattering by an inhomogeneous cylinder with the relative permittivity $\varepsilon^{(i)}(x,y)=1+x^2+y^2$; the wave number $k_0=2$; the number of harmonics $M=20$

| error | DOF = 10 | DOF = 15 | DOF = 20 |
|---|---|---|---|
| $e_{\text{in}}$ | $3.5\cdot10^{-3}$ | $2.2\cdot10^{-4}$ | $1.1\cdot10^{-5}$ |
| $e_{\text{out}}$ | $4.7\cdot10^{-3}$ | $2.4\cdot10^{-4}$ | $1.7\cdot10^{-5}$ |

*3) Variable permittivity. H-polarization.* The relative permittivity inside $\Omega$ is given in (32). In this case Eqs. (4), (5), written in polar coordinates, are:

$$\frac{\partial^2 u^{(e)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(e)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u^{(e)}}{\partial \psi^2} + k_0^2 u^{(e)} = 0, \qquad r > a, \tag{42}$$

$$\frac{\partial^2 u^{(i)}}{\partial r^2} + \frac{1-\beta^2 r^2}{r\left(1+\beta^2 r^2\right)}\frac{\partial u^{(i)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u^{(i)}}{\partial \psi^2} + k_0^2\left(1+\beta^2 r^2\right)u^{(i)} = 0, \qquad r < a, \tag{43}$$

with the boundary conditions:

$$u^{(i)}(a,\psi) - u^{(e)}(a,\psi) = J_0(k_0 a) + 2\sum_{m=1}^{\infty} i^m J_m(k_0 a)\cos m\psi,$$

$$\frac{1}{1+\beta^2 a^2}\frac{\partial u^{(i)}}{\partial r}(a,\psi) - \frac{\partial u^{(e)}}{\partial r}(a,\psi) = k_0 J_0'(k_0 a) + 2k_0\sum_{m=1}^{\infty} i^m J_m'(k_0 a)\cos m\psi,$$

for all $0 \le \psi \le 2\pi$.

Looking for the approximate solution inside the scatterer in the form (36) one gets the equation for $V_m$ :

$$r^2\left(1+\beta^2 r^2\right)\frac{d^2 V_m}{dr^2} + r\left(1-\beta^2 r^2\right)\frac{dV_m}{dr} + \left(1+\beta^2 r^2\right)\left[k_0^2 r^2\left(1+\beta^2 r^2\right) - m^2\right]V_m = 0.$$

Denoting

$$z = \beta r, \qquad c = k_0^2/\beta^2,$$

one gets:

$$z^2\left(1+z^2\right)\frac{d^2 V_m}{dz^2} + z\left(1-z^2\right)\frac{dV_m}{dz} + \left(1+z^2\right)\left[cz^2\left(1+z^2\right) - m^2\right]V_m = 0. \tag{44}$$

An approximate solution of (44) bounded at $z = 0$ is sought in the form of the series:

$$V_m(z,c) = \sum_{n=0}^{\infty} b_n(m,c)\, z^{m+2n}. \tag{45}$$

The following recurrence formulae are obtained using the *Mathematica 3* package:

$$b_n = -\frac{cb_{n-3} + 2cb_{n-2} + \left[4\left(n^2 - 3n + 2\right) + c + (4n-6)m\right]b_{n-1}}{4n(m+n)},$$

$$b_0 = \frac{1}{2^m m!}, \qquad b_1 = \frac{b_0(2m-c)}{4(m+1)}, \qquad b_2 = -\frac{2b_0 c + b_1(c+2m)}{8(m+2)}.$$

As the numerical experiments shown the series (45) converge at least for $z, c$ satisfying $z \le 1$, $c \le 5$. The summation is truncated as $|b_n(m,c)|$ becomes less than $10^{-18}$. The unknown coefficients $A_m, B_m$ are obtained as a solution of the sequence of $2\times 2$ linear systems following from orthogonality of the $\cos m\psi$ basis.

In Table 3 the analytic solution is used to examine the one obtained by the numerical algorithm presented in the previous section. The wave number is: $k_0 = 1$ and $\widetilde{\nu}(x,y) = 1 + 1.5(x^2+y^2)$. The values $e_{\text{in}}$ and $e_{\text{out}}$ are the same as the ones in Table 1. The data presented correspond to the same number of harmonics $M = 20$ in approximate solutions and the two different approximations of $\widetilde{\nu}(x,y)$: with $N = 1$ and $N = 5$.

One can see that for $N = 1$ corresponding to 9-terms expansion (18) increasing of DOF from 10 to 30 does not increase the exactness of the approximate solution. This means that the main error in the solution is caused by the error in approximation of $\widetilde{\nu}(x,y)$. This is the third kind of error of approximation in the method presented. When $N$ increases, the differential operator of (19) approaches the initial one in (5) and the total error begin to decrease with the growth of DOF.

**Table 3.** H-polarization. Scattering by an inhomogeneous cylinder with $\widetilde{\nu}(x,y) = 1 + 1.5(x^2 + y^2)$; the wave number $k_0 = 1$; the number of harmonics $M = 20$

| DOF | $N = 1$ | | $N = 5$ | |
|---|---|---|---|---|
| | $e_{\text{in}}$ | $e_{\text{out}}$ | $e_{\text{in}}$ | $e_{\text{out}}$ |
| 10 | $4.3 \cdot 10^{-3}$ | $6.2 \cdot 10^{-1}$ | $9.6 \cdot 10^{-4}$ | $3.3 \cdot 10^{-2}$ |
| 20 | $4.2 \cdot 10^{-3}$ | $6.2 \cdot 10^{-1}$ | $1.7 \cdot 10^{-5}$ | $2.9 \cdot 10^{-4}$ |
| 30 | $4.2 \cdot 10^{-3}$ | $6.2 \cdot 10^{-1}$ | $1.1 \cdot 10^{-7}$ | $1.5 \cdot 10^{-5}$ |

## 4. Conclusion

The method presented can be regarded as a generalization of MDS onto the problems of scattering by electrical inhomogeneous bodies. When an accurate approximation of the complex permittivity $\varepsilon^{(i)}$ (or of the inverse value $1/\varepsilon^{(i)}$) is known in the form of truncated Fourier series, then it provides solution of the scattering problem with a high precision.

The method presented can be extended to 3D scattering. At least it seems to be quite simple in the scalar acoustic case. These extensions and 3D electromagnetic scattering are topics of the further investigations.

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