Computer Assisted Mechanics and Engineering Sciences, 10: 239–257, 2003.

Topology and shape optimization of structural components with fracture constraints†

Bogdan Wilczyński
Technical University of Koszalin, Department of Mechanical Engineering
ul. Racławicka 15/17, 75-620 Koszalin, Poland

Zenon Mróz
Institute of Fundamental Technological Research, Polish Academy of Sciences
ul. Świętokrzyska 21, 00-049 Warsaw, Poland

(Received October 25, 2002)

In this paper a numerical design algorithm is described which enables the minimization of the stress intensity factor in a machine component by introducing the defense notch system into the component (weakening of the component) or/and by introducing stiffeners into the component (stiffening of the component) and selection of the shape of its boundary. The paper starts with the extensive review of literature devoted to the optimal design of machine parts with fracture constraints. The design procedure used is the combination of mathematical methods of computer graphics, the Boundary Element Method or the Finite Element Method used for the analysis of the stress field, the sensitivity analysis for the response gradient computations assisted by the Sequential Linear Programming. Also the concept of stop holes drilled at the crack tip, to crack arrest, is discussed. That means replacement of singular stress filed problem (cracks) by quasi-singular one (notches) and optimal design of stop holes becomes notch shape optimization problem.

Keywords: stress intensity factor, optimization, BEM, FEM, SLP, defense notch system, stop holes

1. Introduction

Geometrical discontinuities (shoulders, grooves, holes, keyways, cut-outs, fillets and so on), known collectively as notches, and material imperfections (flaws, cracks) are present in almost all engineering structures even though the structure may be "inspected" during fabrications [37]. It is well known that the geometrical discontinuities result in modifications of the simple stress distribution so that localised high stresses occur. This localisation of high stress is known as stress concentration, measured by the stress concentration factor (SCF). A typical example of the stress concentration problem is, for instance, a plate containing an elliptical hole with semi-axes a and b. It is very well known that the stress at the tip of the major axis a is proportional to the a/b ratio. If the width b of the elliptical hole tends to zero the stress tends to infinity and the stress concentration problem looses its meaning and becomes the fracture mechanics problem (a hole becomes a crack). Now the role of the stress concentration factor is replaced by the stress intensity factor (SIF). There are three modes of loading at the crack tip, known as Mode I (opening mode), Mode II (in-plane shear) and Mode III (out-of-plane shear), see Anderson [2]. Each mode of loading produces the $1/\sqrt{r}$ singularity where r is the distance from any point to the crack tip. The stress intensity factor defines the amplitude of the crack tip singularity, that is, stresses near the crack tip increase in proportion to the stress intensity factor (symbol K is used usually). If K is known, it is possible to compute

†This is an extended version of a paper presented at the conference *OPTY-2001, Mathematical and Engineering Aspects of Optimal Design of Materials and Structures*, Poznań, Poland, August 27–29, 2001.

all asymptotic components of stress or displacement. The stress intensity factor K plays also very important role in fatigue crack propagation. The crack growth per cycle (Paris's law, Anderson [2]) is proportional to the range of the stress intensity factor $\Delta K = K_{\max} - K_{\min}$, where $K_{\max}$ and $K_{\min}$ correspond to the loading amplitudes of $\sigma_{\max}$ and $\sigma_{\min}$, respectively. Because the stress intensity factor is important parameter of fracture mechanics in predicting fracture strength and fatigue life of crack components, for this reason, Section 2 contains the review of papers devoted to optimisation of machine parts covering both problems.

There are two ways (among others) of the minimising stress concentration in the machine part: the first one is to change the thickness of the component (sizing optimisation) and the second one is to change the shape of the notch (notch shape optimisation). Tvergaard [53] and Francavilla, Ramakrishnan and Zienkiewicz [20] were the first who published papers in this field. From this time on most of the research on structural shape optimization in the literature aims at minimizing the maximum stress in an elastic element which is subject to external loads and relevant constraints in order to prevent structure from crack initiation or plastic deformation which usually take place in the presence of a notch. The inclusion of fracture constraints in the automated design process was a logical extension of widely used structural optimization methods.

Cracks, which are inherent in structural materials or developed during manufacturing and fabrication or during the service period of the structures, affect strength, fatigue life and integrity of the structures. Because the crack can grow and lead to catastrophic failure of structural components, there is a need for crack arrest (crack stop) to enhance strength and service life. There are many possibilities to stop or delay crack growth (static and dynamic) process: by modification of the component boundary, by using adhesive patches or stiffeners and so on. Some possibilities are the same as for stress concentration reduction, for instance, the use of so called defense hole (notch) system. A very effective repair technique is developed by drilling stop holes at the front of the crack tip. As a result of these techniques the stress intensity factor K is significantly reduced.

It is well known that shape or topological optimisation can significantly improve the strength and stiffness of machine component. The account for singular fields associated with cracks and sharp notches generates a new class of problems in this area.

The aim of this paper is to discuss the numerical algorithm of stress intensity factor minimization by modification of the shape and topology of the machine component. The objective function for the case of existing cracks is the maximal value of K_j, $j = 1, \ldots, m$, or simply K for a single crack. For shape definition of modified boundaries in a machine component the Bézier (standard or modified) curves or superellipses are adopted. A singular quarter-point boundary elements for stress intensity factor analysis and for response gradients finite difference scheme are used. The applied optimization procedure is the sequential linear programming (SLP) with "move limits". Also the shape optimization of stop holes is developed in this paper. In this case we have the stress concentration minimization problem rather than the stress intensity factor minimization problem. The stress analysis in this case is preformed by applying the fictitious stress method, known also as the fictitious load method, and for stress gradient specification the direct differentiation method is used.

The paper is organized as follows: in Section 2 the literature devoted to the structural optimization in fracture conditions is reviewed. In Section 3 the optimization problem is formulated. Components of optimization algorithm are presented in the Section 4. Section 5 contains numerical examples of stress intensity factor minimization. Section 6 discusses the stop holes concept treated as stress concentration problem demonstrating the effectiveness of this approach. The concluding remarks are presented in Section 7.

2. Literature review

The first papers incorporating crack growth constraints in automated minimum weight design using linear fracture mechanics to predict residual strength and crack growth behavior of damaged metal aircraft stiffened panels were published by Toor [50] and Kruse [27]. In the paper by Kruse a nonlinear programming technique was used for determining efficient panel designs with varying stiffener geometry satisfying multiple constraints including fatigue and static strength.

Davis [14, 15] presented optimal design of integrally stiffened two-panel boxes subject to fatigue-crack growth and fracture as well as side constraints, displacements, yielding, local and general buckling. The design variables were skin thickness and the thickness, depth and spacing of integral stiffeners. The fracture constraints included limits on developed stress intensity factors under large static loads as well as limit on crack growth during cyclic loading.

The paper of Dobbs and Nelson [16] presented efficient optimality criteria method for the automated minimum weight design of structural components for which analytic solutions for the developed stress intensity factors are not available. The FEM is used for stress analysis and the strain energy release rate method is used to calculate developed opening mode stress intensity factors.

Gürdal and Haftka [23] used an automated procedure for designing minimum weight composite panels subjected to a local damage constraint (local damage tolerance – resistance to crack or damage propagation in a single component) under tensile loading. A finite element program based on linear elastic fracture mechanics for calculating stress intensity factors was incorporated in the design cycle. Panel fracture toughness was obtained by using the strain-based criterion. A general-purpose mathematical optimisation algorithm was used for the weight minimization with constraints on Mode I SIF (among others). Analytical sensitivity derivatives of the stress intensity factor employing the adjoint variable technique were used to enhance the computational efficiency of the procedure. Design results for both unstiffened and stiffened plates were presented.

Vrbka and Knésl [55] presented the problem of safety optimization of the high pressure compound vessel. The most loaded part of the vessel is the interior layer matrix. The safety of the matrix was taken as objective function. Because of the brittle character of the used hard metal, the combined Mohr's and maximal principal stress limiting theory was applied to express the safety of the vessel. The objective function was constructed for two different loading methods, up to the limiting brittle fracture state (objective function depends on compressive strength, transverse rupture strength and principal stresses). For optimization the variable metric method was used.

Cheng [13] studied the problem of shape optimisation from the viewpoint of structural design philosophy based on durability and damage tolerance. Initial cracks were assumed to exist or to develop at an early stage of the fatigue life. The objective is to minimize the crack propagation rate, or the stress intensity factor range (in fact the stress intensity factor). Quadratic boundary elements were applied to discretize the continuum to be optimized. To obtain the stress intensity factor range, quarter-point singular elements were placed at the tip of the crack. The sensitivity of the stress intensity factor with respect to the structural shape is derived. The circular rotating plate with the central hole containing four cracks (initial square shape of the hole with crack at corners) was presented as example. The reduction of 9% in the stress intensity factor for the smaller crack and 3.2% reduction for the longer crack was reported in this paper. The optimum shapes were nearly circular.

Esping and Holm [17] used OASIS code for structural optimisation with respect to maximum stress intensity factor and maximum effective stress of thick hydraulic cylinder. The cylinder consisted of a cylindrical part and an end plate. The two parts were welded together. The joint was assumed as the weak point with a radial crack. Only the interior of the end part was allowed to change. The significant reduction of the stress intensity factor and the stress were obtained.

Pitukhin [38–40] presented the mechanical fracture method for the optimal design of machine components. Failures of machine parts arise as a result of imperfection or violation of design rules (design failures), the manufacturing process (manufacturing failures), or conditions of exploitation. Cracks were assumed to initiate on defects in the maximum stress action zones. A defect is assumed to generate as a result of manufacturing operations of the machine parts. It is necessary to determine geometrical sizes and limit defect values of machine components during the design stage. A single criterion design is used. Mean annual cost expenses constitute the objective function, depending on the survival function. The methods of statistical fracture mechanics enable the assessment of reliability level dependent on the length of crack-like defect.

The paper of Samuelson et al. [45] is concerned with the optimisation of fatigue-critical part of a hydraulic cylinder. The optimisation problem was formulated as minimization of the stress intensity factor with constraints on the maximal occurring von Mises stress. The design variables in the optimisation process were chosen as the shape of the inner part of the end section. The FE model was simplified to axial symmetry. The stress intensity factor was evaluated by a displacement method using the singular quarter-point finite element. The optimisation reduced the stress intensity factor in the end weld root, by 40%. The fatigue tests verified this reduction.

Keum and Kwak [25] used the boundary integral equation formulation for design sensitivity analysis with varying boundary conditions using the material derivative concept and the direct differentiation method. The variation of boundary conditions is described using the normal component of the velocity field. An arbitrary rigid body motion is considered to remove singularities that occur from differentiation of the fundamental solution, thus avoiding the difficulties associated with their numerical integration. The formulation is then applied to calculate stress intensity factors as a new method of computational fracture mechanics.

Fancello, Taroco and Feijoo [18] treated the growth of an initial crack as a change in shape and demonstrated how shape design sensitivity (SDSA) leads to a well known expression of Rice path-independent integral and how this integral can be computed through domain integration, which gives more accurate results when using FEM.

Knésl [26] adopted the fracture mechanics approach to the optimum design of cracked structures subjected to cyclic loading. The residual fatigue life is considered as objective function. The basic objective of the optimization is to increase the fatigue life of a structure by changing the fatigue crack trajectory by means of modification of the structure geometry. The method of numerical analysis of the suggested approach is based on crack path tracing via the finite element method with all necessary steps of the procedure discussed in detail. If the crack initiation is localized at the notch root, the orientation of the fatigue crack plays a dominant role in the residual life determination. For the determination of the fatigue crack orientation, the energy method has been used. As an example of the presented procedure, the optimization of geometrical sizes of a spherical joint is presented with the crack assumed to initiate at the joining weld.

The paper of Lund [29] is devoted to the problem of designing mechanical components of brittle materials (brittle material has very low strain tolerance and practically exhibits no yielding and the material data are widely scattered) such as ceramics using the Weibull probabilistic treatment (two parameter Weibull distribution) of brittle strength combined with finite element based design optimization and mathematical programming. The analysis of probability of failure using Weibull statistics is introduced with the risk of rupture depending (among others) on an equivalent stress (four different criteria are used: the normal stress averaging criterion, also termed "Mode I failure criterion"; the maximum strain release rate criterion obtained from two-dimensional Griffith cracks; the third criterion is obtained from non-coplanar crack growth of Griffith cracks; and the last criterion assumes that there is no interaction between the principal stresses in the fracture criterion, i.e. the integration over the unit sphere is omitted), and the characteristic mean fracture stress.

Banichuk et al. in a series of papers [3–5] considered the problem of optimal design of quasi-brittle elastic bodies in cases of static and cyclic loading. Optimisation problems considered consist of finding the boundary of a body such that the optimized functional (volume of the body) attains an extremal value, while satisfying prescribed bounds on stress intensity factors and minimum number of cycles (structural durability). These problems are characterized by incomplete information concerning crack size, location and orientation. In this context the paper [5] presented some possible formulations of optimal structural design problems based on probabilistic approaches. The paper [4] displayed relations of sensitivities of fatigue crack growth rate with respect to variation of geometrical parameters defining the shape of the body.

Burczyński et al. [8–11] and Beluch [6] in a series of papers coupled the boundary element method and evolutionary algorithms for optimisation of plane cracked structures. Two main optimisation criteria were considered; minimization of the J-integral (related to stress intensity factor K) and minimization of maximum crack opening. The dual boundary element method was used in analysis of cracked structures. Expressions on sensitivity coefficients (adjoint variable approach) were given by path-independent integrals. The shape variation of the crack controlled a special kind of shape transformation: translation, rotation and scale change, and the external traction-free boundary was modelled by B-spline curves.

Vitali, Haftka and Sankar [54] attempted to solve the weight optimisation problem of a stiffened composite panel, subjected to crack propagation using two optimisation strategies to satisfy the crack propagation requirements. A combination of the high fidelity method of analysis and a low fidelity method were utilized to describe the crack propagation constraint. The first strategy employs correction response surface to relate the high fidelity models. The second method converts a constraint on the stress intensity factor into an equivalent strain constraint and solves the problem through an iterative process.

Gani and Rajan [21] investigated the relationship between structural geometry and number of life cycles to failure to improve the fatigue life of structural components. The linear elastic fracture mechanics approach was integrated with shape optimal design methodology. The primary objective of the design problem is to enhance the life of the structure. Gradient-based nonlinear programming techniques were used with the computed sensitivity information to predict the required shape changes. Relevant issues such as problem formulation, FE modeling, mesh generation, and regeneration were discussed.

Saurin [46] dealed with the shape design sensitivity analysis for quasi-brittle plane bodies and implementation of this analysis in numerical method. Special attention was devoted to basic relations of sensitivity analysis which are derived by means of domain representation of the path-independent J-integral and introduction of adjoint system.

The optimization problems discussed in Serra [47] consisted in finding the optimal shape of beams in such a way that the volume reaches a minimum, while satisfying prescribed bounds on the stress intensity factor or on the fatigue life of the structure. Some of the quantities that describe the model are taken as random while others are deterministic. The analysis has been confined to mode I opening of surface crack.

Chaperon et al. [12] presented recent developments in the optimal design of structural components with fracture constraints. An initial "near optimum" shape obtained from the shape of static optimisation for the non-cracked geometry is used as a starting design in conjunction with the alternating FEM. It was found that an initial "near optimal" shape for the uncracked geometry was in fact an excellent approximation to the optimal solution for the cracked problem. It was also found that, in each case, for a given crack length the stress intensity factors, for cracks emanating at any point near the optimized boundary, were essentially constant along most of the circumference. A similar behaviour was found when considering shape optimisation with durability, i.e., crack growth constraints. In this case it was hypothesised that the optimum shape would be such that all locations around the cut-out would be equally fatigue critical.

3. Problem formulation

Consider a deformed elastic (quasi-brittle) machine component occupying a 2-D region Ω, with surface traction $\mathbf{T}$ on the boundary Γ_σ, and zero displacements on the boundary Γ_u. The machine part contains a single crack (crack can emanate from the notch). The boundary of the notch and the crack are assumed to be traction free.

Examining literature devoted to the analysis of the stress intensity factors it is known that the following ways can be used to decrease the SCF's:

1. by varying the shape of boundary of the structural component,

2. by prestressing the area near the crack tip (for instance by the phase transformation process),

3. by introducing stiffeners into the component,

4. by introducing the defense notch systems,
5. by introducing the protective patches covering the crack area,
6. by using piezoelectric membranes prestressing the cracked area,
7. by introducing of stop holes,
8. other (combined) methods.

In this paper the attention is paid to the stress intensity factor reduction:

1. by changing the shape of the body contour
2. by introducing into the body stiffeners to stop the crack (stiffening of machine part),
3. by introducing the defense hole system (weakening of structural component).

Also the application of stop holes at the crack tip to crack arresting is discussed.

Figure 1 presents schematically the introduced design variables aimed at minimizing the stress intensity factor.

Generally, an optimal design problem of structural elements can be stated mathematically as that of minimising the specific objective (cost) function $f(\mathbf{D})$ subject to certain behavioural constraints $g_j(\mathbf{D})$, $j = 1, \ldots, n_c$, and bounds (side constraints) on the design variables, where $\mathbf{D} \in R^n$ is a vector of design variables, and n_c is the number of constraints.

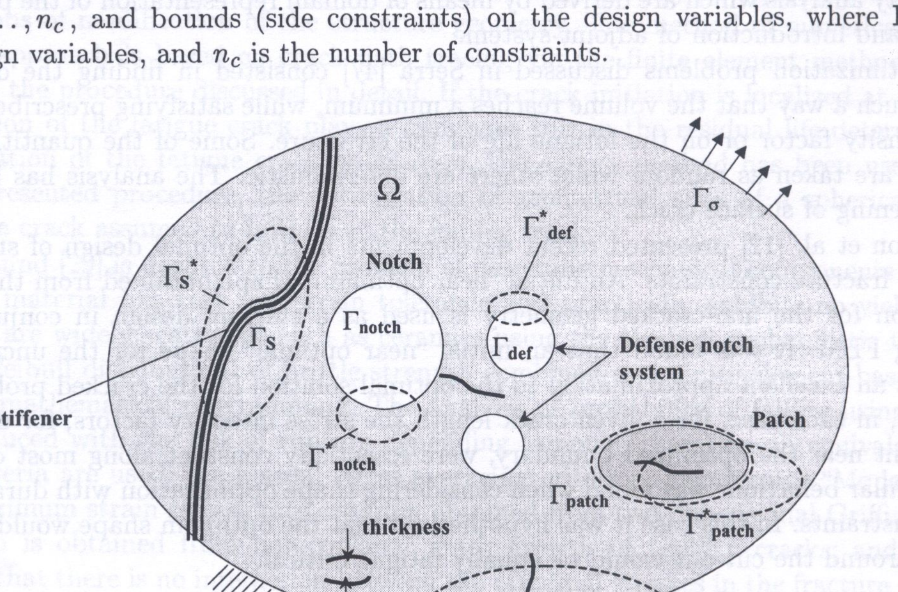

Fig. 1. Machine or structural element with notches, cracks, stiffeners, patches and defense notch systems

3.1. Objective function

The components of the stress tensor nearest to the crack tip have the following asymptotic expression in polar coordinates,

$$\sigma_x = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} f_1(\varphi), \qquad \sigma_y = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} f_2(\varphi), \qquad \tau_{xy} = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} f_3(\varphi),$$

where K_{I} is the mode I stress intensity factor. Non-dimensional functions $f(\varphi)$ depending on the polar angle φ. They can be find in textbook of Anderson [2]. Two remaining shear modes (Mode II and Mode III) can also occur in a cracked body. However the Mode I is the most important as 90% of engineering fracture mechanics problems are of the Mode I type. The combined type modes may account at the beginning of the fracture process but turn next to Mode I immediately after crack growth, Serra [47]. Hence, in this paper our objective is to minimize the opening Mode K_{I}.

3.2. Design variables

In this paper a modified shape of the notch from which crack can emanate, and/or shape of a defense notch, and/or a modified boundary of a structure are defined by Bézier (standard, modified) interpolants, and by an superelliptic shape. The position of some control (Bézier) points and parameters of superellipse are treated as design variables. This description essentially reduces the number of design variables. Topological design variables, that is characteristic dimensions of a defense hole (notch), its location, stiffener location and its geometrical dimensions, are also used.

3.3. Constraints

There are several constraints, which must be satisfied for the design problem of the stress intensity factor minimization must satisfy, such as the notch stress concentration factor, the stress concentration factors in the stiffener, and some volume or area constraints, like, for instance, the notch area or the length of its boundary, the compliance of the machine element, and so on. The value of

Box 2. Stress intensity factor K_{I} minimization

$$\min_{\Gamma_j} K_{\mathrm{I}}(\mathbf{D}) \quad \text{with constraints:} \quad \Gamma_j \subset \Gamma_j^*, \quad V(\mathbf{D}) \le V_{\mathrm{feas}},$$
$$\mathbf{D}_{\mathrm{lower}} \le \mathbf{D} \le \mathbf{D}_{\mathrm{upper}}, \quad C(\mathbf{D}) \le C_{\mathrm{feas}},$$
$$K_{\mathrm{tg}}(\mathbf{D}) \le K_{\mathrm{tg\,feas}}, \quad K_{\mathrm{I}}(\mathbf{D}) \le K_{\mathrm{Ic}},$$
$$K^{\mathrm{s}}_{\mathrm{tg}}(\mathbf{D}) \le K^{\mathrm{s}}_{\mathrm{tg\,feas}},$$

Γ_j – the contour of the free boundary, notch, or/and defense notch, where:
Γ_{notch} – modified boundary of the notch
Γ_{def} – modified boundary of the defense notch
Γ_{free} – modified free boundary
Γ_{patch} – modified shape of the patch
Γ_{s} – modified shape of the axis of the stiffener (including stiffness changes),
Γ_j^* – variation domain of Γ_j, where:
$\Gamma^*_{\mathrm{free}}, \Gamma^*_{\mathrm{notch}}, \Gamma^*_{\mathrm{patch}}, \Gamma^*_{\mathrm{def}}, \Gamma^*_{\mathrm{s}}$ – boundary variation domains,
K_{tg} – theoretical SCF,
$K^{\mathrm{s}}_{\mathrm{tg}}$ – SCF in the stiffener,
V – volume (area) constraints,
C – compliance of the structural element,
K_{Ic} – fracture toughness value,
$\mathbf{D} = [D_1, D_2, \ldots, D_n]^T \subset R^n$ – vector of design variables,
$\mathbf{D}_{\mathrm{lower}}, \mathbf{D}_{\mathrm{upper}}$ – lower and upper limiting values of $\mathbf{D}$.

K_I, specified by a general form can be written as

$$K_I = \beta_1 \beta_2 \sigma \sqrt{\pi a}$$

where a is the half length of the crack, and β_1 and β_2 are corrections to the loading and crack geometry, respectively, Farahmand [19]. This value cannot exceed the critical value K_{Ic}. Box 1 presents the stress intensity factor minimization problem.

4. Components of the optimization algorithm

A general methodology for shape optimization, which is of multidisciplinary nature links: geometric modeling (shape definition) of the boundary in machine component, stress analysis by the finite element method (FEM) or boundary element method (BEM) sensitivity analysis (stress, stress intensity factor and displacement gradients), which is related to the analysis method, and an approximation based design optimization procedure, see Fig. 2.

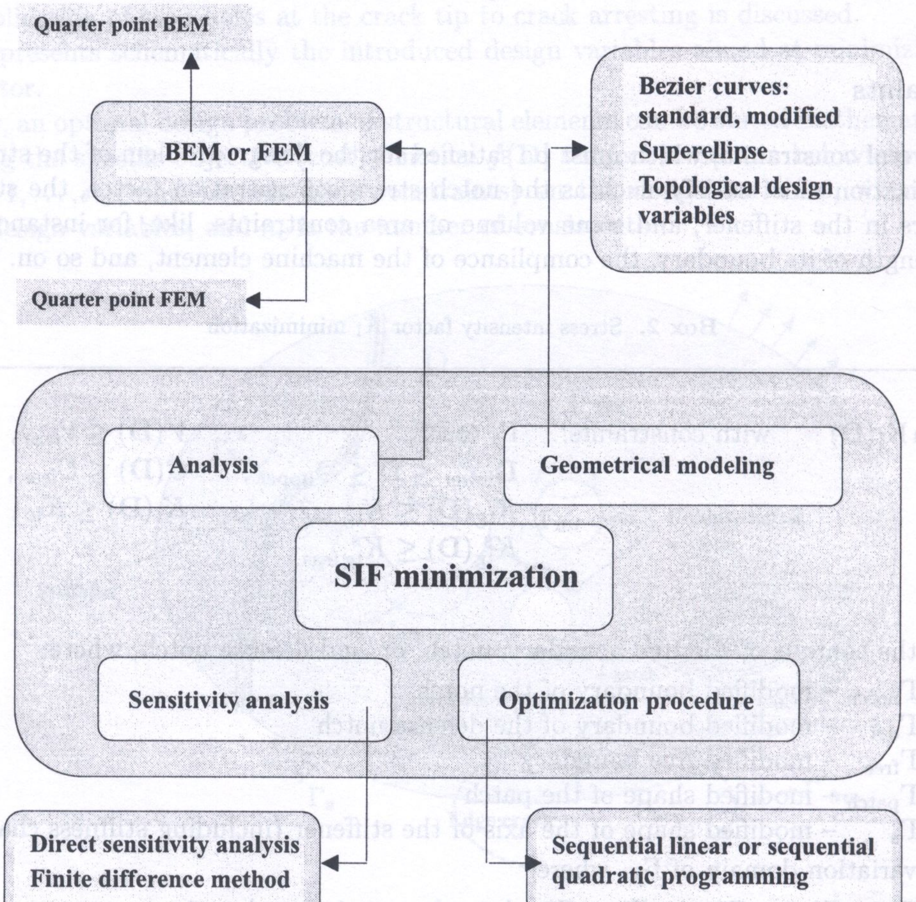

Fig. 2. Components of optimization algorithm

4.1. Shape definition

In general, it is desirable to define the shape of the boundary by means of a reasonably small number of design variables to reduce the dimension of the problem. For computer aided geometric design (CAGD) several tools have been developed for the description of curves for 2D problems. Very efficient in this connection are Bézier, B-spline and Beta-spline curves.

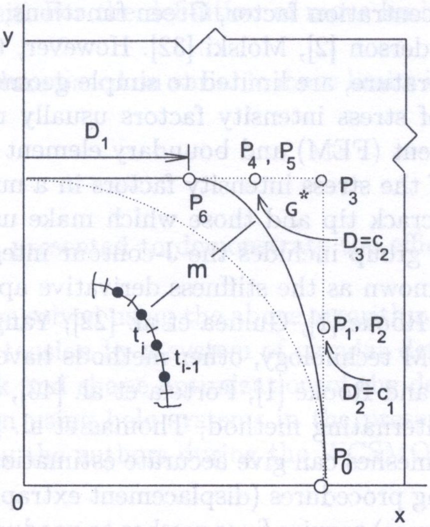

Fig. 3. Example of shape definition of the boundary and/or the notch (hole) by Bézier curves (m – modified boundary); D_1 – the position of the separation point, D_2 and D_3 – shape parameters

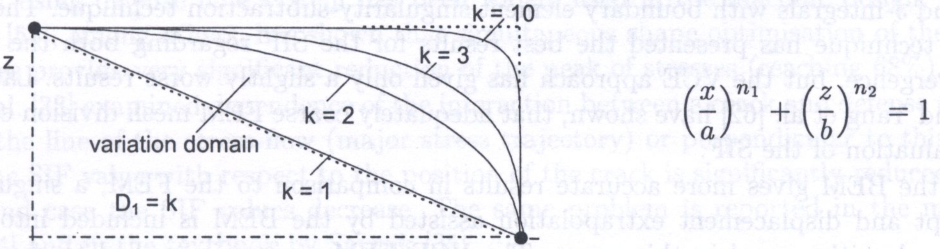

Fig. 4. Shape definition using superellipse; a, b – major and minor semi-axis, $D_1 = n_1$, $D_2 = n_2$ are design variables ($n_1 = n_2 = k$ in the graph above)

In this paper the variable boundary of the notch in a machine component is described by using Bézier (standard or modified) curves, see ref. in Wilczyński [57, 58, 60]. Figure 3 shows the modified variant of Bézier curves. A Bézier curve is defined by the so-called characteristic triangle (CT) $P_0P_3P_6$. The shape parameters c_1 and c_2, which continuously control the shape (by controlling the position of multiple points P_1, P_2 and P_4, P_5), and the position of end points can be treated as design variables. The important feature of Bézier curve is, that it passes through end points, and is tangent to the corresponding edges of the CT. It should be mentioned, that the Bézier curve segment is described by a polynomial of the parameter t, where $t \in [0, 1]$. The subsequent values of parameter t_{j-1} and t_j define the endpoints of the boundary element (see Fig. 3) or corner points of the finite element. The position of some control (Bézier) points (active points) are treated as design variables.

The other class of boundary shapes is generated by introducing the superellipse dependent on the parameters a, b, n_1, n_2. Assuming $n_1 = n_2 = k$, the shape variation depending on k is illustrated in Fig. 4.

4.2. Stress intensity factor and sensitivity analysis

Stress intensity factors are the fundamental parameters of fracture mechanics in predicting fracture strength and fatigue life of crack components and for this reason much effort has been put into their derivation and variety of methods have been developed for their evaluation. Numerous methods of stress intensity factor determination have been developed and published: analytical, superposition, using the stress concentration factor, Green functions, weight functions and numerical, see, Aliabadi and Rooke [1], Anderson [2], Molski [32]. However, the analytical solutions of stress intensity factors, available in literature, are limited to simple geometric and loading conditions. For this reason the determination of stress intensity factors usually require application of numerical methods, such as the finite element (FEM) and boundary element (BEM) methods. There are two groups of estimation methods of the stress intensity factors in a numerical technology, those based on field extrapolation near the crack tip and those which make use of the energy release rate for a propagating crack. This latter group includes the J-contour integral, the virtual crack extension (VCE) technique which is also known as the stiffness derivative approach and energy domain integral formulation (Aliabadi and Rooke [1], Guinea et al. [22], Yang et al. [62]). Using the FEM or the BEM and coupled FEM/BEM technology, other methods have been developed, as singularity-subtraction technique, Aliabadi and Rooke [1], Portela et al. [43], dual boundary element method, Portela and Aliabadi [42] and alternating method, Thomas et al. [49]. The advantage of the VCE method is that relatively coarse meshes can give accurate estimation for the SIF. In contrast, methods based on near-tip fields fitting procedures (displacement extrapolation and stress extrapolation, with the emphasis laid on the former) require finer meshes to produce good numerical representation of crack-tip field. To avoid finer mesh very well known concept of quarter-point element (FEM or BEM) can easily be incorporated into the computer program. Portela and Aliabadi [41] have examined three finite element techniques such as singular quarter-point elements (QPE), virtual crack extension and J-integrals with boundary element singularity-subtraction technique. The singularity subtraction technique has presented the best results for the SIF regarding both the accuracy as well as convergence, but the VCE approach has given only a slightly worse results. Lately, Guinea et al. [22] and Yang et al. [62] have shown, that adequately coarse FEM mesh division can give very accurate evaluation of the SIF.

Because the BEM gives more accurate results in comparison to the FEM, a singular quarter-point concept and displacement extrapolation assisted by the BEM is included into the overall optimisation algorithm used in this paper. The VCE/FEM variant is applied to check the BEM results and is expected to model a component with stiffeners in optimal design for minimization of the stress intensity factor.

The presented algorithm is rather very small scale problem; Bézier or superellipse description of the modified shapes, as was mentioned earlier, essentially reduces the number of design variables and stress computations are not time consuming. For this reason, the stress gradients are computed in the general case by the finite difference method in the first attempt of construction of the optimisation algorithm within the general program. Although the global Finite Difference Method (FDM) is more costly, it is much easier to be adopted to the overall optimisation algorithm. To solve the stop holes concept, considered in this paper, to crack-like defects arrest, the fictitious stress method (FSM) is used. It was proved in the papers of Wilczyński [57, 58] that this indirect variant of the BEM can be effectively used to analyse the stress concentration problems and shape optimisation of machine components with stress constraints. Stress gradients obtained by the direct differentiation method related to the fictitious stress method are provided in the close form, Wilczyński [59]. The FSM method, originally developed by Crouch and Starfield is presented in the paper [57], published in this Journal.

4.3. Optimization procedure

It is well known, that the cost of computations is extremely high when the FEM or BEM is used as component of optimization algorithm. For this reason the implicit original optimization problem is replaced by explicit, approximately equivalent, but much easier to solve. This concept has found wide application in the optimization technology. The most popular approximation is the direct linear (first-order) one based on the Taylor expansion of objective and constraints, leading to the Sequential Linear Programming (SLP). The numerical efficiency of the SLP method depends meaningfully on a proper choice of the move limit. For the definition of move limits see the paper of Lamberti and Pappalettere [28], for instance.

In this paper, the SLP procedure based on variable move limits is used as optimization procedure.

5. NUMERICAL EXAMPLES

Several numerical examples are presented to demonstrate the efficiency and robustness of the presented algorithm.

The specified examples can be solved using the above algorithm: (i) SIF minimization for a single crack in a plate under uniaxial tension by a system of circular defense holes, (ii) simultaneous SIF minimization of the single crack and shape optimization of the defense hole system protecting the crack tip, (iii) SIF minimization using hole systems in the presence of stiffeners. These examples have been partially presented by the authors during the WCSMO-4 Congress [61].

5.1. SIF minimization for a single crack in a plate under uniaxial tension by a system of defense holes

The idea of using defense hole system has been widely used in the machine design, Meguid [31]. Wilczyński [58], among others, has shown that simultaneous shape optimisation of the central and defense holes provide very significant reduction of the peak of stresses (reaching 68%).

Osiv et al. [33] examined dependence of the interaction between a crack and defense notch system located in the line of the stress flow (major stress trajectory) or perpendicular to this line. In the first case the SIF value with respect to the position of the crack is significantly reduced. Obviously, in the second case the SIF values decrease. The same problem is reported in the monograph of Rykaluk [44] and in the textbook by Savruk [35].

The idea of this example results from the paper of Aliabadi and Rooke [1]. A rectangular sheet of width $2W$, length $4W$ and the crack length of $2a$, symmetrically located between two holes of radius R (defense notch system) is subject to a remote uniaxial stress ($\sigma = 100$ stress units). The centers of the holes are on the perpendicular bisector of the crack, at a distance h from the crack. Because of symmetry only the quarter of the sheet needs to be modeled, see Fig. 5a. For the central crack ($a/W = 0.4$) K_{I} is 391.3 SIF units and for the central crack and the regular defense hole system with the starting design variable vector $[D]^T = [R, h]^T = [3, 8]$ K_{I} is 276.6 SIF units. The optimal value of the SIF is $K_{\mathrm{I}} = 172.9$ (the constraints on the upper limit of $R = 4$ and on the lower limit values of $h = 6$ were active) have been obtained, what gives the 56% reduction in SIF

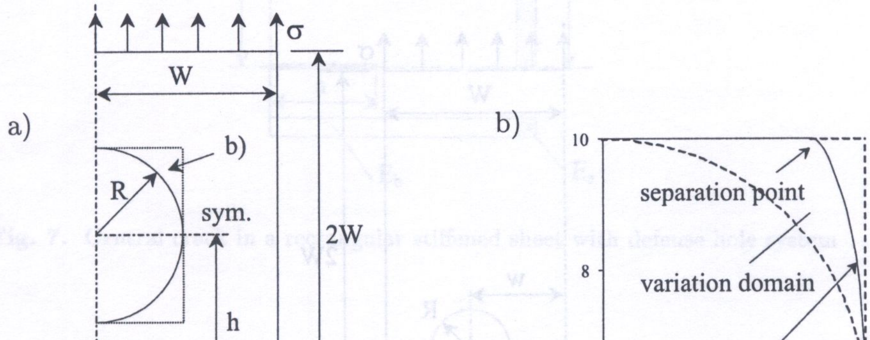

Fig. 5. a) Central crack in a rectangular sheet with defense hole system, b) defense hole definition (optimal shape)

in comparison to the component without defense hole system. These results can be treated as the program test, because this solution was expected, see Aliabadi and Rooke [1].

Further reduction in the SIF is possible by modification of the shape of the defense hole (simultaneous SIF of the single crack and maximum equivalent stress minimization of the defense hole system protecting a crack). Our objective is to minimize K_{I} and also the maximum equivalent stress σ_{eff}. This is a min-max problem: $\min(\max(K_{\mathrm{I}}, \sigma_{\mathrm{eff}}))$. The contour of the hole, which is composed of the straight line and the Bézier curve, should lie between the circle and the square contours, Fig. 5b. A special concept of Bézier interpolants inside the so-called characteristic triangle is adopted. The design variables are: the position of the separation point and two non-dimensional shape parameters, which control the position of some Bézier control points. We have obtained further reduction in the SIF, by 15.6%, with respect to the optimal regular defense hole system. Introducing the optimal defense hole system allows for the significant reduction in the value of the SIF in a machine component.

Because of the influence of the external strip boundary on the SCF factor in the defense notch we have to put the condition constraining the value of this factor. For instance, for the feasible value of the SCF equal 3.0 stress units (there is interaction between the defense notch and the crack) we have obtained the following optimal solution; $[D]^T = [3.583, 6.0]^T$, and the corresponding optimal SIF is 223.3 units. For the SCF equal 3.5 units (we have changed the condition limiting value of the radius from 4.0 to 5.0) the optimal design variables are $[D]^T = [4.492, 6.0]^T$ and the optimal value of the SIF is 125.8 units. The constraints of the limiting value of the SCF and on the hole distance from the crack line were active.

5.2. SIF minimization of a single crack in plate under uniaxial tension by a system of circular defense holes

The problem of using defense holes in the crack problems has been mentioned in the papers of Trevelyan et al. [51], see also [48, 49]. As an example, a rectangular plate with a crack emanating from a small central hole (unsymmetrical case) is presented. It is shown in this paper that the crack tip can be protected to some degree by presence of the other holes which divert the stress flow away from the crack tip.

The idea of this example comes from the mentioned earlier paper of Trevelyan et al. The SIF is minimized using the defense holes located near the crack tip, Fig. 6. The range of changes of the design variables is: $1.0 \le D_1 = R \le 2.0$, $3.0 \le D_2 = h \le 5.0$, and $3.0 \le D_3 = w \le 5.0$. The remaining of data are the same like as in the example 5.1. For starting design variable vector $[D]^T = [1.5, 4.0, 4.0]^T$ the SIF is $K_{\mathrm{I}} = 340.8$. The optimal solution is $[D]^T = [2.0, 3, 03, 0]^T$ and the corresponding value of the SIF, $K_{\mathrm{I}} = 145.7$. This means that this way also significantly reduces the

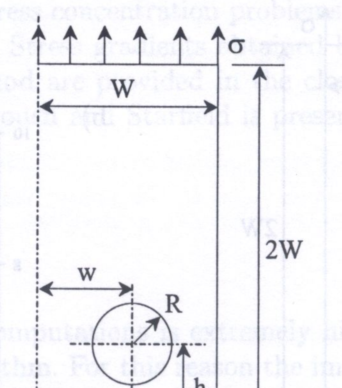

Fig. 6. Central crack in a strip with defense holes located near tip of crack

value of the SIF. Including the condition on the feasible SCF factor $K_{tg} = 4.2$ we have the following results: optimal design variables $[D]^T = [1.86, 3, 03, 0]^T$, and the SIF $K_I = 185.9$.

5.3. SIF minimization using defense hole systems for stopping growth of a single crack in the uniaxial tensioned plate in the presence of stiffeners

In the fail-safe design philosophy, the structure is designed so that the partial failure of a structural component due to crack propagation is localized and safety contained or arrested. An example of a fail-safe designed structure with crack arrest feature is obtained by the use of stiffeners (stringers), attached, for instance, to the plate, Farahmand [19], Tsamasphyros et al. [52]. The use of stiffeners which act as crack arrestors is recommended for the thin-walled structures.

The case considered here is referred to a single central crack in a stiffened finite strip subjected to a tensile edge load (the singular quarter-point FEM is used). Simplifying a bit problem we assume that the stiffener location is given (an edge stiffener) and only its stiffness is assumed as design variable, Fig. 7. The stiffness D_3 is proportional to the ratio of E_s/E_b, where E_s is the Young modulus of the stiffener, and E_b is the Young modulus of the plate. Setting only side constraints, here $1.0 \leq D_3 \leq 5.0$ (test problem), and starting with the $[D]^T = [3.0, 8.02.5]^T$ the following optimal design vector $[D]^T$ has been obtained, $[D]^T = [4.0, 6.0, 5.0]^T$ with the value of the SIF equal $K_I = 137.5$. For the D_3 limiting value equal 9.0 the optimal SIF is 110.0. If we include the constraint on the compliance of the structural element (we assumed the feasible value of maximum displacement is 0.028 units) the optimal solution is $[D]^T = [3.71, 6.0, 5.0]^T$ and the optimal value of SIF is 165.1.

The optimization problem of stress intensity minimization in structural components in presence of stiffeners needs further studies.

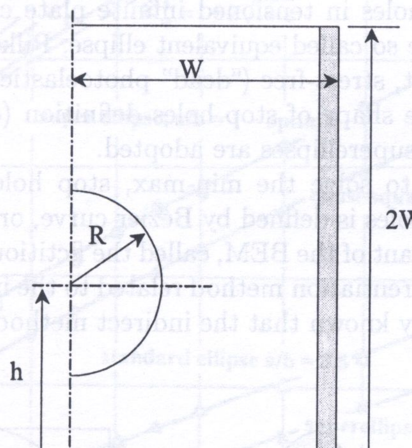

Fig. 7. Central crack in a rectangular stiffened sheet with defense hole system

6. STOP HOLES CONCEPT

In fatigue damage repair the method of hole drilling (stop hole system) at the crack tip to reduce the stress intensity factor has often been used, see Shin et al. [48], Meguid [31]. After drilling, the stop holes may be cold expanded, Vulnić [56]. Meguid discussed this problem from the point of view

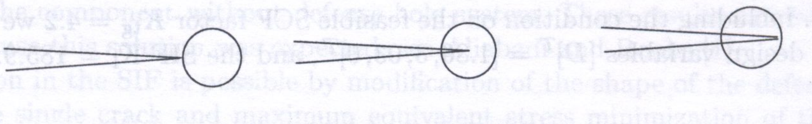

Fig. 8. Position of stop holes

of the SIF with the following conclusions: typically stop hole diameters are the order of 10–15% of the crack length. There are three possible positions for the stop hole, see Fig. 8, (the second is recommended). The introduction of these stop holes results in a reduction in K_I ranging from 8% corresponding to the first position to 18% corresponding to the third position.

But, in fact, for non-sharp crack tips the interpretation of SIF is not clear, see Pedersen [36]. The stop hole system constitutes rather the stress concentration problem. It is very well known, that the circular profile of the hole is not optimal. In the optimal notch contour the stresses are uniformly distributed. Thus, we can select the optimal shape of the stop hole treating this problem as minimization of the maximum stress concentration (tangential stress) around the modified boundary. Hence the notch shape optimization problem is

$$\min_{\Gamma_{AB}} \max \quad \sigma_{\max}, \qquad \text{subject to the constraints } \Gamma_{AB} \subset \Gamma^*$$

where Γ_{AB} is the boundary of the stop hole to be optimized, and Γ^* is the specified variation domain of Γ_{AB}.

6.1. Numerical example

The idea of this example is generated by the paper of Karpow et al. [24] and Panasyuk [34], see also Pilkey [37]. Figure 9 shows two holes in tensioned infinite plate connected by a slit. Because the holes connected by a slit lie inside so called equivalent ellipse, Pilkey [37], the cusps resulting from the enveloping ellipse are, in effect, stress-free ("dead" photoelastically) so it is enough to optimize the contour AB (Fig. 10). For the shape of stop holes definition (contour BC is the offset of AB) Bézier curve and two parametric superellipses are adopted.

The following tools are used to solve the min-max, stop holes shape optimization problem: a modified boundary of the stop holes is defined by Bézier curve, or two parameter superellipse. For the stress analysis the indirect variant of the BEM, called the fictitious stress method, Wilczyński [57, 58]. Stress gradients by direct differentiation method related to the indirect BEM equations are given in a closed form [59]. It is generally known that the indirect method provides incorrect results when

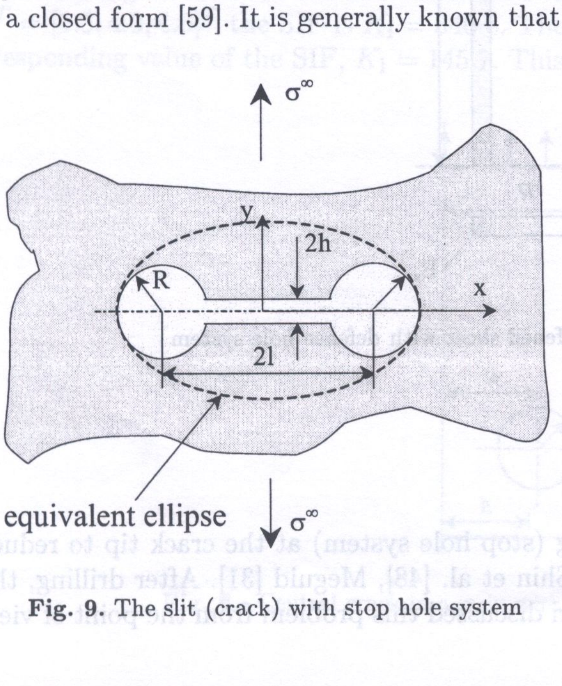

Fig. 9. The slit (crack) with stop hole system

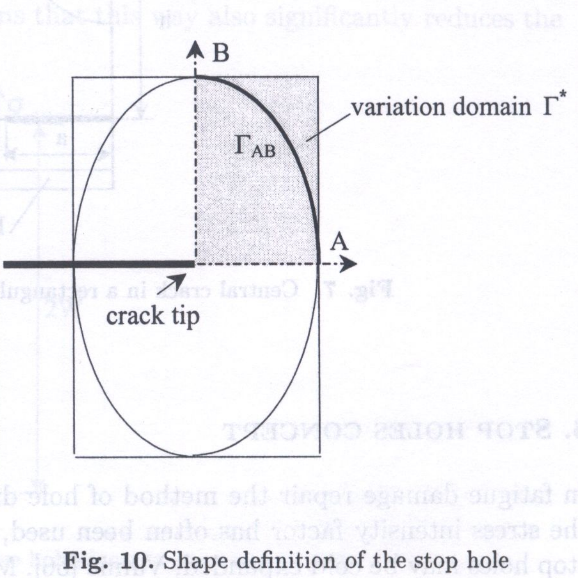

Fig. 10. Shape definition of the stop hole

boundaries with sharp corners are considered. For this reason this method has not been adopted to the SIF minimization problems. In the recent papers by Wilczyński [57, 58] it was shown that the mentioned BEM variant gives the acceptable "exact" result for regions with smooth boundaries and can be recommended for stress minimization. As the optimization procedure, similarly as in previous chapters, the Sequential Linear Programming with move limits is used, after conversion of min-max problem to the simple min problem ("bound formulation").

Figure 11 displays dependence of the stress concentration factors in the stop holes depending on ratio half-length l of the crack to the characteristic dimension R of the circular hole (reference shape) and for optimal shapes defined by Bézier curves, with varying semi-axes ratio of superellipses. This figure presents the analytical and empirical results obtained by Panasyuk [34] (dashed lines). Figure 12 shows stress distribution corresponding to the optimal shapes presented in Fig. 13. Table 1 contains optimal values of superellipse parameters (design variables) for different superellipse axis

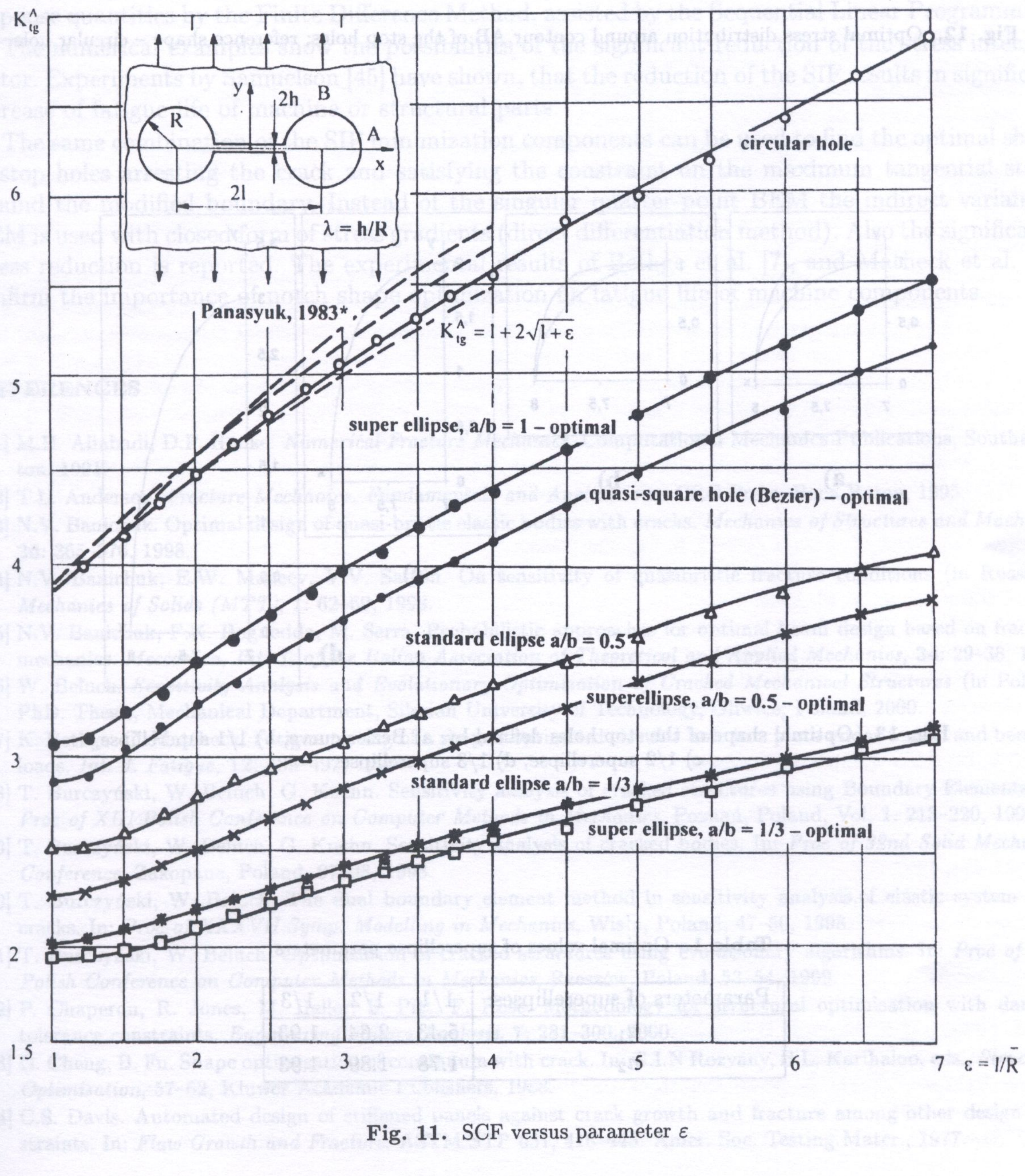

Fig. 11. SCF versus parameter ε

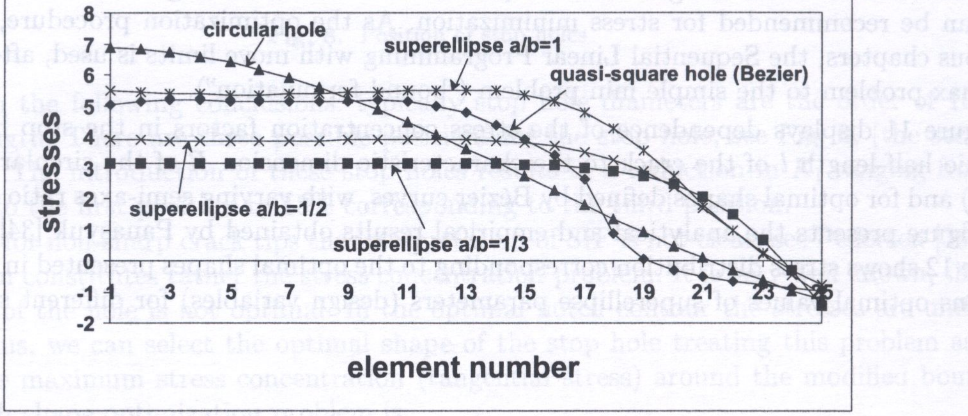

Fig. 12. Optimal stress distribution around contour AB of the stop holes; reference shape – circular hole

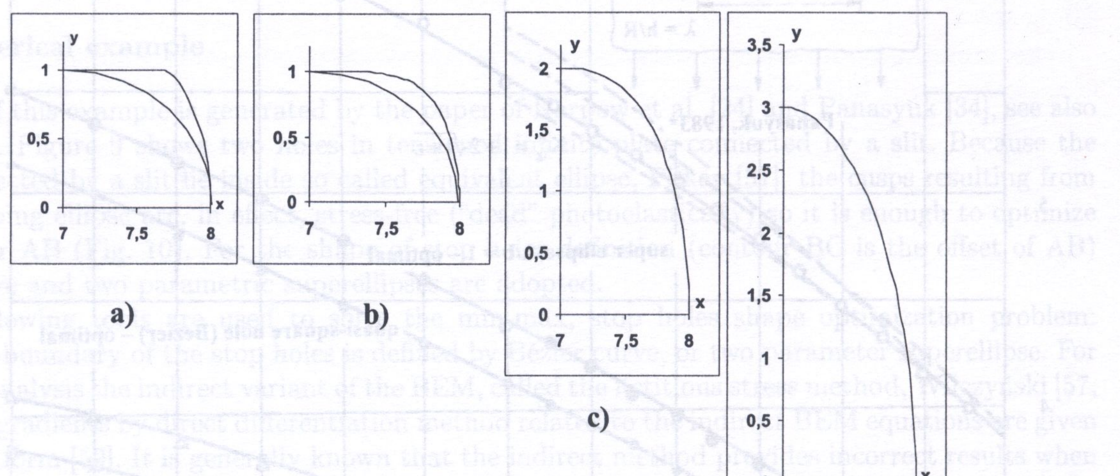

Fig. 13. Optimal shape of the stop holes defined by: a) Bézier curve, b) 1/1 superellipse, c) 1/2 superellipse, d) 1/3 superellipse

Table 1. Optimal values of superellipse parameters

Parameters of superellipses:	1/1	1/2	1/3
n_1	5.43	2.64	1.93
n_2	1.78	1.89	1.93

ratio. It is observed from Figs. 12 and 13, that for parameter $\varepsilon = 7$, the stress reduction is about 72%. The similar problem has been solved by Pedersen [36] with one parameter superellipse defining the stop hole contour and the FEM used for analysis.

7. Concluding remarks

In this paper the numerical design algorithm is described which enables the minimization of the stress intensity factor in the machine or structural components by introducing the defense notch system into the component (weakening of the component) or/ and by introducing stiffeners into the component (stiffening of the component) and selection of the shape of its boundary or/and the defense notch boundary.

The design procedure is the combination of mathematical methods of computer graphics (Bézier's curve or two parameter superellipse), the Boundary Element Method or the Finite Element Method used for the analysis of the stress field and the stress intensity factors, the sensitivity analysis of response quantities by the Finite Difference Method, assisted by the Sequential Linear Programming.

The numerical examples show the possibilities of the significant reduction of the stress intensity factor. Experiments by Samuelson [45] have shown, that the reduction of the SIF results in significant increase of fatigue life of machine or structural parts.

The same combination of the SIF minimization components can be used to find the optimal shape of stop holes arresting the crack and satisfying the constraint on the maximum tangential stress around the modified boundary. Instead of the singular quarter-point BEM the indirect variant of BEM is used with closed form of stress gradients (direct differentiation method). Also the significance stress reduction is reported. The experimental results of Bethge et al. [7], and Matheck et al. [30] confirm the importance of notch shape optimization on fatigue life of machine components.

References

[1] M.H. Aliabadi, D.P. Rooke. *Numerical Fracture Mechanics.* Computational Mechanics Publications, Southampton, 1991.

[2] T.L. Anderson. *Fracture Mechanics. Fundamentals and Applications.* CRC Press, Boca Raton, 1995.

[3] N.V. Banichuk. Optimal design of quasi-brittle elastic bodies with cracks. *Mechanics of Structures and Machines,* **26**: 365–376, 1998.

[4] N.V. Banichuk, E.W. Makeev, V.V. Saurin. On sensitivity of quasibrittle fracture conditions (in Russian). *Mechanics of Solids (MTT),* **1**: 62–69, 1998.

[5] N.V. Banichuk, F.K. Ragnedda, M. Serra. Probabilistic approaches for optimal beam design based on fracture mechanics. *Meccanica, Int. J. of the Italian Association of Theoretical and Applied Mechanics,* **34**: 29–38, 1998.

[6] W. Beluch. *Sensitivity Analysis and Evolutionary Optimization of Cracked Mechanical Structures* (in Polish). PhD. Thesis, Mechanical Department, Silesian University of Technology, Gliwice, Poland, 2000.

[7] K. Bethge, C. Mattheck. Fatigue testing of a shaped-optimized circular hole in a plate under tensile and bending loads. *Int. J. Fatigue,* **12**: 489–492, 1990.

[8] T. Burczyński, W. Beluch, G. Kuchn. Sensitivity analysis of cracked structures using Boundary Elements. In: *Proc of XIII Polish Conference on Computer Methods in Mechanics,* Poznań, Poland, Vol. 1: 213–220, 1997.

[9] T. Burczyński, W Beluch, G. Kuchn. Sensitivity analysis of cracked bodies. In: *Proc of 32nd Solid Mechanics Conference,* Zakopane, Poland, 97–98, 1998.

[10] T. Burczyński, W. Beluch. The dual boundary element method in sensitivity analysis.of elastic system with cracks. In: *Proc of XXXVII Symp. Modelling in Mechanics,* Wisła, Poland, 47–50, 1998.

[11] T. Burczyński, W. Beluch. Optimization of cracked structures using evolutionary algorithms. In: *Proc of XIV Polish Conference on Computer Methods in Mechanics,* Rzeszów, Poland, 53–54, 1999.

[12] P. Chaperon, R. Jones, M. Heller, S. Pitt, F. Rose. Methodology for structural optimisation with damage tolerance constraints. *Engineering Failure Analysis.* **7**: 281–300, 2000.

[13] G. Cheng, B. Fu. Shape optimisation of continuum with crack. In: G.I.N Rozvany, B.L. Karihaloo, eds., *Structural Optimisation,* 57–62, Kluwer Academic Publishers, 1988.

[14] C.S. Davis. Automated design of stiffened panels against crack growth and fracture among other design constraints. In: *Flaw Growth and Fracture,* ASTM STP 631, 416–445. Amer. Soc. Testing Mater., 1977.

[15] C.S. Davis. Line search methods for extended penalty function environments. *Int. J. Num. Meth. Engng.*, **15**: 867–888, 1978.

[16] M.W. Dobbs, R.B. Nelson. Minimum weight design of stiffened panels with fracture constraints. *Computers and Structures*, **8**: 753–759, 1978.

[17] B.J.D. Esping, D. Holm. Structural shape optimization using OASIS, In: G.I.N. Rozvany, B. Karihaloo, eds., *Structural Optimisation*, 299–306, Kluwer Academic Publishers, 1988.

[18] E.A. Fancello, E.O. Taroco, R.A. Feijoo. Shape sensitivity analysis in fracture mechanics. In: J. Herskovits ed., *Proc. of Structural Optimization 93*, Universidade Federale Rio de Janeiro, Vol. **2**, 239–248, 1993.

[19] B. Farahmand. *Fracture Mechanics of Metals, Composites, Welds, and Bolted Joints.* Kluwer Academic Publishers, Boston, 2001.

[20] A. Francavilla, C.V. Ramakrishnan, O.C. Zienkiewicz. Optimization of shape to minimize stress concentration, *J. Strain Anal.*, **10**: 63–69, 1975.

[21] L. Gani, S.D. Rajan. Use of fracture mechanics and shape optimization for component design, *AIAA J.*, **37**: 255–260, 1999.

[22] G.V. Guinea, J. Planas, M. Elices. K_I evaluation by the displacement extrapolation technique. *Engng. Fracture Mechanics.* **66**: 243–255, 2000.

[23] Z. Gürdal, R.T. Haftka. Design of stiffened composite panels with a fracture constraint. *Computers and Structures*, **20**: 457–465, 1985.

[24] G.N. Karpow, N.W. Kurnosow, W.Z. Parton. Application of potential method to 2D elastic domain with irregular boundary (in Russian). *Strength of Materials*, **7**: 35, 1982.

[25] D.J. Keum, B.M. Kwak. Calculation of stress intensity factors by sensitivity analysis with respect to change of boundary conditions. *Computers and Structures*, **32**: 63–69, 1992.

[26] Z. Knésl. A fracture mechanics approach to the optimum design of cracked structures under cyclic loading. In: A. Carpinteri, ed., *Handbook of Fatigue Propagation in Metallic Structures*, Vol. I, 551–577. Elsevier, Amsterdam, 1994.

[27] G.S. Kruse. An automated procedure for preliminary design of primary structure for transport aircraft structures. In: *ASME 76-WA/Aero-9, American Society of Mechanical Engineers*, 1976.

[28] L. Lamberti, C. Pappalettere. Comparison of the numerical efficiency of different sequential linear programming based algorithms for structural optimization problems. *Computers and Structures*, **76**: 713–728, 2000.

[29] E. Lund. Shape optimization using Weibull statistics of brittle failure. *Structural Optimization*, **15**: 208–214, 1998.

[30] C. Mattheck, D. Erb, K. Bethge, U. Begemann. Three-dimensional shape optimization of a bar with a rectangular hole. *Fatigue Fract. Engng. Mater. Struct.*, **15**: 347–351, 1992.

[31] S.A. Meguid. *Engineering Fracture Mechanics*, Elsevier, London, 330–332, 1989.

[32] K. L. Molski. *A Unitary Weight Function Application in the Dimensioning of Structures using Fracture Mechanics Methods* (in Polish). Scientific Reports. Warsaw University of Technology, Mechanics, Monograph, Vol. 185, 2000.

[33] P.N. Osiv, V.V. Panasyuk, M.P. Savruk. Stress state in a plate weakened by two circular holes and crack (in Russian). *Solid Mechanics (MTT)*, **3**: 130–134, 1983.

[34] V.V. Panasyuk. Stress concentration around two ring holes connected by a narrow slot, (in Russian). *Strength of Materials*, **9**: 17–20, 1983.

[35] M.P. Savruk. *Stress Intensity Factors in Cracked Bodies* (in Russian). Vol. 2 of Vols. 1–4, Panasyuk V.V., ed., *Fracture Mechanics and Strength of Materials.* Naukova Dumka, Kyiv, 1988–1990.

[36] P. Pedersen. Optimal shape design of a hole in front of a crack. In: *Extended Abstracts of the Forth IV World Congress of Structural and Multidisciplinary Optimization (WCSMO-4)*, June 4–8, 2001, Dalian, China, 320–321.

[37] W.D. Pilkey. *Peterson's Stress Concentration Factors*, 2-nd edition. John Wiley and Sons, New York, 1997.

[38] A.V. Pitukhin. Teeth gears optimum design with application of fracture mechanics methods (in Russian). *Vestnik mashinostroenija*, **7**: 15–16, 1989.

[39] A.V. Pitukhin. Fracture mechanics and optimal design. *Int. J. Numer. Meth. Engng.*, **34**: 933–940, 1992.

[40] A.V. Pitukhin. Optimal design problems using fracture mechanics methods. *Computers and Structures*, **65**: 621–624, 1997.

[41] A. Portela, M.H. Aliabadi. On the accuracy of boundary and finite element technique for crack problems in fracture mechanics. In: C.A. Brebbia, J.J. Connor, eds., *Advances in Boundary Elements*, Vol. 1, *Computations and Fundamentals.* CMP, Springer-Verlag, Berlin, 123–137, 1989.

[42] A. Portela, M.H. Aliabadi. The dual boundary element method: effective implementation for crack problems. *Int. J. Numer. Meth. Engng.*, **33**: 1269–1297, 1992.

[43] A. Portela, M.H. Aliabadi, D.P. Rooke. Dual boundary element analysis of cracked plates: singularity subtraction technique. *Int. J. Fracture*, **55**: 17–28, 1992.

[44] K. Rykaluk. *Cracks in Steel Structures* (in Polish). Monograph, Dolnośląskie Wydawnictwo Edukacyjne, Wrocław, 2000.

[45] J. Samuelson, D. Holm, B. Esping. Optimization of hydraulic cylinder. *Int. J. Fatigue.* **12**: 493–504, 1990.

[46] V.V. Saurin. Shape design sensitivity analysis for fracture conditions. *Computers and Structures*, **76**: 399–405, 2000.

[47] M. Serra. Optimum beam design based on fracture crack propagation. *Structural and Multi-disciplinary Optimisation*, **19**: 159–163, 2000.

[48] C.S. Shin, C.M. Wang, P.S. Song. Fatigue damage repair: a comparison of some possible methods. *Int. J. Fatigue*, **18**: 535–546, 1996.

[49] S.B. Thomas, M.J. Mhaiskar, R. Sethuraman. Stress intensity factors for circular hole and inclusion using finite element alternating method. *Theoretical and Applied Fracture Mechanics*, **33**: 73–81, 2000.

[50] P.M. Toor. A review of some damage tolerance design approaches for aircraft structures. *Engng. Fracture Mechanics*, **5**: 837–880, 1973.

[51] J. Trevelyan, P.A. Adey, A. Elzein. The use of BEASY to provide a simplified analysis technique in fracture mechanics. In: M.H. Aliabadi. C.A. Brebbia, D.J. Cartwright, eds.,. *Localized Damage Computer Aided Assessment and Control*, Vol. 3, *Advanced Computational Methods*, 307–322. CMP, Southampton, 1990.

[52] G. Tsamasphyros, P. Vouthounis, E.N. Theotokoglu. Reinforced strip weakened by cracks and holes. In: M.H. Aliabadi., D.J. Cartwright, eds., *Localized Damage Computer Aided Assessment and Control*, Vol. 3. *Advanced Computational Methods*, 307–322. CMP, Southampton, 1990.

[53] V. Tvergaard. On the optimum shape of a fillet in a flat bar with restrictions, In: A. Sawczuk, Z. Mróz, eds., *Optimization in Structural Design*, 182–195, IUTAM Symp., Warsaw, 1973. Springer-Verlag, Berlin, 1975.

[54] R. Vitali, R.T. Haftka, B.V. Sankar. Multifidelity design of stiffened composite panel with a crack. In: *Proc. 3rd World Congress of Structural and Multidisciplinary Optimization*, May 17–21, 1999, Buffalo, New York, CD-ROM 51-AAM2-1, pp. 6.

[55] J. Vrbka, Z. Knésl. Optimized design of a high pressure compound vessel by FEM. *Computers and Structures*, **24**: 809–812, 1986.

[56] N. Vulić, S. Jecić, V. Grubišić. Validation of crack arrest technique by numerical modelling. *Int. J. Fatigue*, **19**: 283–291, 1997.

[57] B. Wilczyński. Multi-disciplinary shape optimization of notches in 2-D machine and structural components. *CAMES*, **3**, 245–262, 1996.

[58] B. Wilczyński. Shape optimization for stress reduction around single and interacting notches based on fictitious stress method. *Engng. Analysis with Boundary Elements*, **19**: 117–128, 1997.

[59] B. Wilczyński. Analytical shape sensitivity by direct differentiation using indirect BEM, In: C.A. Brebbia, ed., *Boundary Elements XIX*, 389–398, Computational Mechanics Publications, Southampton, 1997.

[60] B. Wilczyński. Shape optimisation to minimize stress concentration in shell structures. In: *Identification, Control and Optimisation of Engineering Structures*, 119–132. Civil-Comp Press, Edinburgh, 2000.

[61] B. Wilczyński, Z. Mróz. Optimal design for minimization of stress intensity factors with account for stiffness constraints. In: *Extended Abstracts of the Forth World Congress of Structural and Multidisciplinary Optimization*, June 4–8, 2001, Dalian, China, 89–90.

[62] Z.J. Yang, J.F. Chen, G.D. Holt. Efficient evaluation of stress intensity factors using virtual crack extension technique. *Computers and Structures*. **79**: 2705–2715, 2001.