Computer Assisted Mechanics and Engineering Sciences, 9: 447–458, 2002.

On thickness optimization of an unilaterally supported anisotropic plate subjected to buckling[1]

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(Received Februay 9, 2001)

We shall be dealing with the eigenvalue optimization problem for an anisotropic plate. The plate is partly unilaterally supported on its boundary and subjected to longitudinal forces causing its buckling. The state problem has then the form of an eigenvalue variational inequality expressing the deflection of the plate and the maximal possible value of the acting forces keeping its stability which corresponds to the first eigenvalue. The demand of the maximal first eigenvalue with respect to variable thicknesses of the plate means to solve the optimal design problem with eigenvalue variational inequality as the state problem. The existence of a solution in the framework of the general theory will be examined. The necessary optimality conditions will be derived. The convergence of the finite elements approximation will be verified.

1. Introduction

Optimal design of elastic structures involves also the problems connected with eigenfrequencies or eigenvalues of constructions. The geometrical and mechanical characteristics play the role of control variables. The maximization of the first eigenvalue can be expressed in the case of positively definite operators as a max-min problem. The existence of an optimal parameter, the continuity and differentiability properties for the linear eigenvalue problems in an operator form has been first time dealt in [6]. These problems are closely related to the optimal design problems involving stability constraints which have been considered in the monographies [3, 4]. The complete both numerical and theoretical study of the optimization problem for columns against buckling has been performed in [2]. We have dealt with the minimal eigenfrequency of the anisotropic plate with respect to the thickness in [1]. The state problem was reformulated as the eigenvalue problem for the elliptic equation of the 4-th order. We have derived the existence theorem for the optimal thickness. The convergence of the finite elements scheme was verified.

We shall deal here with the plate unilaterally supported on the part of its boundary. If the plate is acting under buckling the state problem is an eigenvalue variational inequality. The state problems of that type were studied by several authors. Problems arose mainly in the study of the bifurcation in variational inequalities depending on the real parameter. The existence of eigenvalues, its characterization and comparing with the corresponding linear problems can be found in [11–14, 16]. The detailed study of the obstacle problems connected with eigenvalues and the bifurcation can be found in [9]. The previous authors investigated mainly the theoretical questions connected

[1] This is an extended version of the article presented at the 8th International Conference on Numerical Methods in Continuum Mechanics, Liptovský Ján, Low Tatras, Slovakia, September 19–24, 2000.

with the unilateral eigenvalue problem. We shall combine it with the question of maximization of the first eigenvalue which can be expressed as the max-min problem.

We formulate and solve the state variational inequality in Section 2. Section 3 will be devoted to a formulation and solving the optimization problem with the convex set of admissible thicknesses. Necessary optimality conditions will be derived in Section 4 using the differentiability properties of the functions defined as the maximum over the compact set in a similar way as in [15] for the case of an eigenvalue equation. The finite elements solving will be investigated in Section 5.

2. Eigenvalue variational inequalities

Let us assume a thin anisotropic plate of a variable thickness $e(x)$, $x = (x_1, x_2) \in \Omega$, where Ω is a bounded region identified with the middle surface of the plate. The third order tensor of material coefficients a_{ijkl}, $i, j, k, l \in \{1, 2\}$ is symmetric and positively definite i.e.

$$\begin{aligned} &a_{ijkl} = a_{jikl} = a_{klij}, \\ &a_{ijkl}\,\tau_{ij}\tau_{kl} \geq \alpha\,\tau_{ij}\tau_{ij} \quad (\alpha > 0) \qquad \text{for all } \tau = (\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}) \in R^4_{\text{sym}}. \end{aligned} \tag{1}$$

The summation convention through the indices $\{i, j, k, l\}$ is considered.

We assume that the Lipschitz boundary $\partial\Omega = \Gamma$ of the middle surface is divided into three parts: $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$, $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $\text{meas}(\Gamma_1) > 0$, $\text{meas}(\Gamma_3) > 0$. The plate is clamped on Γ_1, simple supported on Γ_2 and unilaterally supported by zero on Γ_3. Further we assume that $e \in U = C(\bar\Omega)$ – the Banach space of all continuous functions on $\bar\Omega$ with a norm

$$\|e\|_U = \max_{x \in \bar\Omega} |e(x)|.$$

If the plate is acting under longitudinal forces (buckling) with a proportion coefficients $\lambda(e) > 0$, then its deflection $w(e) : \Omega \to R$ is a solution of the eigenvalue variational inequality

$$\begin{aligned} &\lambda(e) \in R, \quad w(e) \in K, \quad v \neq 0: \\ &\int_\Omega e^3(x) a_{ijkl} \frac{\partial^2 w(e)}{\partial x_i \partial x_j} \frac{\partial^2 (v - w(e)}{\partial x_k \partial x_l}\,\mathrm{d}x \geq \lambda \int_\Omega \nabla w \cdot \nabla (v - w)(x)\,\mathrm{d}x \qquad \text{for all } v \in K, \end{aligned} \tag{2}$$

where

$$\begin{aligned} K &= \{v \in V: \ v \geq 0 \ \text{ on } \Gamma_3\}, \\ V &= \{v \in H^2(\Omega): \ v = \frac{\partial v}{\partial n} = 0 \ \text{ on } \Gamma_1, \ v = 0 \ \text{ on } \Gamma_2\}. \end{aligned}$$

$H^2(\Omega)$ is the Sobolev space of functions with all generalized derivatives up to second order belonging to the space $L_2(\Omega)$. Further we introduce the space

$$H = \{v :\in H^1(\Omega): \ v = 0 \ \text{ on } \Gamma_1 \cup \Gamma_2\}.$$

H and V are Hilbert spaces with the scalar products

$$\begin{aligned} (u, v)_1 &= \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x, \qquad u, v \in H, \\ (u, v)_2 &= \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}\,\mathrm{d}x, \qquad u, v \in V, \end{aligned}$$

and the norms

$$\|u\|_i = (u, u)_i^{1/2}, \quad i = 1, 2.$$

The Dirichlet boundary conditions on the part Γ_1 of the boundary imply the equivalence of the norms in H, V with the usual norms in Sobolev spaces $H^1(\Omega)$, $H^2(\Omega)$.

The set of admissible deflections K is a closed convex cone with a vertex in 0 in the Hilbert space V.

We proceed with the operator formulation of the eigenvalue inequality (2). Let us denote by V^* the dual Banach space of all linear bounded functionals $L : V \to R$ with a norm $\|L\|_*$ and $\langle L, v\rangle = L(v)$ the duality pairing between V and V^*. We introduce the operators $A(e) : V \to V^*, B : V \to V^*$ by the integral identities

$$\langle A(e)u, v\rangle = \int_\Omega e^3(x) a_{ijkl} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_k \partial x_l} \,\mathrm{d}x, \qquad u, v \in V,$$

$$\langle B(e)u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \qquad u, v \in V.$$

The problem (2) can then be expressed in a form

$$\begin{aligned} &\lambda(e) \in R, \quad w(e) \in K \quad w(e) \neq 0 \\ &\langle A(e)w(e)\,, v - w(e)\rangle \geq \lambda(e)\langle (Bw(e)\,, v - w(e))\rangle \qquad \text{for all } v \in K. \end{aligned} \tag{3}$$

In the case of the set $K = V$ we have the eigenvalue equality problem

$$\lambda(e) \in R, \quad w(e) \in V, \quad w(e) \neq 0 : \qquad A(e)w(e) = \lambda(e)Bw(e),$$

or in the classical formulation

$$\frac{\partial^2}{\partial x_i \partial x_j}\left[e^3(x)\, a_{ijkl}\, \frac{\partial^2 w}{\partial x_k \partial x_l}\right] + \lambda(e)\Delta w = 0 \quad \text{in } \Omega,$$

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma_1\,, \qquad w = M(w) = 0 \quad \text{on } \Gamma_2\,, \qquad M(w) = T(w) = 0 \quad \text{on } \Gamma_3\,,$$

where $M(w)$ and $T(w)$ are the bending moment and the effective shear force of the plate respectively.

The operators $A(e)$, $B : V \to V^*$ are linear continuous symmetric and positively definite, B is moreover compact. It is well known from the spectral theory of the linear operators that there exists the nondecreasing sequence of positive eigenvalues $\{\lambda_n(e)\}$ and the corresponding eigenelements $\{w_n(e)\}$ such that

$$\lim_{n\to\infty} \lambda_n(e) = \infty$$

and the eigenelements $\{w_n(e)\}$ form the total orthogonal system in the Hilbert space V. The first (the least) eigenvalue solves the minimization problem

$$\lambda_1(e) = \min_{v\in V,\, v\neq \bar{0}} \frac{\langle A(e)v\,, v\rangle}{\langle Bv\,, v\rangle} = \frac{\langle A(e)w(e)\,, w(e)\rangle}{\langle Bw(e)\,, w(e)\rangle}. \tag{4}$$

In the case of a closed convex cone K we formulate the minimization problem

$$\lambda(e) = \min_{v\in K,\, v\neq \bar{0}} \frac{\langle A(e)v\,, v\rangle}{\langle Bv\,, v\rangle} = \frac{\langle A(e)w(e)\,, w(e)\rangle}{\langle Bw(e)\,, w(e)\rangle}. \tag{5}$$

The following theorem due to [11, 12] plays the crucial role in the analyzing of eigenvalue variational inequalities

Theorem 1. *For every $e \in U$ there exists a solution $\{\lambda(e), w(e)\}$ of the problem (5).*

(i) The set of elements $\{w(e)\}$ minimizing the functional in (5) has the form $K(e) \setminus 0$, where $K(e) \subset K$ is the closed convex cone with a vertex at zero.

(ii) $\lambda(e)$ is the least positive number with a nontrivial solution $w(e)$ of a variational inequality

$$\begin{aligned} &\lambda(e) \in R, \quad w(e) \in K, \quad w(e) \neq \bar{0}: \\ &\langle A(e)w(e)\,,\, v - w(e)\rangle \geq \lambda(e)\langle Bw(e)\,,\, v - w(e)\rangle \qquad \text{for all } v \in K. \end{aligned} \tag{6}$$

(iii) Following relations are equivalent with (6)

$$\langle A(e)w(e)\,,\, v\rangle \geq \lambda(e)\langle Bw(e)\,,\, v\rangle \qquad \text{for all } v \in V, \tag{7}$$

$$\langle A(e)w(e)\,,\, w(e)\rangle = \lambda(e)\langle Bw(e),\, w(e)\rangle. \tag{8}$$

The characterization of the value $\lambda(e)^{-1}$ as the maximal value of the functional defined on the whole cone K is suitable from the approximation point of view.

Theorem 2.

$$\lambda(e)^{-1} = \max_{v \in V} L(e,v), \qquad L(e,v) = \sqrt{2}\sqrt{\langle Bv, v\rangle} - \frac{1}{2}\langle A(e)v, v\rangle. \tag{9}$$

The functional $v \to L(e,v)$ attains its maximum only on those eigenelements $w(e)$ belonging to $\lambda(e)$ for which $\sqrt{\langle Bw(e)\,,\, w(e)\rangle} = \sqrt{2}\lambda(e)^{-1}$.

Proof. Applying the minimum property (5) we obtain for every $v \in K$ the inequalities

$$\begin{aligned} L(e,v) &\leq \sqrt{2}\sqrt{\langle B(e)v\,,\, v\rangle} - \frac{1}{2}\lambda(e)\langle B(e)v\,,\, v\rangle \\ &= -\frac{1}{2}\lambda(e)\Big(\langle Bv\,,\, v\rangle - 2\sqrt{2}\sqrt{\langle Bv\,,\, v\rangle}\lambda(e)^{-1}\Big) \\ &= -\frac{1}{2}\lambda(e)\Big(\sqrt{\langle Bv\,,\, v\rangle} - \sqrt{2}\lambda(e)^{-1}\Big)^2 + \lambda(e)^{-1} \leq \lambda(e)^{-1}. \end{aligned}$$

We have the equality $L(e, w(e)) = \lambda(e)^{-1}$ for the eigenelements $w(e) \in K$ corresponding to $\lambda(e)$ and fulfilling the relation

$$\sqrt{\langle Bw(e)\,,\, w(e)\rangle} = \sqrt{2}\lambda(e)^{-1}.$$

3. OPTIMAL CONTROL PROBLEM

One of the basic control problems for eigenvalues is to determine the control parameters in such a way that the first eigenvalue is maximal what corresponds to the minimal possible first eigenfrequency of the construction. In the case of a state eigenvalue problem for a variational inequality we are looking for a maximal value of a force causing the buckling of a construction in contact with an obstacle.

We assume that the set of admissible thicknesses has the form

$$\begin{aligned} U_{ad} = \Big\{ &e \in C^{(0),1}(\bar{\Omega}) \mid 0 < e_{\min} \leq e(x) \leq e_{\max} \ \text{ for all } x \in \bar{\Omega}, \\ &\left|\frac{\partial e}{\partial x_i}\right| \leq C_i \quad \text{for all } x \in \Omega, \quad i = 1, 2, \qquad \int_\Omega e(x)\,\mathrm{d}x = C_3 \Big\}. \end{aligned}$$

We remember that $C^{(0),1}(\bar{\Omega})$ is the set of Lipschitz continuous functions on the set $\bar{\Omega}$. The admissible set U_{ad} is convex and compact in the Banach space $U = C(\bar{\Omega})$.

The operator $A(e) : V \to V^*$ fulfils the lower and upper estimates

$$\alpha_1 \|v\|_2^2 \le \langle A(e)v, v\rangle \le \alpha_2 \|v\|_2^2 \quad \text{for all } e \in U, \quad v \in V, \quad \alpha_1 > 0. \tag{10}$$

Moreover it is continuous with respect to the thicknesses $e \in U$ i.e.

$$e_n \to e \text{ in } U \implies A(e_n) \to A(e) \text{ in } \mathcal{L}(V, V^*), \tag{11}$$

where $\mathcal{L}(V, V^*)$ is the Banach space of all linear continuous operators mapping V to V^*. We formulate now

Optimal Control Problem 1. *To find $e^* \in U_{ad}$ such that*

$$\lambda(e^*) = \max_{e \in U_{ad}} \lambda(e) = \max_{e \in U_{ad}} \min_{v \in V,\, v \ne \bar{0}} \frac{\langle A(e)v, v\rangle}{\langle Bv, v\rangle}. \tag{12}$$

Using the method of compactness we obtain the following existence result.

Theorem 3. *There exists a solution $e^* \in U_{ad}$ of the Optimal Control Problem 1.*

Proof. Applying the estimates (10) and the relation (6) we obtain the upper estimate

$$\lambda(e) \le \frac{\langle A(e)v, v\rangle}{\langle Bv, v\rangle} \le \alpha_2 \frac{\|v\|_2^2}{\|v\|_1^2} \quad \text{for all } e \in U_{ad}, \quad v \in K \setminus \bar{0}. \tag{13}$$

Let $\{e_n\}$ be a maximizing sequence for $\lambda(e)$ on U_{ad}:

$$\lim_{n\to\infty} \lambda(e_n) = \sup_{e \in U_{ad}} \lambda(e). \tag{14}$$

There exists its convergent subsequence (again denoted by $\{e_n\}$) with a limit element

$$e^* \in U_{ad} : e_n \to e^* \text{ in } U. \tag{15}$$

Let us denote

$$w_n = w(e_n), \qquad u_n = \langle Bw_n, w_n\rangle^{-1/2} w_n, \tag{16}$$

a normalized solution of the state eigenvalue inequality corresponding to the thickness e_n, $n = 1, 2, \ldots$ We obtain from Eq. (8) the relation

$$\lambda(e_n) = \langle A(e_n)u_n, u_n\rangle. \tag{17}$$

The uniform coercivity of $\{A(e)\}$ due to Eq. (10) implies the boundedness of the sequence $\{u_n\}$ in V. Then there exists its subsequence (again denoted by $\{u_n\}$) and the element $u \in K \setminus \bar{0}$ such that

$$u_n \rightharpoonup u \text{ (weakly) in } V, \qquad u_n \to u \text{ (strongly) in } H. \tag{18}$$

The strong convergence in H is due to the compact imbedding $H \subset\subset V$. We have $u \ne \bar{0}$ as a consequence of

$$\langle Bu, u\rangle = \lim_{n\to\infty} \langle Bu_n, u_n\rangle = 1. \tag{19}$$

The functional

$$v \to \langle A(e^*)v, v\rangle$$

is weakly lower semicontinuous on V. The element u_n minimizes the functional

$$v \to \frac{\langle A(e_n)v, v\rangle}{\langle Bv, v\rangle}$$

over the set $K \setminus \bar{0}$, $n = 1, 2, \ldots$ The convergence (11), (15) and (18) then imply

$$\langle A(e^*)u, u\rangle \leq \lim_{n\to\infty} \langle A(e_n)u_n, u_n\rangle \leq \lim_{n\to\infty} \frac{\langle A(e_n)v, v\rangle}{\langle Bv, v\rangle} = \frac{\langle A(e)v, v\rangle}{\langle Bv, v\rangle} \quad \text{for all } v \in K \setminus \bar{0}. \tag{20}$$

Using Eqs. (19), (20) we have

$$\lambda(e^*) = \langle A(e^*)u, u\rangle. \tag{21}$$

It holds

$$\langle A(e_n)u_n, u_n\rangle \leq \frac{\langle A(e_n)u, u\rangle}{\langle Bu, u\rangle}.$$

Applying Eqs. (14), (15), (17), (21) we arrive at

$$\sup_{e\in U_{ad}} \lambda(e) \leq \lambda(e^*)$$

and hence

$$\lambda(e^*) = \max_{e\in U_{ad}} \lambda(e)$$

what completes the proof.

4. Necessary optimality conditions

In order to derive optimality conditions for the Optimal Control Problem 1 we formulate a theorem of Zolesio [17] (see also [15]) on differentiability of a functional defined as an infimum over a compact set.

Theorem 4. *Let U be a Banach space, W a compact topological space and $F(\cdot,\cdot) : U \times W \to R$ a mapping fulfilling the assumptions*

(i) $F(\cdot,\cdot)$ is lower semicontinuous on $U \times W$,

(ii) $F(\cdot, w)$ is continuous for every element $w \in W$,

(iii) $F(\cdot, w)$ is Gâteaux differentiable on U for every element $w \in W$, i.e. there exists the limit

$$\mathrm{d}F(e, w; \xi) = \lim_{t\to 0} \frac{F(e + t\xi, w) - F(e, w)}{t}$$

such that the mapping $\mathrm{d}F(e, w; \cdot) : U \to R$ is linear and continuous for every pair $(e, w) \in U \times W$;

(iv) convergence property:
For every sequence $\{e_n\} \subset U$, $\{w_n\} \subset W$, $\{\xi_n\} \subset U$ such that

$$F(e_n; w_n) = \inf_{w\in W} F(e_n; w),$$

$$e_n \to e \text{ in } U, \qquad w_n \to w \text{ in } W, \qquad \xi_n \to \xi \text{ in } U,$$

it follows

$$\mathrm{d}F(e, w; \xi) \leq \lim_{n\to\infty} \mathrm{d}F(e_n, w_n; \xi_n). \tag{22}$$

Then the functional $f : U \to R$ defined by

$$f(e) = \inf_{w \in W} F(e,w) \tag{23}$$

is Gâteaux differentiable and for every $\xi \in U$

$$\mathrm{d}f(e;\xi) = \inf\{\mathrm{d}F(e,w;\xi) \mid w \in W, \quad f(e) = F(e,w)\}. \tag{24}$$

In order to apply Theorem 4. to the max-min problem (12) we recall that the operator-function $A(\cdot) : U \to \mathcal{L}(V, V^*)$, is continuously Gâteaux differentiable i.e. its Gâteaux derivative fulfils the assumption

$$e_n \to e \text{ in } U \Longrightarrow \mathrm{d}A(e_n,\cdot) \to \mathrm{d}A(e,\cdot) \quad \text{in } \mathcal{L}(U, \mathcal{L}(V, V^*)). \tag{25}$$

The derivative has the form

$$\langle \mathrm{d}A(e;\xi)u, v\rangle = 3\int_\Omega e^2(x)\,\xi(x)\,a_{ijkl}\,\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_k \partial x_l}\,\mathrm{d}x, \qquad e,\xi \in U, \quad u,v \in V.$$

The following lemma expresses the eigenvalues $\lambda(e)$ from (6) as minimizing the functional

$$F(e,v) = \langle A(e)v, v\rangle \tag{26}$$

over a weakly compact set $W \subset V$.

Lemma 1. *There exists a weakly compact set $W \subset V$ such that*

$$\lambda(e) = \inf_{w \in W} F(e,w) \text{ for every } e \in U_{ad}\,. \tag{27}$$

Proof. Let $e \in U_{ad}$ be arbitrary. We introduce the set

$$M(e) = \{u \in V \mid \lambda(e) = F(e,u), \quad \langle Bu, u\rangle = \|u\|_1^2 = 1\} \tag{28}$$

The set $M(e)$ is nonempty due to Theorem 1. The assumption (10) implies the estimates

$$\alpha_1\|v\|_2^2 \le F(e,v) \quad \text{for all } e \in U_{ad}\,, \quad v \in V \cap S_1\,, \tag{29}$$

where

$$S_1 = \{v \in H \mid \|v\|_1 = 1\}.$$

Let $u_0 \in K \cap S_1$. We have due to Eqs. (10), (6), (8) the estimates

$$F(e,v) \le F(e,u_0) \le \alpha_2\|u_0\|_2^2 \quad \text{for all } v \in M(e).$$

The inequality (29) then implies

$$\|v\|_2 \le r \quad \text{for all } v \in M(e) \tag{30}$$

with

$$r = \left(\frac{\alpha_2}{\alpha_1}\right)^{1/2} \|u_0\|_2^2\,.$$

The relation (27) then holds with a weakly compact set $W \subset V$:

$$W = \{v \in K \cap S_1 \mid \|v\|_2 \le r\}. \tag{31}$$

The next lemma describes the differentiability properties of the functional F from (26).

Lemma 2.

(i) The function $F(\cdot;w) : U \to R$ is Gâteaux differentiable in $e \in U$ for every $w \in W$ and

$$\mathrm{d}F(e,w;\xi) = \langle \mathrm{d}A(e;\xi)w, w\rangle. \tag{32}$$

(ii) The functional $\mathrm{d}F(\cdot,\cdot;\cdot) : U \times W \times U \to R$ fulfils the assumption (iv) of Theorem 4 with respect to a weak topology in $U \times W \times U$.

Proof. The relation (32) is a consequence of the differentiability assumption (25). The assumption $F(e_n;w_n) = \inf_{v\in W} F(e_n,v)$ means

$$F(e_n;w_n) = \lambda(e_n) = \langle A(e_n)w_n\,,\, w_n\rangle. \tag{33}$$

Let

$$e_n \to e \text{ in } U, \qquad \xi_n \to \xi \text{ in } U, \qquad w_n \rightharpoonup w \text{ in } W, \qquad F(e_n,w_n) = \inf_{v\in W} F(e_n,v). \tag{34}$$

The last relation is equivalent with the eigenvalue inequality

$$\langle A(e_n)w_n\,,\, v-w_n\rangle \geq \lambda(e_n)\langle Bw_n\,,\, v-w_n\rangle \quad \text{for all } v \in K. \tag{35}$$

There exists the subsequence of $\{\lambda(e_n)\}$ (again denoted by $\{\lambda(e_n)\}$) such that

$$\lambda(e_n) \to \lambda^*. \tag{36}$$

Inserting $v \equiv w_n$ in (36) we obtain

$$\langle A(e_n)(w_n-w)\,,\, w_n-w\rangle \leq \lambda(e_n)\langle Bw_n\,,\, w_n-w\rangle - \langle A(e_n)w\,,\, w_n-w\rangle. \tag{37}$$

The relations (36), (37), the properties of the operator family $\{A(e)\}$ and the strong convergence $w_n \to w$ in H imply the strong convergence also in the space V:

$$w_n \to w \quad \text{in } V. \tag{38}$$

Using the expression (32) and the assumption (25) we obtain

$$\mathrm{d}F(e,w;\xi) = \lim_{n\to\infty} \mathrm{d}F(e_n,w_n;\xi_n) \tag{39}$$

which is even the stronger result as (22) and the proof is complete.

Lemma 1 and Theorem 4 imply the necessary optimality condition in a form of a quasi-variational inequality as expressed in the following theorem.

Theorem 5. *Let e^* be a solution of the Optimal Control Problem 1 over the convex and compact set $U_{ad} \subset U$. Then the necessary optimality condition has the form of the quasi-variational inequality*

$$\inf_{v\in M(e^*)} \langle \mathrm{d}A(e^*;e-e^*)v\,,\, v\rangle \leq 0 \quad \text{for all } e \in U_{ad} \tag{40}$$

where

$$M(e^*) = \{v \in K \mid \lambda(e^*) = \langle A(e^*)v,v\rangle\},\ \ \langle Bv,v\rangle = 1\}. \tag{41}$$

Remark. Previous results can be generalized to the case of the Optimal Control Problem

$$\lambda(e_0) = \max_{e\in U_{ad}} \lambda(e) = \max_{e\in U_{ad}} \min_{v\in K\setminus\bar{0}} \frac{\langle A(e)v,v\rangle}{\langle B(e)v,v\rangle}\,. \tag{42}$$

If the operator family $\{B(e)\} \subset \mathcal{L}(U,\mathcal{L}(H,H^*))$ is uniformly bounded, coercive and Gâteaux differentiable, then there exists a solution e_0 of the problem (42) and it fulfils the optimality condition

$$\inf_{v\in M(e_0)} \langle \mathrm{d}A(e_0;e-e_0)v,v\rangle - \langle \lambda(e_0)\mathrm{d}B(e_0;e-e_0)v,v\rangle \leq 0 \quad \text{for all } e \in U_{ad} \tag{43}$$

where

$$M(e_0) = \left\{ v \in V \mid \ \lambda(e_0) = \frac{\langle A(e_0)v,v)\rangle}{\langle B(e_0)v,v\rangle},\quad \|v\|_1 = 1 \right\}. \tag{44}$$

5. FINITE ELEMENTS APPROXIMATION

We suppose for simplicity the rectangular middle plane of the plate

$$\Omega = (0,a)\times(0,b), \quad a>0, \quad b>0.$$

Let

$$T_h := \{R_{ij}\}, \qquad R_{ij} = [(i-1)h_1\,,\, ih_1]\times[(j-1)h_2\,,\, jh_2], \qquad i=1,\dots,N_1(h_1), \quad j=1,\dots,N_2(h_2), \tag{45}$$

denote a partition of the middle surface $\overline{\Omega}$ into $N_1(h_1)\,N_2(h_2)$ rectangles R_{ij}. We define $h=\sqrt{h_1^2+h_2^2}$. We assume further that T_h is consistent with the partition of the boundary $\partial\Omega = \overline{\Gamma}_1\cup\overline{\Gamma}_2\cup\overline{\Gamma}_3$, i.e. the sets $\overline{\Gamma}_1$, $\overline{\Gamma}_2$, $\overline{\Gamma}_3$ intersect only in the nodal points and the unilaterally supported part of the boundary can be written in the form

$$\Gamma_3 = \bigcup_{i=1}^{n(h)} \overline{A_{i-1}A_i}\,,$$

where $A_1, \dots, A_{n(h)}$ also belong to the nodal points of the partition T_h.

We introduce the sets $Q_1(R_{ij})$ and $Q_3(R_{ij})$ of bilinear and bicubic polynomials over the rectangle R_{ij}. Further we set

$$U_{ad}^h = \{e\in U_{ad} \mid \; e|_{R_{ij}}\in Q_1(R_{ij}) \;\forall i=1,\dots,N_1(h_1),\; j=1,\dots,N_2(h_2)\},$$
$$V_h \;= \{v\in V\cap C^1(\bar\Omega) \mid \quad v|_{R_{ij}}\in Q_3(R_{ij}) \quad \forall i=1,\dots,N_1(h_1), \quad j=1,\dots,N_2(h_2)\},$$
$$K_h = X\cap V_h\,.$$

Let $\{\lambda_h(e), w_h(e)\}\in R\times[K_h\setminus\bar 0]$ be a couple solving the finite dimensional minimization problem

$$\lambda_h(e) = \min_{v\in K_h\setminus\bar 0}\frac{\langle A(e)v,v\rangle}{\langle Bv,v\rangle} = \frac{\langle A(e)u_h(e)\,,\,u_h(e)\rangle}{\langle Bu_h(e)\,,\,u_h(e)\rangle}\,. \tag{46}$$

We formulate the approximate

Optimal Control Problem 2_h. *To find $e_h^*\in U_{ad}^h$ such that*

$$\lambda_h(e_h^*) = \max_{e\in U_{ad}^h}\lambda_h(e) = \max_{e\in U_{ad}^h}\min_{v\in K_h\setminus\bar 0}\frac{\langle A(e)v,v\rangle}{\langle Bv,v\rangle}\,. \tag{47}$$

In order to verify the main convergence we introduce a following lemma verified in [8] which expresses the approximation properties of the convex sets $\{K_h\}$, $\{U_{ad}^h\}$, $0<h<h_0$.

Lemma 3. *For every $v\in K$ and $e\in U_{ad}$ there exist sets $\{v_h\}$, $v_h\in K_h$ and $\{e_h\}$, $e_h\in U_{ad}^h$ such that for $h\to 0+$*

$$v_h\to v \quad \text{in } V, \tag{48}$$
$$e_h\to e \quad \text{in } C(\bar\Omega) \text{ (uniformly on) } \bar\Omega. \tag{49}$$

We formulate the theorem about the convergence of the finite elements method. We denote S_1 the unit sphere in the space H.

Theorem 6. *For every $h\in(0,h_0)$ with sufficiently small h_0 there exists a solution $e_h^*\in U_{ad}^h$ of the Optimal Control Problem 2_h.*

If $\{e_h^*\}$, $h \to 0+$, is a sequence of solutions of the Problem 2_h, then a subsequence $\{e_k^*\}$ of $\{e_h^*\}$ exists such that for $k \to 0+$

$$e_k^* \to e^* \quad \text{in } C(\bar{\Omega}) \text{ (uniformly on } \bar{\Omega}), \tag{50}$$

$$\lambda_k(e_k^*) \to \lambda(e^*) \quad \text{in } R, \tag{51}$$

$$w_k(e_k^*) \to w(e^*) \quad \text{strongly in } V, \tag{52}$$

where $e^* \in U_{ad}$ is a solution of the Optimal Control Problem 1 and a couple $\{\lambda(e^*), w(e^*)\} \in R \times K \cap S_1$ solves the corresponding state eigenvalue problem

$$\lambda(e^*) = \min_{v \in K \setminus \bar{0}} \frac{\langle A(e)v, v\rangle}{\langle Bv, v\rangle} = \langle A(e)w(e^*), w(e^*)\rangle. \tag{53}$$

Each subsequence $\{e_k^*\}$ of $\{e_h^*\}$, which converges uniformly on $\bar{\Omega}$, has the properties (51), (52).

Proof. The existence of the approximate optimal thicknesses e_h^* and the corresponding couples $\{\lambda_h(e_h^*), w_h(e_h^*)\} \in R \times K_h \cap S_1$ is assured due to Theorem 3 with $K_h \subset V_h$ instead of $K \subset V$. The sequence $\{e_h^*\}$ fulfils the assumptions of the Ascoli–Arzela theorem due to the uniform boundedness of the derivatives. Then there exists its subsequence $\{e_k^*\}$ fulfilling the uniform convergence

$$e_k^* \to e_0 \text{ in } C(\bar{\Omega}) \quad \text{for } k \to 0+. \tag{54}$$

Further there exists the subsequence of $\{e_k^*\}$ (again denoted by $\{e_k^*\}$) weakly star convergent in $W^{1,\infty}(\Omega)$ which implies that e_0 fulfils the bounds of partial derivatives from the admissible set U_{ad}. All the other restrictions of the set U_{ad} are also satisfied by due to the uniform convergence (54). Let a couple $\{\lambda(e_0), w(e_0)\} \in R \times K \cap S_1$ be a solution of the problem

$$\lambda(e_0) = \min_{v \in K \setminus \bar{0}} \frac{\langle A(e_0)v, v\rangle}{\langle Bv, v\rangle} = \langle A(e_0)w(e_0), w(e_0)\rangle. \tag{55}$$

If $v \in K \setminus \bar{0}$, then there exists due to Lemma 3 the sequence $\{v_k\}$ fulfilling

$$v_k \in K_k \setminus \bar{0}, \quad v_k \to v \text{ (strongly) in } V \text{ for } k \to 0+. \tag{56}$$

We have the inequalities

$$\lambda_k(e_k^*) \le \frac{\langle A(e_k^*)v_k, v_k\rangle}{\langle Bv_k, v_k\rangle}, \quad k > 0. \tag{57}$$

The convergences (53), (56) imply

$$\frac{\langle A(e_k^*)v_k, v_k\rangle}{\langle Bv_k, v_k\rangle} \to \frac{\langle A(e_0)v, v\rangle}{\langle Bv, v\rangle} \quad \text{for } k \to 0+. \tag{58}$$

Hence, the sequence $\{\lambda_k(e_k)\}$ is bounded and contains the convergent subsequence fulfilling

$$\lambda_k(e_k^*) \to \lambda^* \quad \text{for } k \to 0+. \tag{59}$$

Let $e \in U_{ad}$, $\omega_k \in K_k$ and $\xi_k \in U_{ad}^k$ be sequences fulfilling

$$\omega_k \to w(e_0) \text{ in } V, \qquad \xi_k \to e \text{ in } C(\bar{\Omega}). \tag{60}$$

Their existence is due to the Lemma 3. Applying (53), (60) we arrive at the following relations

$$\begin{aligned}
\lambda(e_0) &= \langle A(e_0)w(e_0), w(e_0)\rangle = \lim_{k\to 0+} \frac{\langle A(e_k^*)\omega_k, \omega_k\rangle}{\langle B\omega_k, \omega_k\rangle} \\
&\ge \limsup_{k\to 0+} \langle A(e_k^*)w_k(e_k^*), w_k(e_k^*)\rangle_2 = \limsup_{k\to 0+} \lambda_k(e_k^*) \\
&\ge \limsup_{k\to 0+} \lambda_k(\xi_k) = \limsup_{k\to 0+} \langle A(\xi_k)w_k(\xi_k), w_k(\xi_k)\rangle \\
&= \limsup_{k\to 0+} \langle A(e)w_k(\xi_k), w_k(\xi_k)\rangle \ge \langle A(e)w(e), w(e)\rangle = \lambda(e)
\end{aligned} \tag{61}$$

Hence $e_0 = e^*$, $\lambda(e_0) = \lambda(e^*) = \max_{e \in U_{ad}} \lambda(e)$ and the convergence (50) holds.

We proceed with verifying (51), (52). The inequalities (61) imply that the sequences $\{\lambda_k(e_k^*)\}$ and $\{w_k(e_k^*)\}$ are bounded in R and V respectively. We have for their subsequences the relations

$$\begin{aligned}\lambda^* = \lim_{k\to 0+} \lambda_k(e_k^*) &= \lim_{k\to 0+} \langle A(e_k^*) w_k(e_k^*)\,,\, w_k(e_k^*)\rangle \\ &= \lim_{k\to 0+} \langle A(e^*) w_k(e_k^*)\,,\, w_k(e_k^*)\rangle \geq \langle A(e^*) w^*, w^*)\rangle = \lambda(e^*),\end{aligned} \tag{62}$$

where $w^* \in K \cap S_1$ is a weak limit in V of $\{w_k(e_k^*)\}$.

Further we have the relations

$$\lambda^* = \lim_{k\to 0+} \langle A(e^*) w_k(e_k^*),\, w_k(e_k^*)\rangle \leq \lim_{k\to 0+} \frac{\langle A(e_k^*)\omega_k\,,\, \omega_k\rangle}{\langle B\omega_k\,,\, \omega_k\rangle} = \langle A(e^*) w(e^*)\,,\, w(e^*)\rangle = \lambda(e^*).$$

Comparing with Eq. (62) we have $\lambda^* = \lambda(e^*)$ and the convergence (51) holds. Simultaneously we obtain

$$\lambda^* = \langle A(e^*) w^*\,,\, w^*\rangle \quad \text{and} \quad w^* = w(e^*). \tag{63}$$

It remains us to verify the strong convergence (52). We have the variational inequalities

$$\langle A(e^*) w(e^*)\,,\, v - w(e^*)\rangle \geq \lambda(e^*) \langle Bw(e^*)\,,\, v - w(e^*)\rangle \quad \text{for all } v \in K, \tag{64}$$

$$\langle A(e_k^*) w_k(e_k^*)\,,\, v - w_k(e_k^*)\rangle \geq \lambda_k(e_k^*) \langle Bw_k(e_k^*)\,,\, v - w_k(e_k^*)\rangle \quad \text{for all } v \in K_k\,. \tag{65}$$

We set $v := w_k(e_k^*)$ in Eq. (64) and $v := \omega_k$ in Eq. (65), where the sequence $\{\omega_k\}$ fulfils the convergence (60). After adding the inequalities we obtain

$$\begin{aligned}&\langle [A(e_k^*) - A(e^*)] w_k(e_k^*)\,,\, \omega_k - w_k(e_k^*)\rangle \\ &\quad + \langle A(e^*) w_k(e_k^*)\,,\, \omega_k - w(e^*)\rangle + \langle A(e^*) [w(e^*) - w_k(e_k^*)]\,,\, w_k(e_k^*) - w(e^*)\rangle \\ &\qquad \geq \lambda(e^*) \langle Bw(e^*)\,,\, w_k(e_k^*) - w(e^*)\rangle + \lambda_k(e_k^*) \langle Bw_k(e_k^*)\,,\, \omega_k - w_k(e_k^*)\rangle\,.\end{aligned}$$

Using the coercivity of $A(e^*)$ we arrive at the estimate

$$\begin{aligned}\alpha_1 \|w_k(e_k^*) - w(e^*)\|_2^2 \leq\ & \langle [A(e_k^*) - A(e^*)] w_k(e_k^*)\,,\, \omega_k - w_k(e_k^*)\rangle \\ & + \langle A(e^*) w_k(e_k^*)\,,\, \omega_k - w(e^*)\rangle \\ & + \lambda(e^*) \langle Bw(e^*)\,,\, w(e^*) - w_k(e_k^*)\rangle \\ & + \lambda_k(e_k^*) \langle Bw_k(e_k^*)\,,\, w_k(e_k^*) - \omega_k w_k(e_k^*)\rangle\end{aligned}$$

and the strong convergence (52) follows after considering (54), (59), (60) and the strong convergence of $\{w_k(e_k^*)\}$ in H.

It results from the course of the whole proof that each subsequence $\{e_k^*\}$ of $\{e_h^*\}$ converging uniformly on $\bar{\Omega}$ fulfils the convergence (51), (52).

Remark. The numerical algorithm of the above problem deserves an individual study. A lot of problems arise due to nonconvexity and nondifferentiability of the state eigenvalue variational inequality. In order to overcome these problems it is inevitable to transform the max–min to the min–max problem using Theorem 1 and the bundle method due to Lemaréchal [7, 10], see also [5] (Appendix) and references therein.

ACKNOWLEDGEMENTS

This work was supported by Grant 1/8263/01 of the Grant Agency of the Slovak Republic.

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