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Semi-analytical method for the dynamic analysis[1]

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The paper presents a semi-analytical method for the study of a linear differential system with variable coefficients. The solution is given in terms of real positive integer powers; it is obtained in terms of independent functions which are computed numerically. The paper extended the semi-analytical method from [5] (for one differential equation only), to the study of a linear differential system. The differential system became a system with recurrent expressions between the coefficients of the power series in a matrix form. The strength of this method is shown by application to the dynamic analysis of typical rotor blades. The frequencies and mode shapes are calculated. The results are compared with theoretical results for the degenerate cases and with results obtained through other methods.

NOMENCLATURE

E	Young's modulus,
G	tangential elasticity modulus,
GJ	torsion rigidity,
I_2, I_3	geometrical inertial moments with respect to the principal axes passing through the elastic centre,
$\overline{[EI]}$	non-dimensional matrix of the rigidity due to bending,
k_{2E}, k_{3E}	gyration radius of the cross section with respect to the principal axes passing through the elastic centre,
$\overline{[K]}$	non-dimensional matrix of gyration radius,
m	mass of the unit length of the beam,
l	beam length,
$\bar{M}_x$, $\bar{M}_y$, $\bar{M}_z$	non-dimensional components of the moment in cross section with respect to the rotating coordinate system,
r_s, r_m	geometrical and mass asymmetry section coefficients,
r_g	gyration radius dimensionless by l,
$\bar{S}_y$, $\bar{S}_z$	non-dimensional shear stress resultants,
T	centrifugal stress,

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x_G , $\bar{x}_G$	the distance between GC and EC of the cross section, dimensionless by l,
x_0^* , $\bar{x}_0^*$	x_0 cross section coordinate where pre-twisted angle is θ^*,
$[z(\xi)]$	state vector used in the transfer matrix method,
ξ	non-dimensional coordinate,
θ	pre-twist angle of the beam,
θ^*	θ angle at x_0^* position,
ω	natural frequencies, (rad/sec),
$()_{,x} = \dfrac{\partial()}{\partial x_0}$, $()' = \dfrac{\partial()}{\partial \xi}$.	

1. Introduction

The paper presents a semi-analytical method for the study of a linear differential system with variable coefficients (polynomial coefficients). The solution is given in terms of real positive integer powers; it is obtained in terms of independent functions, which are computed numerically.

V. Giurgiuţiu presented a similarly semi-analytical method, but for one differential equation only in [5]. The present paper extended the semi-analytical method from [5], to the study of a linear differential system with variable coefficients. The differential system became a system with recurrent expressions between the coefficients of the power series in a matrix form. The mutual influence between the unknown functions (terms of coupling) can be evaluated. The frequencies and mode shapes can be calculated.

The strength of this method is shown by application to the dynamic analysis of typical rotor blades. The frequencies and mode shapes are calculated. To the dynamic analysis of partial coupled movements, of helicopter blades, the semi-analytical method has been applied in [1–3]. The results are compared with theoretical results for the degenerate cases and with results obtained trough others methods: Transmission Matrix Method (TMM), Integrating Matrix Methods (IMM) and Structural Influence Functions Method (SIFM) respectively

2. Description of the semi-analytical method

A system in a matrix form with three differential equations with polynomial coefficients is considered, where $\xi \in [\xi_0, \xi_1]$ is the variable,

$$[M]_{(4)} \begin{bmatrix} x \\ y \\ z \end{bmatrix}^{(4)} + [M]_{(3)} \begin{bmatrix} x \\ y \\ z \end{bmatrix}^{(3)} + [M]_{(2)} \begin{bmatrix} x \\ y \\ z \end{bmatrix}^{(2)} + [M]_{(1)} \begin{bmatrix} x \\ y \\ z \end{bmatrix}^{(1)} + [M]_{(0)} \begin{bmatrix} x \\ y \\ z \end{bmatrix}^{(0)} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} . \quad (1)$$

The boundary conditions are

$$x(\xi_0) = x_0 , \qquad y(\xi_0) = y_0 , \qquad z(\xi_0) = z_0 ,$$
$$\frac{\mathrm{d}}{\mathrm{d}\xi}[x(\xi_0)] = v_{0x} , \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}[y(\xi_0)] = v_{0y} , \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}[z(\xi_0)] = v_{0z} , \quad (2a)$$

$$x(\xi_1) = x_1 , \qquad y(\xi_1) = y_1 , \qquad z(\xi_1) = z_1 ,$$
$$\frac{\mathrm{d}}{\mathrm{d}\xi}[x(\xi_1)] = v_{1x} , \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}[y(\xi_1)] = v_{1y} , \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}[z(\xi_1)] = v_{1z} , \quad (2b)$$

$[M]_{(j)}$ is a 3×3 matrix with polynomial coefficients (fourth degree polynomials are considered), $\begin{bmatrix} x \\ y \\ z \end{bmatrix}^{(j)}$; $j = 0, 1, 2, 3, 4$ is the ξ-derivative of j order of the $x(\xi)$, $y(\xi)$, $z(\xi)$ functions, $\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$ is the absolute term of differential system (1) (fourth degree polynomials are considered).

In the differential system (1), the solution $x(\xi)$, $y(\xi)$, $z(\xi)$ will be approximated by power series,

$$[\ x \quad y \quad z\ ] = \sum_{k=0}^{\infty} [\ A_k \quad B_k \quad C_k\ ]\, \xi^k\,, \tag{3}$$

resulting in

$$[\ x \quad y \quad z\ ]^{(j)} = \sum_{k=0}^{\infty} [\ A_{k+j} \quad B_{k+j} \quad C_{k+j}\ ] \frac{(k+j)!}{k!} \xi^k\,, \qquad j = 1, 2, 3, 4. \tag{4}$$

After calculations the differential system (1) will be transformed through (3) and (4) in a system with recurrent expressions between the coefficients of the power series. These expressions written in a matrix form

$$\sum_{k=0}^{\infty} \xi^k \left\{ \sum_{i=4}^{-3} \left[([N]_i) \begin{bmatrix} A_{k+i} \\ B_{k+i} \\ C_{k+i} \end{bmatrix} \right] + \begin{bmatrix} a_k \\ b_k \\ e_k \end{bmatrix} \right\} = 0\,. \tag{5}$$

Since the relation (5) is valid $(\forall)\ \xi \in [\xi_0\,,\, \xi_1]$, the following recurrent matrix expression results,

$$\begin{bmatrix} A_{k+4} \\ B_{k+4} \\ C_{k+4} \end{bmatrix} = -[N]_4^{-1} \begin{bmatrix} a_k \\ b_k \\ e_k \end{bmatrix} - \sum_{i=3}^{-3} \left\{ [([N]_4^{-1})\,[N]_i] \begin{bmatrix} A_{k+i} \\ B_{k+i} \\ C_{k+i} \end{bmatrix} \right\}, \qquad (\forall)\ k \geq 0, \tag{6}$$

where $[\ A_{k+i} \quad B_{k+i} \quad C_{k+i}\ ] = 0$ if $k + i < 0$.

The analysis of the recurrent expressions shows that there are only 12 independent coefficients: $A_0\,, A_1\,, A_2\,, A_3\,, B_0\,, B_1\,, B_2\,, B_3\,, C_0\,, C_1\,, C_2\,, C_3\,.$

The mutual influence between the unknown functions $x(\xi)$, $y(\xi)$, $z(\xi)$ can be evaluated as follows:

- 36 independent functions are defined

$$\begin{aligned}
X_n(\xi) &= \xi^{n-1} + \sum_{k=4}^{\infty} A_k^n \xi^k\,, && n \in \{1, 2, 3, 4\}, \\
X_n(\xi) &= \sum_{k=4}^{\infty} A_k^n \xi^k\,, && n \in \{5, 6, 7, 8, 9, 10, 11, 12\}, \\
Y_n(\xi) &= \xi^{n-5} + \sum_{k=4}^{\infty} B_k^n \xi^k\,, && n \in \{5, 6, 7, 8\}, \\
Y_n(\xi) &= \sum_{k=4}^{\infty} B_k^n \xi^k\,, && n \in \{1, 2, 3, 4, 9, 10, 11, 12\}, \\
Z_n(\xi) &= \xi^{n-5} + \sum_{k=4}^{\infty} C_k^n \xi^k\,, && n \in \{9, 10, 11, 12\}, \\
Z_n(\xi) &= \sum_{k=4}^{\infty} C_k^n \xi^k\,, && n \in \{1, 2, 3, 4, 5, 6, 7, 8\}.
\end{aligned} \tag{7}$$

- The coefficients A_k^n, B_k^n, C_k^n are determined by using the recurrent expression (6) for the following initial sets of values,

$$\begin{array}{llllll}
(S1) & A_0^1=1, & A_k^1=0, & k\in\{1,2,3\}, & B_k^1=C_k^1=0, & k\in\{0,1,2,3\},\\
(S2) & A_1^2=1, & A_k^2=0, & k\in\{0,2,3\}, & B_k^2=C_k^2=0, & k\in\{0,1,2,3\},\\
(S3) & A_2^3=1, & A_k^3=0, & k\in\{0,1,3\}, & B_k^3=C_k^3=0, & k\in\{0,1,2,3\},\\
(S4) & A_3^4=1, & A_k^4=0, & k\in\{0,1,2\}, & B_k^4=C_k^4=0, & k\in\{0,1,2,3\},\\
(S5) & B_0^5=1, & B_k^5=0, & k\in\{1,2,3\}, & A_k^5=C_k^5=0, & k\in\{0,1,2,3\},\\
(S6) & B_1^6=1, & B_k^6=0, & k\in\{0,2,3\}, & A_k^6=C_k^6=0, & k\in\{0,1,2,3\},\\
(S7) & B_2^7=1, & B_k^7=0, & k\in\{0,1,3\}, & A_k^7=C_k^7=0, & k\in\{0,1,2,3\},\\
(S8) & B_3^8=1, & B_k^8=0, & k\in\{0,1,2\}, & A_k^8=C_k^8=0, & k\in\{0,1,2,3\},\\
(S9) & C_0^9=1, & C_k^9=0, & k\in\{1,2,3\}, & A_k^9=B_k^9=0, & k\in\{0,1,2,3\},\\
(S10) & C_1^{10}=1, & C_k^{10}=0, & k\in\{0,2,3\}, & A_k^{10}=B_k^{10}=0, & k\in\{0,1,2,3\},\\
(S11) & C_2^{11}=1, & C_k^{11}=0, & k\in\{0,1,3\}, & A_k^{11}=B_k^{11}=0, & k\in\{0,1,2,3\},\\
(S12) & C_3^{12}=1, & C_k^{12}=0, & k\in\{0,1,2\}, & A_k^{12}=B_k^{12}=0, & k\in\{0,1,2,3\}.
\end{array} \tag{8}$$

- The general solution of the system (1) can be described as

$$\begin{bmatrix} x(\xi) \\ y(\xi) \\ z(\xi) \end{bmatrix} = \begin{bmatrix} X_1(\xi) & X_2(\xi) & X_3(\xi) & \dots & \dots & X_{12}(\xi) \\ Y_1(\xi) & Y_2(\xi) & Y_3(\xi) & \dots & \dots & Y_{12}(\xi) \\ Z_1(\xi) & Z_2(\xi) & Z_3(\xi) & \dots & \dots & Z_{12}(\xi) \end{bmatrix} \left(\begin{bmatrix} A_0 & A_1 & A_2 & A_3 & B_0 & B_1 & B_2 & B_3 & C_0 & C_1 & C_2 & C_3 \end{bmatrix}^T \right) \tag{9}$$

3. Dynamic analysis of rotating blades

This work develops the problem of coupled free linear vibrations for the blade of a lifting propeller modelled as a rotating beam. In accordance with the general theory from [4, 6, 9, 10, 12, 13], the differential equations for the coupled vibrations were determined. A semi-analytical method is used for the study of coupled linear vibrations. The displacements in a cross section of beam are approximated with power series. Until now power series have been used only to the study the uncoupled linear vibrations.

3.1. Hypothesis: The linear equation of motion

The following assumptions are made in order to develop the model of a long blade of a lifting propeller.

The blade is modelled as a straight beam, contained in the plane of rotation; one of its ends is embedded off the axis of rotation (e_0); the centre of gravity (GC) and the centre of elongation (TC) of any cross section of beam are generally different. The beam can be simultaneously bent in two planes and can have a rotation around the elastic axis (EA).

The elastic axis of the blade (EA), is assumed initially to be a straight line; the angle between EA and the plane of rotation is β. By construction, the blade is pre-twisted by with angle θ around EA. The pre-twisting angle varies linearly along the blade. We consider the Taylor series expansion around θ^*, truncated to the first four terms for the trigonometric functions *sin* and *cos*.

Hooke laws and Euler–Bernoulli hypotheses regarding the rotation of the cross section are considered for the case of small displacements. Also, moderate rotations are taken into consideration.

The reference system considered (see Fig. 1) has the following features: it is a rotating system; Ω-constant angular velocity; the origin is the embedded end of the blade, located at the distance e_0

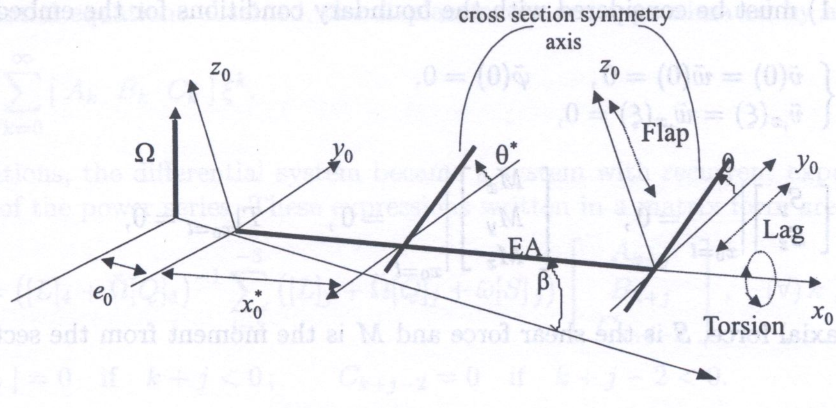

Fig. 1. Geometry of the undeformed blade

from the rotation axis (Ox_0 – the undeformed elastic axis); a reference section located at the distance x_0^* from the origin is pre-twisted with the angle θ^*.

The displacements of the elastic axis (v, w) are measured on the Oy_0 respective on the Oz_0 directions; φ is the twisting displacement around the elastic axis.

The displacements have harmonic variations in time

$$[\,v\ \ w\ \ \varphi\,] = [\,\tilde{v}\ \ \tilde{w}\ \ \tilde{\varphi}\,]\cdot e^{i\cdot\omega t}, \qquad i=\sqrt{-1}, \tag{10}$$

considering $\tilde{v}$, $\tilde{w}$, $\tilde{\varphi}$ as functions of only x_0 coordinate.

The differential equations for the coupled free linear vibrations, having in mind all the above considerations, are

$$\begin{aligned}
&\left\{[EI]\begin{bmatrix}\tilde{v}\\ \tilde{w}\end{bmatrix}_{,xx} - x_T T\tilde{\varphi}\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\right\}_{,xx}\\
&\quad + \Omega^2\left\{m\,\tilde{\varphi}\left(x_G(x_0+e)\begin{bmatrix}\sin\theta\\ -\cos\theta\end{bmatrix} + \beta\begin{bmatrix}k_{22E}-k_{33E}\\ 2k_{23E}\end{bmatrix}\right)\right\}_{,x} - \left\{T\begin{bmatrix}\tilde{v}\\ \tilde{w}\end{bmatrix}_{,x}\right\}_{,x} +\\
&\quad + \Omega^2 x_G\left\{m\beta\,\tilde{w}_{,x}(x_0+e)\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} + m\beta\,\tilde{w}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\right\}_{,x} + \left\{m[K]\begin{bmatrix}\omega^2\tilde{v}\\ (\omega^2+\Omega^2)\tilde{w}\end{bmatrix}_{,x}\right\}_{,x}\\
&\quad - m\begin{bmatrix}(\omega^2+\Omega^2)\tilde{v}\\ \omega^2\tilde{w}\end{bmatrix} + m x_G\begin{bmatrix}(\omega^2+\Omega^2)\sin\theta\\ \omega^2\cos\theta\end{bmatrix}\tilde{\varphi} = 0,
\end{aligned} \tag{11a}$$

$$\begin{aligned}
&\left\{(GJ + r_{ig}^2 T)\tilde{\varphi}_{,x}\right\}_{,x} + m\left\{\omega^2(k_{22E}+k_{33E}) - \Omega^2[k_{22E} - k_{33E} - \beta x_G(x_0+e)\sin\theta]\right\}\tilde{\varphi}\\
&\quad + T x_T(\tilde{w}_{,xx}\cos\theta - \tilde{v}_{,xx}\sin\theta) - m\Omega^2 x_G(x_0+e)(\tilde{w}_{,x}\cos\theta - \tilde{v}_{,x}\sin\theta)\\
&\quad + m\Omega^2\beta\left[(k_{22E}-k_{33E})\tilde{v}_{,x} + 2k_{23E}\tilde{w}_{,x}\right] - m x_G\left[(\Omega^2+\omega^2)\tilde{v}\sin\theta - \omega^2\tilde{w}\cos\theta\right] = 0,
\end{aligned} \tag{11b}$$

where

$$[EI] = \begin{bmatrix}EI_{22} & EI_{23}\\ EI_{23} & EI_{33}\end{bmatrix} = \frac{1}{2}(EI_2+EI_3)\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{EI_2-EI_3}{EI_2+EI_3}\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}\right\}, \tag{12a}$$

$$[K] = \begin{bmatrix}k_{22E} & k_{23E}\\ k_{23E} & k_{33E}\end{bmatrix} = \frac{1}{2}(k_{2E}+k_{3E})\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{k_{2E}-k_{3E}}{k_{2E}+k_{3E}}\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}\right\}. \tag{12b}$$

Equations (11) must be considered with the boundary conditions for the embedded blade,

$$\underline{x_0 = 0}: \quad \begin{cases} \tilde{v}(0) = \tilde{w}(0) = 0\,, & \tilde{\varphi}(0) = 0\,, \\ \tilde{v}_{,x}(\xi) = \tilde{w}_{,x}(\xi) = 0, & \end{cases}$$

$$\underline{x_0 = l}: \quad \begin{bmatrix} S_y \\ S_z \end{bmatrix}\Bigg|_{x_0=l} = 0\,, \quad \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}\Bigg|_{x_0=l} = 0\,, \qquad T|_{x_0=l} = 0, \tag{13}$$

where T is the axial force, S is the shear force and M is the moment from the section.

3.2. Semi-analytical method for the dynamic analysis of the blade

The differential equations (11) are processed step by step as presented below:

- the change of variable $x_0 \to \xi$ leads to the following expression for the law of the beam pre-twisting $\theta(\xi)$,

$$\xi = \frac{x_0}{l} - \frac{x_0^*}{l}\,, \quad \xi \in \left[-\frac{x_0^*}{l}\,, 1 - \frac{x_0^*}{l}\right], \qquad \theta(x_0) = \theta(\xi) = \theta^* + h\xi\,; \tag{14}$$

- we consider the Taylor series expansion around θ^* for the trigonometric functions *sin* and *cos*; the series are truncated to the first four terms;

- the differential equations (11) written in a non-dimensional form using specific coefficients:

  – rotation speed: $\bar{\Omega} = \dfrac{m\,\Omega^2 l^4}{E(I_2 + I_3)}\,,$

  – frequency: $\bar{\omega} = \dfrac{m\,\omega^2 l^4}{E(I_2 + I_3)}\,,$

  – axial force: $f = \dfrac{T}{m\,\Omega^2 l^2} = a\xi^2 + b\xi + c\,,$

  – inertia: $\mu_2 = \dfrac{k_{2E}}{l^2}\,, \quad \mu_3 = \dfrac{k_{3E}}{l^2}\,, \quad \mu = \mu_2 + \mu_3\,,$

  – rigidity ratio: $\Lambda = \dfrac{GJ}{E(I_2 + I_3)}\,,$

  – asymmetry coefficients for cross section (mass and geometry):

$$r_s = \frac{I_2 - I_3}{I_2 + I_3}\,, \qquad r_m = \frac{\mu_2 - \mu_3}{\mu_2 + \mu_3}\,,$$

  – coefficients for geometrical dimensions:

$$\bar{x}_G = \frac{x_G}{l}\,, \quad \bar{x}_0^* = \frac{x_0^*}{l}\,, \quad e_1 = \frac{e_0}{l} + \bar{x}_0^*\,, \quad \dddot{v} = \frac{\tilde{v}}{l}\,, \quad \dddot{w} = \frac{\tilde{w}}{l}\,.$$

In the differential equations obtained, the displacements are approximated by a power series

$$[\,\bar{v}\ \ \bar{w}\ \ \bar{\varphi}\,] = \sum_{k=0}^{\infty} [\,A_k\ \ B_k\ \ C_k\,]\,\xi^k. \tag{15}$$

After calculations, the differential system became a system with recurrent expressions between the coefficients of the power series. These expressions written in a matrix form are:

$$\begin{bmatrix} A_{k+4} \\ B_{k+4} \\ C_{k+2} \end{bmatrix} = \left([L]_4 + \bar{\Omega}[Q]_4\right)^{-1} \sum_{l=3}^{-3} \left([L]_j + \bar{\Omega}[Q]_j + \bar{\omega}[S]_j\right) \begin{bmatrix} A_{k+j} \\ B_{k+j} \\ C_{k+j-2} \end{bmatrix}, \quad (\forall)\, k \ge 0\,;$$
$$[\,A_{k+j}\ \ B_{k+j}\,] = 0 \quad \text{if} \quad k+j < 0\,; \qquad C_{k+j-2} = 0 \quad \text{if} \quad k+j-2 < 0. \tag{16}$$

The matrices $[L]_j$, $[S]_j$, $[Q]_j$, $j \in \{-3,-2,-1,0,1,2,3,4\}$, were obtained after analytical calculations with power series coefficients.

The analysis of the recurrent expressions concluded that there are only 10 independent coefficients: A_0 , A_1 , A_2 , A_3 , B_0 , B_1 , B_2 , B_3 , C_0 , C_1 .

Thirty independent functions are defined:

$$\begin{aligned}
V_n(\xi) &= \xi^{n-1} + \sum_{k=4}^{\infty} A_k^n \xi^k\,, && n \in \{1,2,3,4\},\\
V_n(\xi) &= \sum_{k=4}^{\infty} A_k^n \xi^k\,, && n \in \{5,6,7,8,9,10\},\\
W_n(\xi) &= \xi^{n-5} + \sum_{k=4}^{\infty} B_k^n \xi^k\,, && n \in \{5,6,7,8\},\\
W_n(\xi) &= \sum_{k=4}^{\infty} B_k^n \xi^k\,, && n \in \{1,2,3,4,9,10\},\\
\Phi_n(\xi) &= \xi^{n-5} + \sum_{k=2}^{\infty} C_k^n \xi^k\,, && n \in \{9,10\},\\
\Phi_n(\xi) &= \sum_{k=2}^{\infty} C_k^n \xi^k\,, && n \in \{1,2,3,4,5,6,7,8\}.
\end{aligned} \tag{17}$$

The coefficients A_k^n , B_k^n , C_k^n will be determined by using the recurrent expressions starting with 10 initial sets of values, selected for these independent coefficients (see relation (8)).

The general solution of the system is

$$\begin{aligned}
\begin{bmatrix} \bar{v}(\xi) \\ \bar{w}(\xi) \\ \bar{\varphi}(\xi) \end{bmatrix} &= \begin{bmatrix} V_1(\xi) & V_2(\xi) & V_3(\xi) & \dots & \dots & V_{10}(\xi) \\ W_1(\xi) & W_2(\xi) & W_3(\xi) & \dots & \dots & W_{10}(\xi) \\ \Phi_1(\xi) & \Phi_2(\xi) & \Phi_3(\xi) & \dots & \dots & \Phi_{10}(\xi) \end{bmatrix} \cdot [D]\,,\\
[D]^T &= [\,A_0\ \ A_1\ \ A_2\ \ A_3\ \ B_0\ \ B_1\ \ B_2\ \ B_3\ \ C_0\ \ C_1\,]\,.
\end{aligned} \tag{18}$$

The ξ derivative of j order, $j \in \{1,2,3,4\}$ of $\bar{v}(\xi)$, $\bar{w}(\xi)$, $\bar{\varphi}(\xi)$ functions can be calculated.

The boundary conditions for the embedded blade are:

$$\underline{\xi = \xi_0 = -\bar{x}_0^*}: \qquad \begin{cases} \bar{v}(\xi) = \bar{w}(\xi) = 0\,, & \bar{\varphi}(\xi) = 0\,, \\ \bar{v}'(\xi) = \bar{w}'(\xi) = 0\,, \end{cases}$$

$$\underline{\xi = \xi_1 = 1 - \bar{x}_0^*}: \qquad \left.\begin{bmatrix} \bar{S}_y \\ \bar{S}_z \end{bmatrix}\right|_{\xi=\xi_1} = 0\,, \qquad \left.\begin{bmatrix} \bar{M}_x \\ \bar{M}_y \\ \bar{M}_z \end{bmatrix}\right|_{\xi=\xi_1} = 0\,, \qquad f\big|_{\xi=\xi_1} = 0\,. \tag{19}$$

The transfer matrix method will be used to determine the frequencies and mode shapes. The state vector is

$$[z](\xi) = \left[\ \bar{v}(\xi)\ \ \bar{w}(\xi)\ \ \bar{\varphi}(\xi)\ \ \bar{v}'(\xi)\ \ \bar{w}'(\xi)\ \ \bar{S}_y(\xi)\ \ \bar{S}_z(\xi)\ \ \bar{M}_z(\xi)\ \ -\bar{M}_y(\xi)\ \ \bar{M}_x(\xi)\ \right]^T . \tag{20}$$

The relation between ξ_0 and ξ sections is

$$[z](\xi) = [F(\xi)] \cdot [D] = [F(\xi)] \cdot [F(\xi_0)]^{-1} \cdot [z(\xi_0)] = [G(\xi, \xi_0)] \cdot [z(\xi_0)] . \tag{21}$$

The $[F(\xi)]$ matrix is computed by substituting the relations (18) into the expressions of the state vector elements. The elements of the $[F(\xi)]$ matrix are:

$$\begin{aligned}
&F_{1n}(\xi) = V_n(\xi)\,, \qquad F_{2n}(\xi) = W_n(\xi)\,, \qquad F_{3n}(\xi) = \Phi_n(\xi)\,,\\
&F_{4n}(\xi) = V_n'(\xi)\,, \qquad F_{5n}(\xi) = W_n'(\xi)\,,\\
&\begin{bmatrix} F_{6n}(\xi) \\ F_{7n}(\xi) \end{bmatrix} = -\overline{[EI]}\begin{bmatrix} V_n(\xi) \\ W_n(\xi) \end{bmatrix}''' - \overline{[EI]}'\begin{bmatrix} V_n(\xi) \\ W_n(\xi) \end{bmatrix}'' - \bar{\omega}\overline{[K]}\begin{bmatrix} V_n(\xi) \\ W_n(\xi) \end{bmatrix}',\\
&\begin{bmatrix} F_{8n}(\xi) \\ F_{9n}(\xi) \end{bmatrix} = \overline{[EI]}\begin{bmatrix} V_n(\xi) \\ W_n(\xi) \end{bmatrix}'',\\
&F_{10n}(\xi) = \Lambda\Phi_n'(\xi)\,.
\end{aligned} \tag{22}$$

The transfer matrix 10×10 $[G(\xi, \xi_0)]$ can be numerically computed for a fixed value ξ.

The partition matrix of the $[G(\xi, \xi_0)]$ matrix, which contains the elements between i and j rows and k and l columns respectively, is denoted by

$$\left[G\begin{pmatrix} i\,,\,k & \\ & j\,,\,l \end{pmatrix}(\xi, \xi_0)\right].$$

The following relations are obtained for a cantilever blade:

- the frequency equation is

$$\det\left(\left[G\begin{pmatrix} 6\,,\,6 & \\ & 10\,,\,10 \end{pmatrix}(\xi_1, \xi_0)\right]\right) = 0; \tag{23}$$

- the mode shapes relation is

$$\begin{aligned}
\begin{bmatrix} \bar{v}(\xi) \\ \bar{w}(\xi) \\ \bar{\varphi}(\xi) \end{bmatrix} = &\left[G\begin{pmatrix} 1\,,\,6 & \\ & 3\,,\,6 \end{pmatrix}(\xi, \xi_0)\right]\\
&- \left[G\begin{pmatrix} 1\,,\,7 & \\ & 3\,,\,10 \end{pmatrix}(\xi, \xi_0)\right] \cdot \left[G\begin{pmatrix} 7\,,\,7 & \\ & 10\,,\,10 \end{pmatrix}(\xi_1\,, \xi_0)\right]^{-1}\\
&\cdot \left[G\begin{pmatrix} 7\,,\,6 & \\ & 10\,,\,6 \end{pmatrix}(\xi_1\,, \xi_0)\right] .
\end{aligned} \tag{24}$$

The relations (23) and (24) may be particularized for uncoupled and partially coupled cases of calculus.

The present method can be extended for study of non-uniform blade (structural or inertial discontinuous of blade).

3.3. Numerical applications

The matrix form of this method makes possible a simple computational implementation.

For the circular cantilever beam in Fig. 2 the relative error for the first six natural bending frequencies, is presented with regard to theoretical results, reported by $k_{\max}$ the maximum degree of the polynomial resulted from the truncated power series. For $k_{\max} > 50$ the error is very small.

For the circular cantilever beam in Fig. 3, the influence of the rotation speed on the natural bending frequencies and a comparison with results from [14] are presented.

The present semi-analytical method is applied to the hypothetical rotor blade, which was studied in [7, 8, 11] with the Transmission Matrix Method (TMM), Integrating Matrix Methods (IMM) and Structural Influence Functions Method (SIFM) respectively. The characteristics of this blade are the following, according to [7, 8, 11]:

length	$l = 40$ in $= 1.016$ m
mass/length	$m = 0.0015$ slugs/m $= 0.86185$ kg/m
section	$x_G = 0.4$ in $= 0.01016$ m, $x_T = 0$ m $=$ const,
characteristics:	$k_{2E} = 0.71$ in$^2 = 4.5806 \cdot 10^{-4}$ m^2, $k_{3E} = 0.18$ in$^2 = 1.1613 \cdot 10^{-4}$ m^2,
	$EI_2 = EI_3 = 25000$ lb·in$^2 = 71.745$ Nm^2, $GJ = 9000$ lb·in$^2 = 25.8282$ Nm^2.

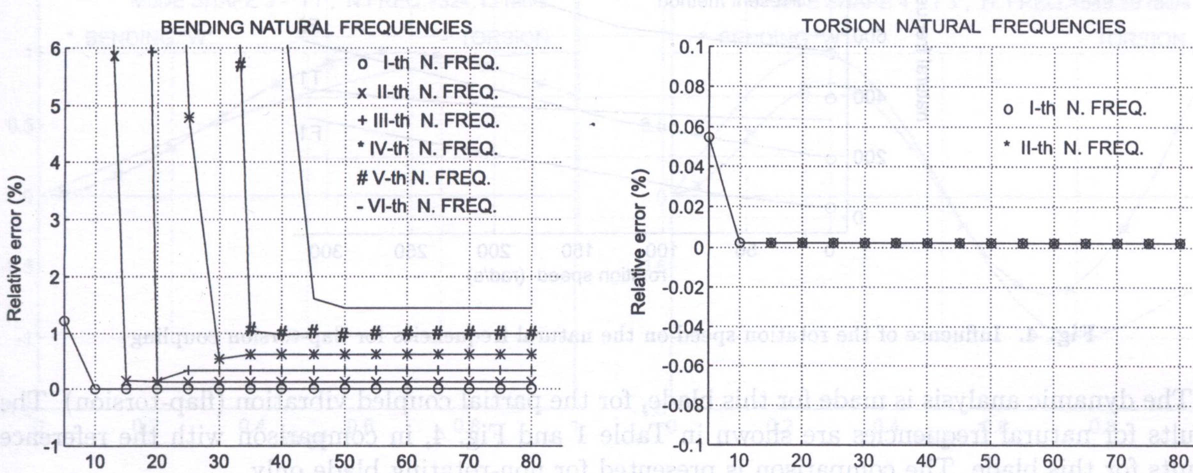

Fig. 2. Relative error of natural frequencies reported by $k_{\max}$

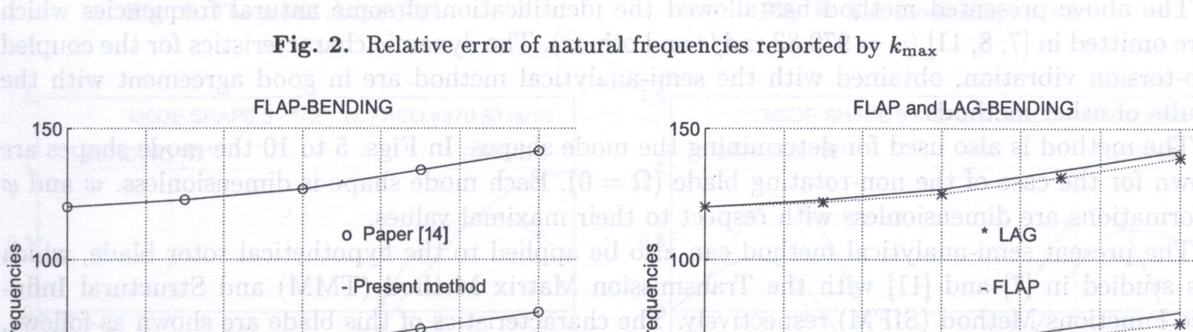

Fig. 3. Influence of the rotation speed on the natural frequencies for circular cantilever beam

Table 1. Comparison of results for natural frequencies for flap-torsion coupled movements

Mode shapes	Natural frequencies (rad/s)			
	Reference [7] M.M.T.	Reference [8] M.M.I.	Reference [11] F.I.G.	Present paper (S.M.)
1-F1	31.05	31.05	31.06	31.089
2-F2	193.74	193.74	193.79	195.53
3-T1	390.87	390.87	390.91	324.13
4-F3	539.54	539.54	539.64	549.59
5-T2	–	–	–	970.83
6-F4	1041.72	1043.94	1043.87	1086.20

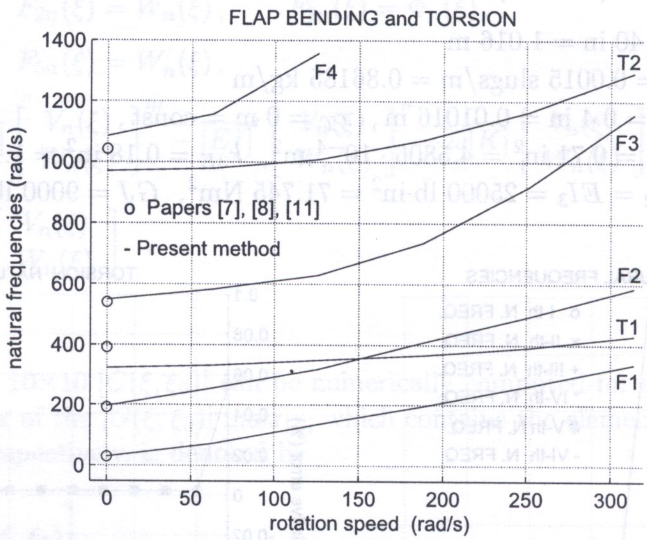

Fig. 4. Influence of the rotation speed on the natural frequencies for flap-torsion coupling

The dynamic analysis is made for this blade, for the partial coupled vibration (flap-torsion). The results for natural frequencies are shown in Table 1 and Fig. 4, in comparison with the reference results for this blade. The comparison is presented for non-rotating blade only.

The above presented method has allowed the identification of some natural frequencies which were omitted in [7, 8, 11] ($\omega = 970.83$ rad/s and others). The dynamic characteristics for the coupled flap-torsion vibration, obtained with the semi-analytical method are in good agreement with the results of other methods.

The method is also used for determining the mode shapes. In Figs. 5 to 10 the mode shapes are shown for the case of the non-rotating blade ($\Omega = 0$). Each mode shape is dimensionless. w and φ deformations are dimensionless with respect to their maximal values.

The present semi-analytical method can also be applied to the hypothetical rotor blade, which was studied in [7] and [11] with the Transmission Matrix Method (TMM) and Structural Influence Functions Method (SIFM) respectively. The characteristics of this blade are shown as follows, according to [7].

length	$l = 40$ in $= 1.016$ m
mass/length	$m = 0.0015$ slugs/m $= 0.86185$ kg/m
pre-twist	$\theta = 45°$
section	$x_G = \sqrt{2}$ in $= 0.035921$ m , $x_T = 0$ m $=$ const ,
characteristics:	$EI_2 = 75000$ lb·in^2 $= 215.235$ Nm^2, $EI_3 = 25000$ lb·in^2 $= 71.745$ Nm^2,
	$GJ = 9000$ lb·in^2 $= 25.8282$ Nm^2.

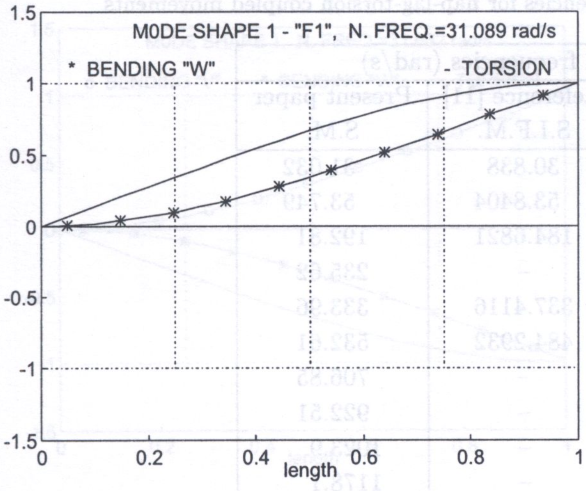

Fig. 5. The mode shape 1-F1

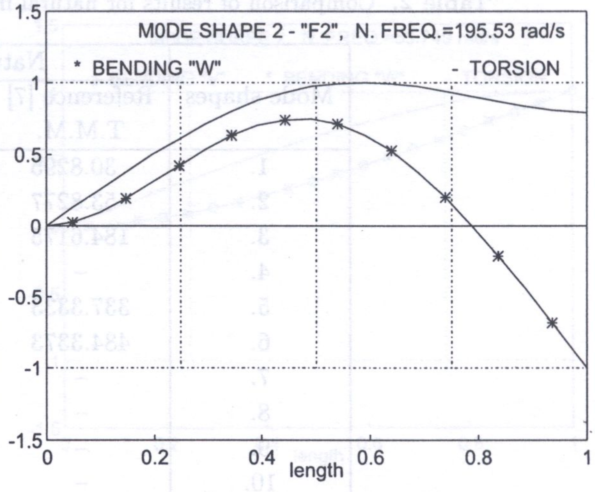

Fig. 6. The mode shape 2-F2

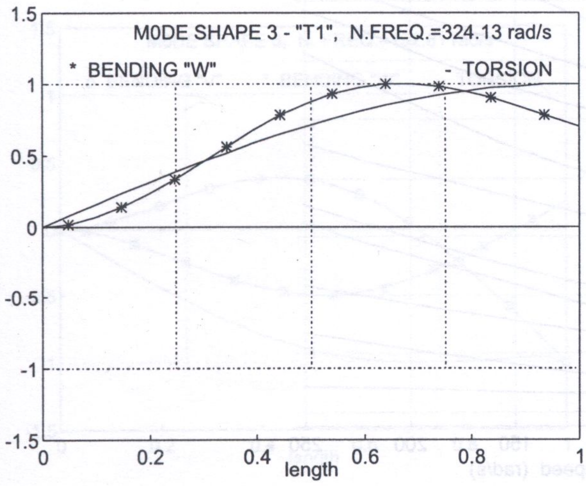

Fig. 7. The mode shape 3-T1

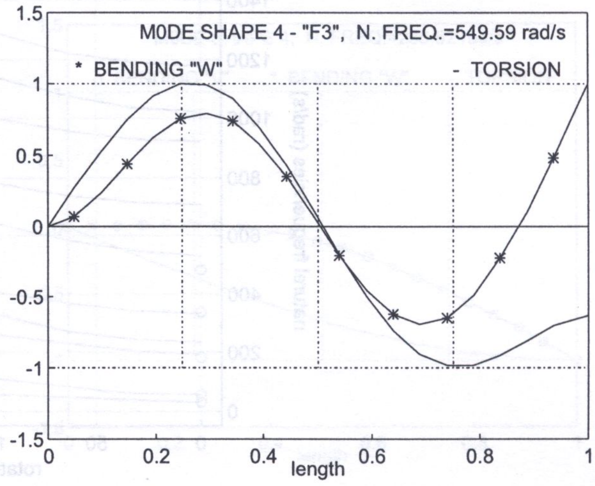

Fig. 8. The mode shape 4-F3

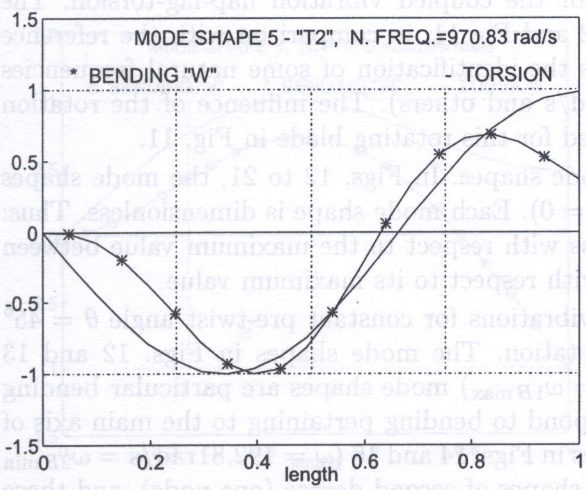

Fig. 9. The mode shape 5-T2

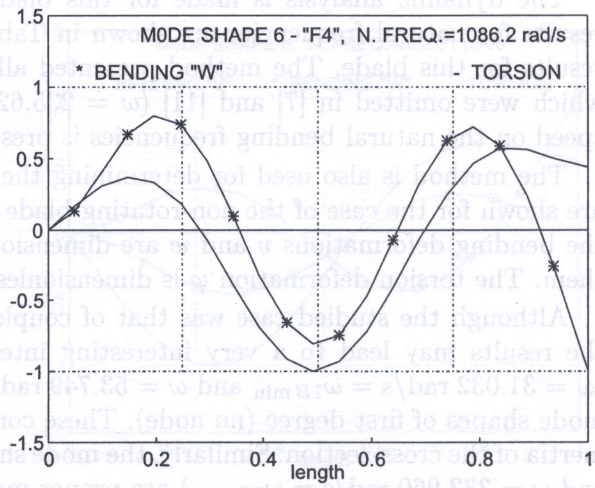

Fig. 10. The mode shape 6-F4

Table 2. Comparison of results for natural frequencies for flap-lag-torsion coupled movements

Mode shapes	Natural frequencies (rad/s)		
	Reference [7] T.M.M.	Reference [11] S.I.F.M.	Present paper S.M.
1.	30.8295	30.838	31.032
2.	53.8277	53.8404	53.749
3.	184.6175	184.6821	192.81
4.	–	–	235.62
5.	337.3333	337.4116	333.96
6.	484.3373	484.2932	532.61
7.	–	–	706.85
8.	–	–	922.51
9.	–	–	1023.9
10.	–	–	1178.1

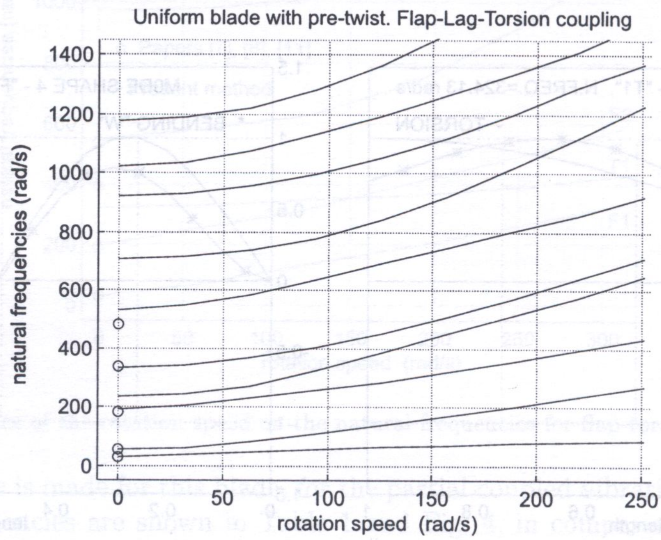

Fig. 11. Influence of the rotation speed on the natural frequencies

The dynamic analysis is made for this blade, for the coupled vibration flap-lag-torsion. The results for natural frequencies are shown in Table 2 and Fig 11, in comparison with the reference results for this blade. The method presented allows the identification of some natural frequencies which were omitted in [7] and [11] ($\omega = 235.62$ rad/s and others). The influence of the rotation speed on the natural bending frequencies is presented for this rotating blade in Fig. 11.

The method is also used for determining the mode shapes. In Figs. 12 to 21, the mode shapes are shown for the case of the non-rotating blade ($\Omega = 0$). Each mode shape is dimensionless. Thus: the bending deformations v and w are dimensionless with respect to the maximum value between them. The torsion deformation φ is dimensionless with respect to its maximum value.

Although the studied case was that of coupled vibrations for constant pre-twist angle $\theta = 45°$ the results may lead to a very interesting interpretation. The mode shapes in Figs. 12 and 13 ($\omega = 31.032$ rad/s $= \omega_{1B\,\min}$ and $\omega = 53.749$ rad/s $= \omega_{1B\,\max}$) mode shapes are particular bending mode shapes of first degree (no node). These correspond to bending pertaining to the main axis of inertia of the cross-section. Similarly, the mode shapes in Figs. 14 and 16 ($\omega = 192.81$rad/s $= \omega_{2B\,\min}$ and $\omega = 333.960$ rad/s $= \omega_{2B\,\max}$) are proper mode shapes of second degree (one node), and those in Figs. 17 and 19 ($\omega = 532.61$ rad/s $= \; = \omega_{3B\,\min}$ and $\omega = 922.51$ rad/s $= \omega_{3B\,\max}$) are proper modes of third degree (two nodes).

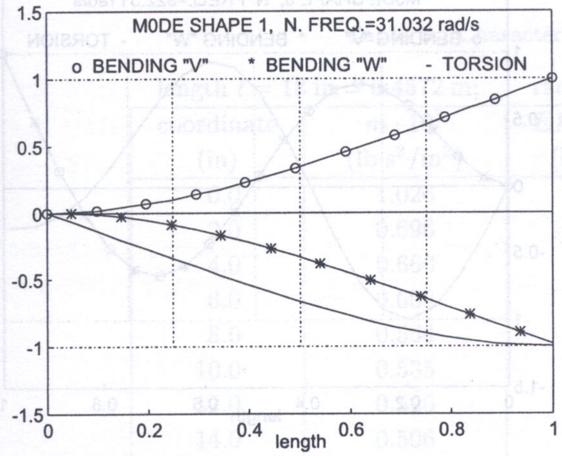

Fig. 12. The mode shape 1

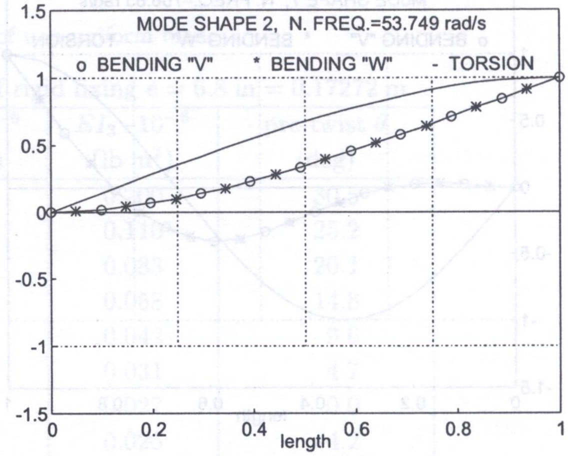

Fig. 13. The mode shape 2

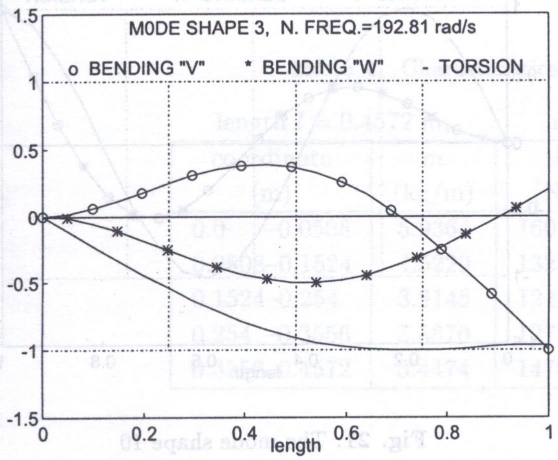

Fig. 14. The mode shape 3

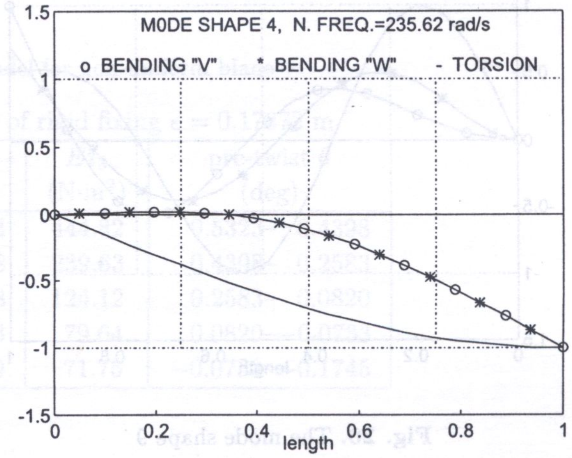

Fig. 15. The mode shape 4

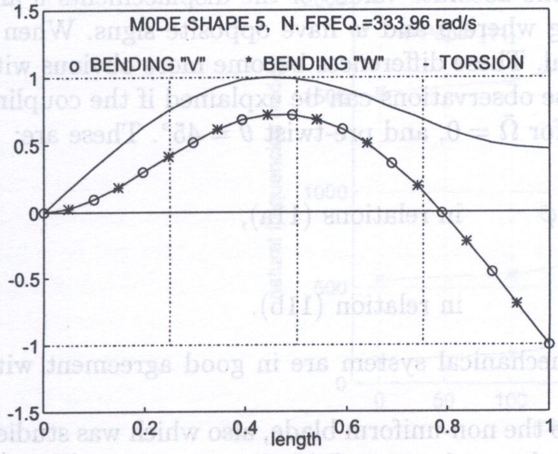

Fig. 16. The mode shape 5

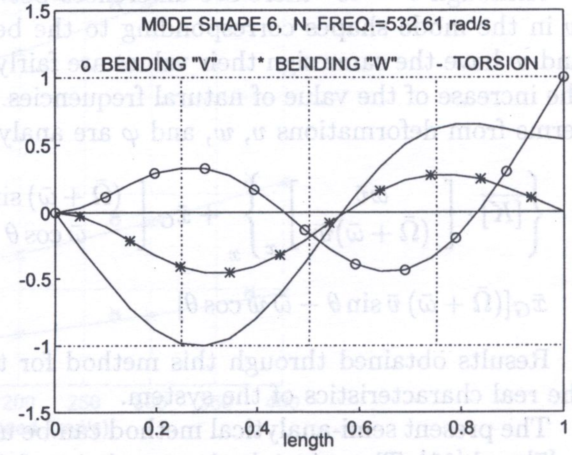

Fig. 17. The mode shape 6

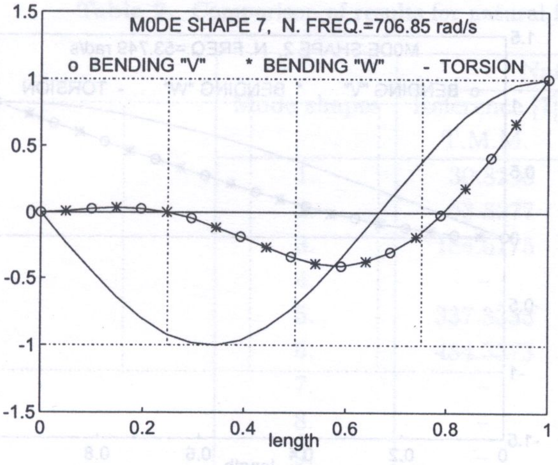

Fig. 18. The mode shape 7

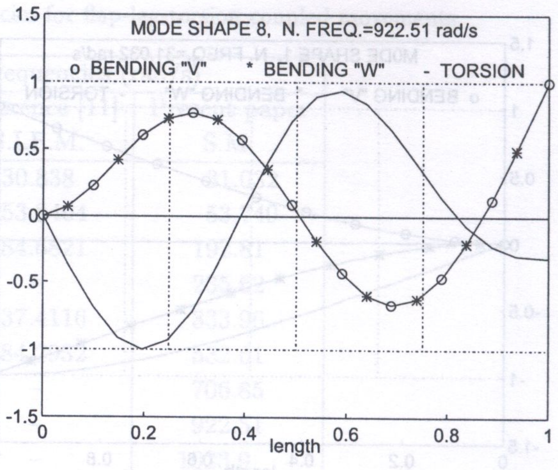

Fig. 19. The mode shape 8

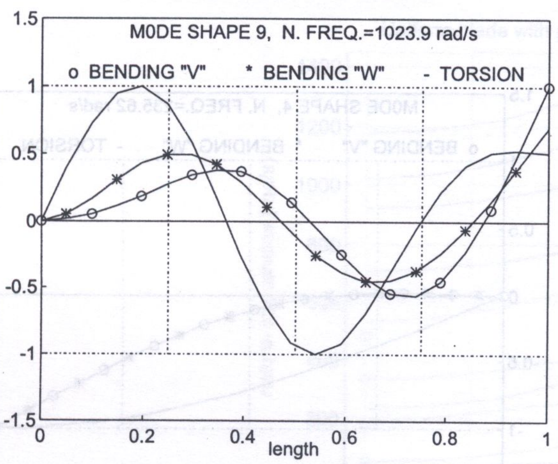

Fig. 20. The mode shape 9

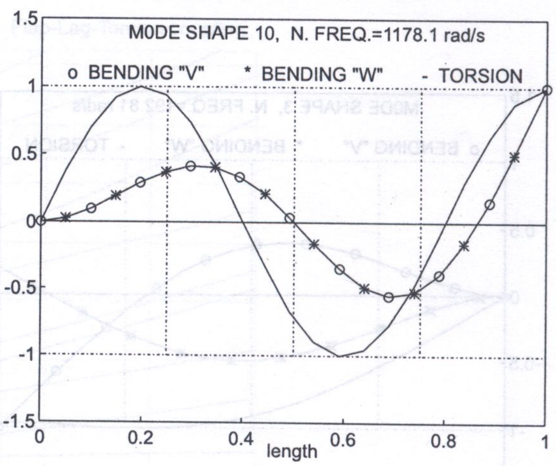

Fig. 21. The mode shape 10

The mode shapes in Figs. 12, 13, 14, 16, 17, 19 are "mostly" bending mode shapes. The mode shapes in Fig. 15 ($\omega = 235.62\,\mathrm{rad/s}$), Fig. 18 ($\omega = 706.85\,\mathrm{rad/s}$) and Fig. 21 ($\omega = 1178.1\,\mathrm{rad/s}$) are "mostly" torsion mode shapes.

Although $\theta = 45°$ there are differences between the absolute values of the displacements v and w in the mode shapes corresponding to the bending where v and w have opposite signs. When v and w have the same sign their values are fairly equal. These differences become more obvious with the increase of the value of natural frequencies. These observations can be explained if the coupling terms from deformations v, w, and φ are analysed, for $\bar{\Omega} = 0$, and pre-twist $\theta = 45°$. These are:

$$\left\{ [K] \cdot \begin{bmatrix} \bar{\omega}\bar{v} \\ (\bar{\Omega}+\bar{\omega})\bar{w} \end{bmatrix}_{,x} \right\}_{x} + \bar{x}_G \begin{bmatrix} (\bar{\Omega}+\bar{\omega})\sin\theta \\ \bar{\omega}\cos\theta \end{bmatrix} \bar{\varphi} \qquad \text{in relations (11a),}$$

$$\bar{x}_G[(\bar{\Omega}+\bar{\omega})\,\bar{v}\sin\theta - \bar{\omega}\,\bar{w}\cos\theta] \qquad \text{in relation (11b).}$$

Results obtained through this method for this mechanical system are in good agreement with the real characteristics of the system.

The present semi-analytical method can be used to the non-uniform blade, also which was studied in [7] and [11]. The principle characteristics of this blade are shown in Table 3, according to [7]. This blade was modelled as a blade made of five uniform segments. The characteristics of this model are shown in Table 4.

Table 3. Characteristics of non-uniform blade

length $l = 18$ in $= 0.4572$ m; radius of rigid fixing $e = 6.8$ in $= 0.17272$ m

coordinate (in)	$m \cdot 10^3$ (lb·s^2/in^2)	$EI_2 \cdot 10^{-6}$ (lb·in^2)	$EI_3 \cdot 10^{-6}$ (lb·in^2)	pre-twist θ (deg)
0.0	1.026	63.0	0.200	30.5
2.0	0.696	49.0	0.110	25.2
4.0	0.660	46.0	0.083	20.1
6.0	0.608	44.0	0.058	14.8
8.0	0.564	43.0	0.042	9.6
10.0	0.535	43.0	0.031	4.7
12.0	0.520	44.0	0.027	0.0
14.0	0.506	47.0	0.026	−4.2
16.0	0.498	51.0	0.025	−7.5
18.0	0.498	56.0	0.024	−10.0

Table 4. Characteristics of model for non-uniform blade

length $l = 0.4572$ m; radius of rigid fixing $e = 0.17272$ m

coordinate (m)	m (kg/m)	EI_2 (N·m^2)	EI_3 (N·m^2)	pre-twist θ (deg)
0.0 –0.0508	5.9364	160708.84	444.82	0.5323– 0.4398
0.0508–0.1524	4.5229	132728.28	239.63	0.4398– 0.2583
0.1524–0.254	3.9145	124118.88	124.12	0.2583– 0.0820
0.254 –0.3556	3.5870	127706.13	79.64	0.0820–−0.0733
0.3556–0.4572	3.4474	147077.29	71.75	−0.0733–−0.1745

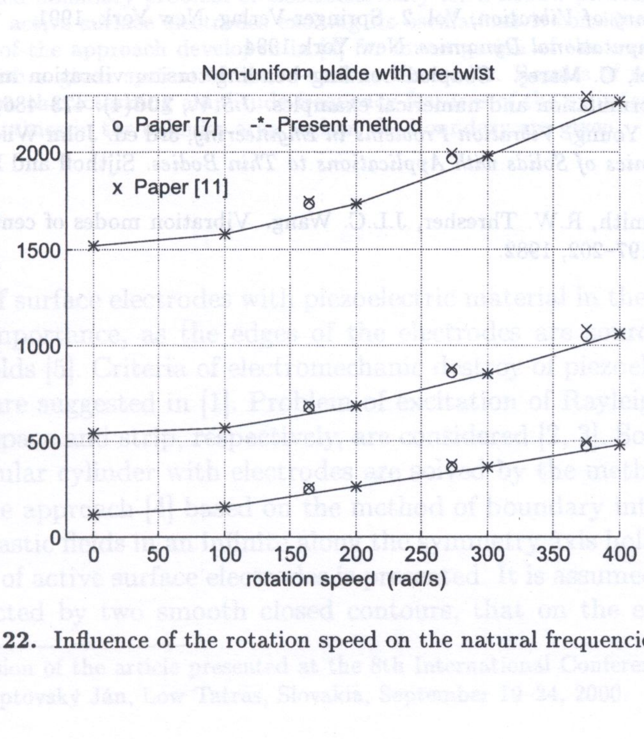

Fig. 22. Influence of the rotation speed on the natural frequencies

The results for natural frequencies are shown in Fig. 22, in comparison with the reference results for this blade [7, 11]. Results obtained for this mechanical system are in good agreement with the references.

4. Conclusions

All results obtained and your comparison with the reference results concluded:

The exact representation of centrifugal forces and their influence on the phenomena are the main advantage of this semi-analytical method. The possibility to accurately calculate higher derivatives for complex blades is the second advantage.

Another advantage of the method is that it points out the influences of all functional and constructive parameters of the blade. The convergence of the method was analysed with respect to the maximum degree of the polynomial resulted from the truncated power series.

The dynamic characteristics for the coupled flap-lag-torsion vibration, obtained with this method are in good agreement with the results of other methods. The method may be extended for discontinuous blades, or blades with concentrated inertial loads.

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