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Sensitivity analysis for variable dynamic load parameters†

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The paper is concerned with a class of generalized structural optimization problems for which not only stiffness, damping and mass parameters but also loading and support parameters are unspecified and subject to sensitivity analysis and optimization. Both, viscous and complex modulus damping models are used. Single concentrated force and coupling of a force with a concentrated moment, which lags by $\pi/2$, are considered. The latter case corresponds to an excitation induced by a rotational machine with eccentricity. Steady-state periodic vibrations are studied. Response functionals in the form of displacement amplitudes are discussed. Numerical examples of beam and plate structures illustrate the theory and demonstrate the accuracy of the derived formulae for sensitivity operators.

Keywords: sensitivity analysis, optimal design, structural dynamics, vibrations

1. Introduction

The class of problems where position of support was unspecified and subject to optimization was first formulated in [7] and [9] and the optimality conditions were derived for prescribed static load. The problem when the loading distribution was subject to optimization was first considered in [6]. Since then many papers have appeared, where optimal position and stiffness of supports were studied. Position of supports providing optimal response in eigenvibrations was discussed in [10]. Position and stiffness of intermediate support resulting in bimodal eigenvibrations was studied in [1]. Optimal placement of supports of beams subjected to harmonic load was solved in [4] accounting for complex modulus damping. Minimum weight design of planar trusses and frameworks under multiple dynamic loads was studied in [8], where modal decomposition and proportional damping approach were used. Sensitivity analysis with respect to variation of parameters of dynamic loading allowing for viscous and complex modulus damping was presented in [3]. Considerations were confined there to transverse loads and displacements.

The present paper further extends the study presented in [3] by introduction of two dynamic forces, vertical and horizontal. The latter one lags by $\pi/2$ in order to model the action of a rotating machine. Section 2 contains derivation of sensitivity operators allowing for viscous and complex modulus damping. Continuous formulation and the adjoint variable method are used. The methods known in the literature require that in the case of viscous damping the integration of motion equations of the adjoint problem is carried out for prescribed terminal conditions, employing time integration in inverse direction. In the approach presented in the paper we avoid this inconvenient inverse integration. The sensitivity operators are expressed in an explicit form of amplitudes and phase angles of dynamic quantities, which can be obtained by the use of professional FEM programs. Numerical examples solved with the use of ABAQUS FEM system are presented in Section 3. Section 4 contains concluding remarks.

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2. Sensitivity operators

Consider a beam or plate type base structure subjected to dynamic loading of a machine. The machine is modelled as a rigid body with the center of gravity at the height h_m (Fig. 1).

The vertical and horizontal components of the dynamic force are induced in the point at the height h_P and at the vertical symmetry line of the machine body. We assume a point-wise periodic action of the machine on the base structure in the form of concentrated force R_{yr} and moment $R_{\varphi r}$ at the point x_r. We assume that the machine is connected with the base structure by a visco-elastic hinge. We also allow for viscous damping in the base structure. Complex modulus damping will be discussed, too. The considerations will be referred to thin beams and plates following Bernoulli and Kirchhoff assumptions, respectively. Let $\mathbf{Q}$ and $\mathbf{q}$ denote generalized stress and strain, respectively. In the case of a plate $\mathbf{Q} = [M_{11}, M_{22}, M_{12}]$ and $\mathbf{q} = [u_{,11}, u_{,22}, u_{,12}]$, whereas in the case of a beam they simply represent the bending moment and curvature. Linear physical law is assumed $\mathbf{Q} = \mathbf{kq}$. In the case of a plate,

$$k_{ijkl} = \frac{Eh^3}{12(1-\nu^2)}[(1-\nu^2)\delta_{ik}\delta_{jl} + \nu\delta_{ij}\delta_{kl}], \tag{1}$$

and in case of a beam $k = EI$. For the simplicity of presentation we derive the sensitivity operators using the beam model of the base structure, then x_r is the 1D point vector along the beam. Figure 2a presents a close-up of the beam at the point of action of the machine. (Figure 2a can also represent a section of a plate type base structure.)

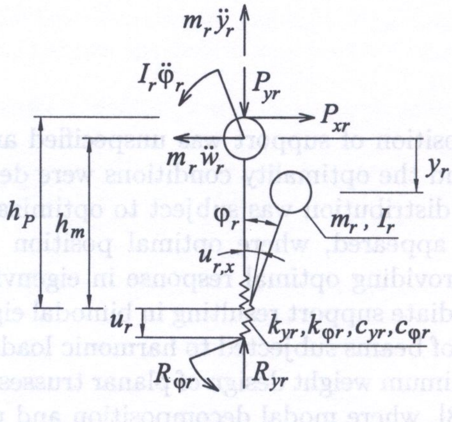

Fig. 1. Model of the machine

We introduce the following control parameters specifying the dynamic subsystem in the point r: position x_r, mass m_r, moment of inertia I_r stiffness parameters k_{yr} and $k_{\varphi r}$ in vertical and angular directions, viscous damping parameters c_{yr} and $c_{\varphi r}$ (Fig. 1). The subscript r tells that a quantity refers to the point r at $x = x_r$. We also introduce the distributed control parameters: mass $m(x)$, bending stiffness $k(x)$ and viscous damping $c(x)$ in the beam. All quantities denoted by star in Fig. 2a refer to the structure with perturbed values of control parameters. Hence, $u(x)$ and $u^\star(x)$ denote the displacement field of the beam for original and for perturbed values of control parameters, respectively. Furthermore, u_r denotes the displacement of the original beam at the point x_r, whereas $u_r^\star$ and $u_{r^\star}^\star$ denote the displacements of the perturbed beam at points x_r, and $x_{r^\star}$, respectively.

Assume the response functional as a function of vertical and angular displacements of the machine y_r and φ_r, respectively, and integrated displacements of the beam $u(x)$,

$$G = \int_0^T \left\{ f_1[y_r(t)] + f_2[\varphi_r(t)] + \int_0^L f_3[u(x,t)]\,\mathrm{d}x \right\} \mathrm{d}t, \tag{2}$$

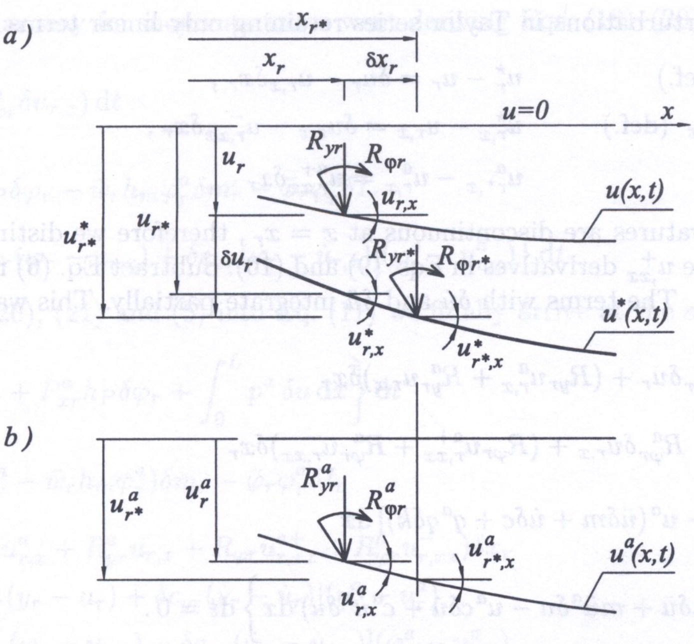

Fig. 2. Close-up of the beam structure; a) primary structure, b) adjoint structure

where f_1, f_2 and f_3 are arbitrary, differentiable functions and T is a time period. Variation of Eq. (2) is

$$\delta G = \int_0^T \left\{ \frac{\mathrm{d}f_1}{\mathrm{d}y_r}\,\delta y_r + \frac{\mathrm{d}f_2}{\mathrm{d}\varphi_r}\,\delta\varphi_r + \int_0^L \frac{\partial f_3}{\partial u}\,\delta u\,\mathrm{d}x \right\} \mathrm{d}t. \tag{3}$$

Note that δy_r and $\delta\varphi_r$ represent implicit variations of vertical and rotational displacements of the machine due to variations of all control parameters, among others due to the displacement of the machine from x_r to $x_{r^\star}$. To transform Eq. (3) to explicit form we introduce an adjoint structure, which is identical with the primary one (Fig. 2a without star) with the exceptions that the damping parameters are

$$c^a_{yr} = -c_{yr}\,, \qquad c^a_{\varphi r} = -c_{\varphi r}\,, \qquad c^a(x) = -c(x), \tag{4}$$

and the adjoint loadings are

$$P^a_{yr} = \frac{\mathrm{d}f_1}{\mathrm{d}y_r}\,, \qquad P^a_{\varphi r} = P^a_{xr}\cdot h_P = \frac{\mathrm{d}f_2}{\mathrm{d}\varphi_r}\,, \qquad p^a(x) = \frac{\partial f_3}{\partial u}\,. \tag{5}$$

We can apply either the moment $P^a_{\varphi r}$ or the horizontal force $P^a_{xr} = P^a_{\varphi r}/h_P$ (Figs. 1, 2b).

Let us write the virtual work equation using adjoint forces (Fig. 2b) and variation of kinematic fields of the primary structure (Fig. 2a),

$$\int_{t_0}^{t_1} \Big\{ R^a_{yr}(u_{r^\star} - u_r) + R^a_{\varphi r}(u_{r^\star,x} - u_{r,x}) + \int_0^L \big[(p^a - m\ddot{u}^a - c\dot{u}^a)(u^\star - u) - kq^a(q^\star - q)\big]\,\mathrm{d}x \Big\}\,\mathrm{d}t = 0\,, \tag{6}$$

where q denotes generalized strains and $u_{r,x} = \mathrm{d}u/\mathrm{d}x$ for $x = x_r$. In the case of a beam $q = \mathrm{d}^2u/\mathrm{d}x^2 = u_{,xx}$. Conversely, using variation of primary forces and adjoint kinematics we obtain

$$\int_{t_0}^{t_1} \Big\{ R^\star_{yr^\star} u^a_{r^\star} - R_{yr} u^a_r + R^\star_{\varphi r^\star} u^a_{r^\star,x} - R_{\varphi r} u^a_{r,x} + \int_0^L \big\{\big[(p - m^\star\ddot{u}^\star - c^\star\dot{u}^\star) - (p - m\ddot{u} - c\dot{u})\big]u^a - (k^\star q^\star - kq)q^a\big\}\,\mathrm{d}x \Big\}\,\mathrm{d}t = 0\,. \tag{7}$$

We develop the perturbations in Taylor series retaining only linear terms (Fig. 2). Hence

$$u_{r^\star}^\star - u_r = \delta u_r \quad \text{(def.)} \qquad u_r^\star - u_r = \delta u_r - u_{r,x}\delta x_r\,, \tag{8}$$

$$u_{r^\star,x}^\star - u_{r,x} = \delta u_{r,x} \quad \text{(def.)} \qquad u_{r,x}^\star - u_{r,x} = \delta u_{r,x} - u_{r,xx}^- \delta x_r\,, \tag{9}$$

$$u_{r^\star}^a - u_r^a = u_{r,x}^a \delta x_r \qquad u_{r^\star,x}^a - u_{r,x}^a = u_{r,xx}^{a+} \delta x_r\,. \tag{10}$$

Note that the curvatures are discontinuous at $x = x_r$, therefore we distinguished between left-side $u_{,xx}^-$ and right-side $u_{,xx}^+$ derivatives in Eqs. (9) and (10). Subtract Eq. (6) from (7) and introduce Eqs. (8), (9) and (10). The terms with $\delta\dot{u}$ and $\delta\ddot{u}$ integrate partially. This way one arrives at

$$\begin{aligned}\int_0^T \Big\{ & \delta R_{yr} u_r^a - R_{yr}^a \delta u_r + (R_{yr} u_{r,x}^a + R_{yr}^a u_{r,x})\delta x_r \\ & + \delta R_{\varphi r} u_{r,x}^a - R_{\varphi r}^a \delta u_{r,x} + (R_{\varphi r} u_{r,xx}^{a+} + R_{\varphi r}^a u_{r,xx}^-)\delta x_r \\ & - \int_0^L [p^a \delta u + u^a(\ddot{u}\delta m + \dot{u}\delta c + q^a q \delta k)]\,\mathrm{d}x \\ & + \int_0^L (-u^a m \delta\ddot{u} + m\ddot{u}^a \delta u - u^a c \delta\dot{u} + c^a \dot{u}^a \delta u)\,\mathrm{d}x \Big\}\,\mathrm{d}t = 0\,.\end{aligned} \tag{11}$$

The last integral vanishes. To prove it we integrate by parts the terms containing variation of velocity and acceleration keeping in mind Eq. (4),

$$\int_0^T (-u^a m\delta\ddot{u} + m\ddot{u}^a \delta u)\,\mathrm{d}t = [-u^a m \delta\dot{u} + \dot{u}^a m \delta u]_0^T + \int_0^T (-\ddot{u}^a m \delta u + m\ddot{u}^a \delta u)\,\mathrm{d}t = 0\,, \tag{12}$$

$$\int_0^T (-u^a c\delta\dot{u} + c^a \dot{u}^a \delta u)\,\mathrm{d}t \;= [-u^a c \delta u]_0^T + \int_0^T (\dot{u}^a c \delta u + c^a \dot{u}^a \delta u)\,\mathrm{d}t = 0\,. \tag{13}$$

The balance equations for the mass (Fig. 1) are

$$P_{yr} - m_r \ddot{y}_r - R_{yr} = 0\,, \tag{14}$$

$$(P_{xr} - m_r \ddot{w}_r)h - I_r \ddot{\varphi}_r - R_{\varphi r} = 0\,. \tag{15}$$

The force and moment in visco-elastic spring are

$$R_{yr} = k_{yr}(y_r - u_r) + c_{yr}(\dot{y}_r - \dot{u}_r)\,, \tag{16}$$

$$R_{\varphi r} = k_{\varphi r}(\varphi_r - u_{r,x}) + c_{\varphi r}(\dot{\varphi}_r - \dot{u}_{r,x})\,. \tag{17}$$

To transform implicit variations δR_r and δu_r to explicit forms we write again the virtual work equations, this time for the concentrated mass. First we write the work of adjoint vertical forces on variations of primal displacements,

$$(P_{yr}^a - m_r \ddot{y}_r^a)\delta y_r - R_{yr}^a \delta u_r - [k_{yr}(y_r^a - u_r^a) + c_{yr}^a(\dot{y}_r^a - \dot{u}_r^a)](\delta y_r - \delta u_r) = 0\,, \tag{18}$$

and conversely, using variations of primal forces and adjoint displacements,

$$\begin{aligned}&(-\delta m_r \ddot{y}_r - m_r \delta\ddot{y}_r) y_r^a - \delta R_{yr} u_r^a \\ &\quad - [\delta k_{yr}(y_r - u_r) + k_{yr}(\delta y_r - \delta u_r) + \delta c_{yr}(\dot{y}_r - \dot{u}_r) + c_{yr}(\delta\dot{y}_r - \delta\dot{u}_r)](y_r^a - u_r^a) = 0\,.\end{aligned} \tag{19}$$

Subtract Eq. (18) from (19) and integrate by parts the terms with $\delta\ddot{u}$ and $\delta\dot{u}$. We arrive at

$$\begin{aligned}&\int_0^T (\delta R_{yr} u_r^a - R_{yr}^a \delta u_r)\,\mathrm{d}t \\ &\quad = \int_0^T \{-P_{yr}^a \delta y_r - \ddot{y}_r y_r^a \delta m_r - [\delta k_{yr}(y_r - u_r) + \delta c_{yr}(\dot{y}_r - \dot{u}_r)](y_r^a - u_r^a)\}\,\mathrm{d}t\,.\end{aligned} \tag{20}$$

Following the similar way for moments as we went deriving Eqs. (18)–(20) we obtain

$$\int_0^T (\delta R_{\varphi r} u^a_{r,x} - R^a_{\varphi r}\delta u_{r,x})\,\mathrm{d}t = \int_0^T [P^a_{xr} h_P \delta\varphi_r - \ddot{w}_r h_m \varphi^a_r \delta m_r - \ddot{\varphi}_r \varphi^a_r \delta I_r - [\delta k_{\varphi r}(\varphi_r - u_{r,x}) + \delta c_{\varphi r}(\dot{\varphi}_r - \dot{u}_{r,x})](\varphi^a_r - u^a_{r,x})\}\,\mathrm{d}t\,. \tag{21}$$

Introducing Eqs. (20), (21) and (5) into Eq. (11) we finally arrive at the sensitivity operator

$$\begin{aligned}\delta G &= \int_0^T \left\{ P^a_{yr}\delta y_r + P^a_{xr} h_P \delta\varphi_r + \int_0^L p^a\,\delta u\,\mathrm{d}x \right\}\mathrm{d}t \\ &= \int_0^T \Big\{ (-\ddot{y}_r y^a_r - \ddot{w}_r h_m \varphi^a_r)\delta m_r - \ddot{\varphi}_r \varphi^a_r \delta I_r \\ &\quad + (R_{yr} u^a_{r,x} + R^a_{yr} u_{r,x} + R_{\varphi r} u^{a+}_{r,xx} + R^a_{\varphi r} u^-_{r,xx})\delta x_r \\ &\quad - [\delta k_{yr}(y_r - u_r) + \delta c_{yr}(\dot{y}_r - \dot{u}_r)](y^a_r - u^a_r) \\ &\quad - [\delta k_{\varphi r}(\varphi_r - u_{r,x}) + \delta c_{\varphi r}(\dot{\varphi}_r - \dot{u}_{r,x})](\varphi^a_r - u^a_{r,x}) \\ &\quad - \int_0^L (u^a \ddot{u}\delta m + u^a \dot{u}\delta c + q^a q \delta k)\,\mathrm{d}x \Big\}\,\mathrm{d}t\,. \end{aligned} \tag{22}$$

The operator (22) expresses the variation (3) of the response functional (2) as an explicit function of variations of all control parameters. It has quite general form and can be used to many special cases. The functions f_1 and f_2 in Eq. (2) represent structural response expressed by the vertical and angular displacements of the machine. If we are interested in displacements of the base structure, we need only to apply the adjoint force P^a_{yr} and/or moment $P^a_{\varphi r}$ at the base structure. The operator (22) will still take into account that the point x_r, where displacements are measured, moves together with the force by δx_r. If we are interested in the displacement in a fixed point x_0, we can still use the form (22). We only need to apply the adjoint force or moment at this point x_0, and neglect in Eq. (22) the terms: $R^a_{yr} u_{r,x}$ and $R^a_{\varphi r} u_{r,xx}$.

Let us assume now that the loads are harmonic with the period T, namely $P_{yr} = \hat{P}_{yr}\cos(\omega t)$ and $P_{xr} = \hat{P}_{xr}\cos(\omega t - \phi)$ with $\omega = 2\pi/T$.

Consider the steady state vibrations. Then the displacements u, y and w and their derivatives are harmonic functions with the period T, too. However, due to viscous damping, all terms in Eq. (22) have different phase angles ϕ. In fact, even the displacements of different points of the beam have different phase angles, hence $u(x,t) = \hat{u}(x)\cos[\omega t - \phi(x)]$. This is a serious difficulty. One cannot simply express Eq. (22) in amplitudes of forces and displacements, as was presented in the literature for undamped structures, when the phase angles were equal to zero. Nevertheless, we can integrate analytically all terms with respect to time when the time domain is equal to the period T, as was shown in [3] and we arrive at the formula where all terms in Eq. (22) are expressed in amplitudes of forces and displacements multiplied by the cosines of the difference of the respective phase angles. For brevity we rewrite only the first part of Eq. (22), denoting amplitudes by $\hat{\ }$:

$$\delta G = (-\hat{\ddot{y}}_r \hat{y}^a_r \cos(\phi_1 - \phi_2) - \hat{\ddot{w}}_r h_m \hat{\varphi}^a_r \cos(\phi_3 - \phi_4))\delta m_r - \hat{\ddot{\varphi}}_r \hat{\varphi}^a_r \cos(\phi_5 - \phi_6)\delta I_r + \cdots \tag{23}$$

The form (23) is very convenient for numerical application using professional FEM codes, because all quantities can be easily computed. The numerical examples presented in Section 3 illustrate the theory, prove the correctness of sensitivity operator (23) and demonstrate its accuracy when it is applied with typical FEM program.

Let us now briefly discuss the case when the damping is introduced in the form of complex modulus. This model of damping is suitable for harmonic vibrations. One can express the amplitudes of displacements and forces in the time-independent form, where the phase angles are represented in complex plane. Defining properly scalar products we can follow precisely the same way writing virtual work equations as above and arrive at the sensitivity operator very similar to Eq. (22) with the exception that there is no time integral and there appear multipliers ω and ω^2. Compare [3–5] where the sensitivity operators for different problems using complex modulus damping were derived.

Let us focus our attention on sensitivity of displacement amplitude. We start from the functional

$$G = \int_0^T \frac{1}{2} y_r^2 \, dt \tag{24}$$

where

$$y_r = \hat{y}_r \cos(\omega t - \phi) . \tag{25}$$

Henceforth $\hat{}$ will denote amplitude. Introducing Eq. (25) into (24) and integrating it we obtain

$$G = \int_0^T \frac{1}{2} \hat{y}_r^2 \cos^2(\omega t - \phi) \, dt = \frac{T}{4} \hat{y}_r^2 . \tag{26}$$

Variations of Eqs. (24) and (26) take the form

$$\delta G = \int_0^T y_r \delta y_r \, dt = \frac{T}{4} 2\hat{y}_r \delta\hat{y}_r . \tag{27}$$

Hence

$$\delta\hat{y}_r = \frac{2}{T\hat{y}_r} \delta G . \tag{28}$$

Basing on Eqs. (3), (5) and (27) we assume the adjoint force equal to y_r, namely

$$P^a = y_r = \hat{y}_r \cos(\omega t - \phi) , \tag{29}$$

and substitute Eq. (22) for δG in Eq. (28) to obtain the variation $\delta\hat{y}_r$. According to Eqs. (24) and (27) the domain of time integration was the period T.

3. Examples

3.1. Example 1

Consider a simply supported plate excited by vertical concentrated dynamic force. Technical parameters of the system are as follows: length×width of the plate 6.00×4.00 [m], thickness of the plate 0.16 [m], Young modulus $E = 27e6$ [kPa], Poisson ratio $\nu = 0.1667$ [-], mass $m = 720$ [kg/m^2], damping $c = 0.0c_{cr}$.

The first two eigenfrequencies of the plate are: $f_1 = 16.219$ [1/s], $f_2 = 31.098$ [1/s]. The dynamic load with frequency $f = 28$ [1/s] is $P(t) = 10\cos(2\pi 28t)$ [kN].

The amplitude of displacement of the point under the force was computed for different force locations and plotted in the form of contour lines in Fig. 3.

The aim of the example was to demonstrate, that the amplitudes of vibrations strongly depend on the force position. Small variation in the position of the machine can improve or deteriorate the structural response. Directional sensitivity derivatives for this structure can be computed from Eq. (23). For example we compute directional sensitivity derivatives of displacement amplitude for the point $A(x_1 , x_2) = (1.5 ; 1.0)$, due to variation of force position. In this case Eq. (23) reduces to

$$\delta\hat{u}_s = (\hat{P}\hat{u}^a_{,s} + \hat{P}^a\hat{u}_{,s})\delta s \tag{30}$$

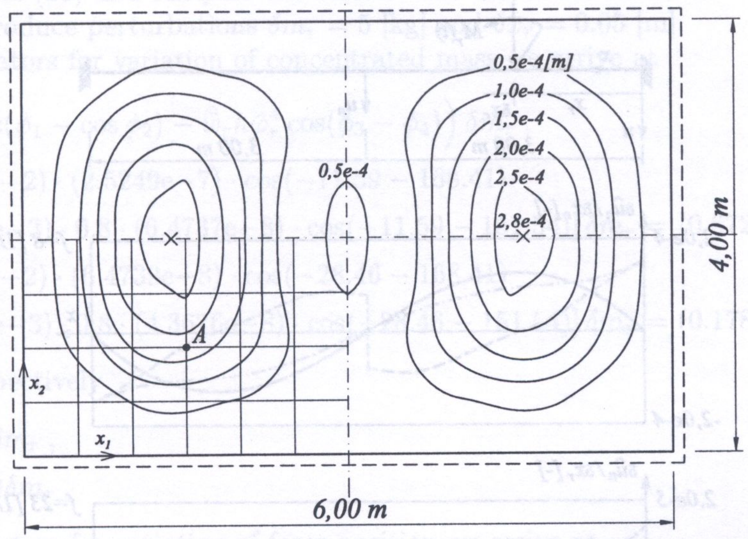

Fig. 3. Contour lines of displacement amplitudes under the force

where s is a scalar measure in arbitrary considered direction in the plane x_1, x_2.

Variations of displacement amplitude were computed for three particular cases: s_1 aligned with x_1, s_2 aligned with x_2 and s_3 in diagonal direction. Using sensitivity operators (30) for these directions we arrive at

$$\begin{aligned}\delta\hat{u}_{s_1} &= (-0.640\text{e}{-4})\,\delta s_1\ [\text{m}],\\ \delta\hat{u}_{s_2} &= (1.914\text{e}{-4})\,\delta s_2\ [\text{m}],\\ \delta\hat{u}_{s_3} &= (0.860\text{e}{-4})\,\delta s_3\ [\text{m}].\end{aligned} \tag{31}$$

For evaluation of Eq. (30) we introduce the adjoint load (29), setting $\phi = 0$.

We compare the results with the total Finite Difference Method, for the perturbations: $\delta s_1 = 0.10$ [m], $\delta s_2 = 0.10$ [m], $\delta s_3 = 0.10\sqrt{2}$ [m]. FDM provided respectively

$$\begin{aligned}\delta\hat{u}_{s_1} &= (-0.596\text{e}{-4})\,\delta s_1\ [\text{m}],\\ \delta\hat{u}_{s_2} &= (2.002\text{e}{-4})\,\delta s_2\ [\text{m}],\\ \delta\hat{u}_{s_3} &= (0.916\text{e}{-4})\,\delta s_3\ [\text{m}].\end{aligned} \tag{32}$$

The agreement can be considered as satisfactory.

3.2. Example 2

Consider a clamped-clamped beam excited by different dynamic forces. The parameters of the system are as follows: length of the beam 6.00 [m], bending stiffness $EI = 16\,398$ [kNm^2], stiffness for tension $EA = 1\,063\,540$ [kN], mass $m = 2011.54$ [kg/m], damping $c = 0.10c_{cr}$.

The first two eigenfrequencies of the beam are: $f_1 = 8.930$ [1/s], $f_2 = 24.616$ [1/s]. We consider the following specific cases of dynamic loading for different frequencies:

- vertical load $P(t) = 1.0\cos(2\pi f t)$ [kN],
- bending moment $M(t) = 0.8\cos(2\pi f t - \pi/2)$ [kNm],
- simultaneous action of $P(t)$ and $M(t)$.

Our aim is to find variation of displacement amplitude in the middle point of the beam $\delta u(x = l/2)$ due to variation of the force position δx_r. We will solve the problem for different force locations x_r. The solutions are shown in Fig. 4.

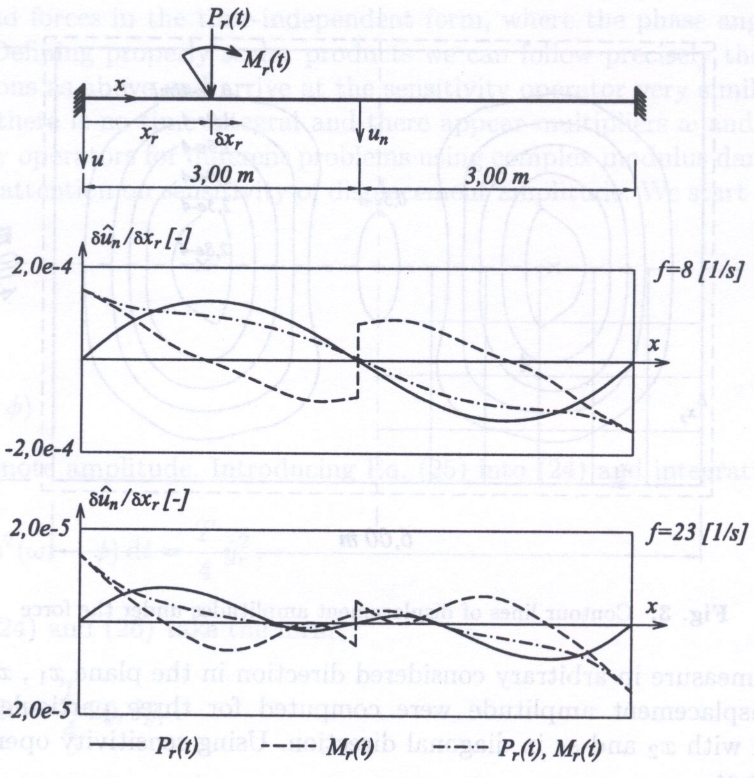

Fig. 4. Displacement amplitudes in the middle point of the beam $\hat{u}_n$, as function force position

3.3. Example 3

Consider a beam structure shown in Fig. 5. The structure is subjected to concentrated harmonic loads: $P_{yr}(t) = 0.1\cos(10\pi t)$ [kN] and $P_{xr}(t) = 0.1\cos(10\pi t - \pi/2)$ [kN]. Following parameters of the system are assumed: concentrated mass $m_r = 100$ [kg], Young modulus $E = 205e6$ [kPa], Poisson ratio $\nu = 0.3$ [-], damping $c = 0.0c_{cr}$.

- Element 1: bending stiffness $EI = 16\,398$ [kNm^2], stiffness for tension $EA = 1\,063\,540$ [kN], mass $m = 155.6$ [kg/m].
- Element 2: $EI = 1\,590\,552$ [kNm^2], $EA = 3\,537\,890$ [kN], $m = 5.2e{-}7$ [kg/m].

Our aim is to compute the sensitivity of the displacement amplitude at the point $x = x_r$, due to variations of concentrated mass δm_r and forces position δx_r . Note that both, the force position and the point of considered displacement move simultaneously by δx_r . For the sensitivity $\delta\hat{u}_r$ we apply adjoint vertical force $P^a_{yr} = 1\cdot\cos(10\pi t - \phi_1)$. For the sensitivity $\delta\hat{u}_{r,x}$ we apply adjoint moment $P^a_{\varphi r} = 1\cdot\cos(10\pi t - \phi_2)$, where ϕ_1 and ϕ_2 are respective phase angles of primary structure. We use

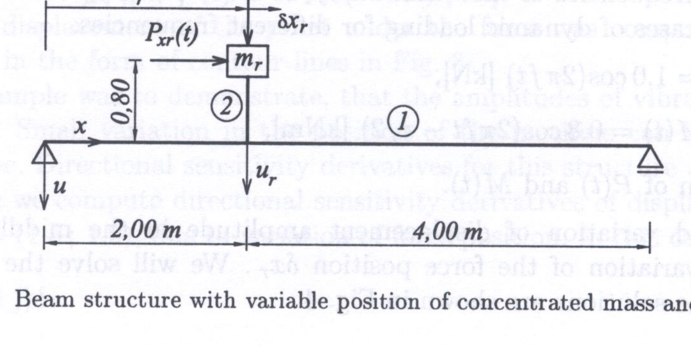

Fig. 5. Beam structure with variable position of concentrated mass and forces

the sensitivity operators (23) and compare the results with the total Finite Difference Method. For the total FDM we introduce perturbations $\delta m_r = 5$ [kg] and $\delta x_r = 0.05$ [m].

Using derived operators for variation of concentrated mass we arrive at

$$\delta G = \Big(-\hat{y}_r \hat{y}_r^a \cos(\phi_1 - \cos\phi_2) - \hat{w}_r h \hat{\varphi}_r^a \cos(\phi_3 - \phi_4)\Big)\delta m_r\,, \tag{33}$$

$$\begin{aligned}\delta\hat{u}_r &= [(-2.5439\text{e}{-2})\cdot(2.5249\text{e}{-7})\cdot\cos(-11.59-168.41)\\ &\quad -(5.8142\text{e}{-3})\cdot 0.8\cdot(6.4737\text{e}{-8})\cdot\cos(-11.59-151.54)]\,\delta m_r = (0.672\text{e}{-8})\delta m_r\,,\end{aligned} \tag{34}$$

$$\begin{aligned}\delta\hat{u}_{r,x} &= [(-2.5439\text{e}{-2})\cdot(6.4739\text{e}{-8})\cdot\cos(-28.46-168.41)\\ &\quad -(5.8142\text{e}{-3})\cdot 0.8\cdot(4.3676\text{e}{-8})\cdot\cos(-28.46-151.54)]\,\delta m_r = (0.178\text{e}{-8})\delta m_r\,.\end{aligned} \tag{35}$$

FDM provided respectively

$$\delta\hat{u}_r = (0.68\text{e}{-8})\delta m_r\,, \tag{36}$$

$$\delta\hat{u}_{r,x} = (0.182\text{e}{-8})\delta m_r\,. \tag{37}$$

Using derived operators for variation of force position we arrive at

$$\begin{aligned}\delta G = \Big[&\hat{P}_{yr}\hat{u}^a_{r,x}\cos(\cdot) + \hat{P}^a_{yr}\hat{u}_{r,x}\cos(\cdot) + \hat{P}_{\varphi r}\hat{u}^{a+}_{r,xx}\cos(\cdot) + \hat{P}^a_{\varphi r}\hat{u}^{-}_{r,xx}\cos(\cdot)\\ &- m_r(\hat{u}_r\hat{u}^a_{r,x}\cos(\cdot) + \hat{u}^a_r\hat{u}_{r,x}\cos(\cdot)) - h^2 m_r(\hat{u}_{r,x}\hat{u}^{a+}_{r,xx}\cos(\cdot) + \hat{u}^a_{r,x}\hat{u}_{r,xx}\cos(\cdot))\Big]\delta x_r\,.\end{aligned} \tag{38}$$

After substitution of the quantities obtained from ABAQUS program,

$$\delta\hat{u}_r = (12.226\text{e}{-6})\delta x_r\,, \tag{39}$$

$$\delta\hat{u}_{r,x} = (-4.925\text{e}{-6})\delta x_r\,. \tag{40}$$

FDM provided respectively

$$\delta\hat{u}_r = (11.96\text{e}{-6})\delta x_r\,, \tag{41}$$

$$\delta\hat{u}_{r,x} = (-5.154\text{e}{-6})\delta x_r\,. \tag{42}$$

The agreement can be considered as satisfactory.

4. Concluding remarks

Sensitivity derivatives accounting for variations of design parameters of a loading subsystem and a basic structure were derived allowing for viscous damping and complex modulus damping. The sensitivity is expressed by amplitudes of stress and kinematic quantities of the primary and adjoint structures and by the cosines of the respective phase angles. All these quantities are easily obtainable from professional FEM programs. For steady state vibrations of primary structure the adjoint problem is a steady state type, too. Numerical examples solved with ABAQUS program illustrate the practical use of derived formulae and the accuracy. The sensitivity operators derived in the paper can be used in optimal design or in identification problems. The optimal loading conditions correspond to minimal vibrations and may by used in designing dynamic systems composed of a dynamic machine, vibro-isolation and basic structure. In structural identification the optimal load position may correspond to maximum structural response.

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References

[1] B. Åakesson, N. Olhoff. Minimum stiffness of optimally located supports for maximum value of beam eigenfrequencies. *J. Sound Vibr.*, **120**: 457–463, 1988.

[2] *ABAQUS/Standard*. Hibbitt, Karlsson and Sorensen, Inc., USA, 1995.

[3] A. Garstecki, Z. Pozorski. Structural sensitivity analysis with respect to dynamic load conditions. In: Z. Waszczyszyn, J. Pamin, eds., *Proc. 2nd ECCM, Cracow*, 436–437, CD. Vesalius Publisher, Cracow, 2000.

[4] T. Lekszycki, Z. Mróz. On optimal support reaction in viscoelastic vibrating structures. *J. Struct. Mech.*, **11**: 67–79, 1983.

[5] T. Lekszycki, N. Olhoff. Optimal design of viscoelastic structures under forced steady-state vibration. *J. Struct. Mech.*, **9**: 363–387, 1981.

[6] Z. Mróz, A. Garstecki. Optimal design of structures with unspecified loading distribution. *J. of Optim. Theory and Applications*, **20**: 359–380, 1976.

[7] Z. Mróz, G.I.N. Rozvany. Optimal design of structures with variable support conditions. *J. of Optim. Theory and Applications*, **15**: 85–101, 1975.

[8] T. Ohno, G.J.E. Kramer, D.E. Grierson. Least-weight design of frameworks under multiple dynamic loads. *Structural Optimization*, **1**: 181–191, 1989.

[9] G.I.N. Rozvany, Z. Mróz. Optimal design taking cost of joints into account. *J. Eng. Mech. Div., Proc. ASCE*, **101**: 917–922, 1975.

[10] D. Szeląg, Z. Mróz. Optimal design of elastic beams with unspecified support conditions. *ZAMM*, **58**: 501–510, 1978.