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Finite displacements in reciprocity-based FE formulation[1]

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In this paper, Trefftz polynomials are used for the development of FEM based on the reciprocity relations. Such reciprocity principles are known from the Boundary Element formulations, however, using the Trefftz polynomials in the reciprocity relations instead of the fundamental solutions yields the non-singular integral equations for the evaluation of corresponding sub-domain (element) relations. A weak form satisfaction of the equilibrium is used for the inter-domain connectivity relations. For linear problems, the element stiffness matrices are defined in the boundary integral equation form. In non-linear problems the total Lagrangian formulation leads to the evaluation of the boundary integrals over the original (related) domain evaluated only once during the solution and to the volume integrals containing the non-linear terms.

Also, Trefftz polynomials can be used in the post-processing phase of the FEM computations for small strain problems. By using the Trefftz polynomials as local interpolators, smooth fields of the secondary variables (strains, stresses, etc.) can be found in the whole domain (if it is homogeneous). This approach considerably increases the accuracy of the evaluated fields while maintaining the same rate of convergence as that of the primary fields. Stress smoothing for large displacements will be the aim of further research.

Considering the examples of simple tension, pure bending and tension of fully clamped rectangular plate (2D stress/strain problems) for large strain–large rotation problems, the use of the initial stiffness, the Newton–Raphson procedure, and the incremental Newton–Raphson procedure will be discussed.

1. Introduction

The Trefftz (T-) functions [26] are those which satisfy all the governing equations of the problem, i.e. the differential equations inside the domain and the boundary conditions on the domain boundaries. The polynomial Trefftz functions can be found in a simple way for many important 2D and 3D solid and fluid mechanics problems by the methods of symbolic algebra or by numerical methods [12, 14]. Also, the fundamental solutions of the governing equations in all domain (except the source point if lying outside the domain) can be included into the T-interpolation functions. For general problems, obtaining the solution satisfying all governing equations is difficult; only an approximate solution can be found. The simplest way to satisfy the boundary conditions is the collocation method, where the boundary conditions are satisfied in the discrete points of the boundary only [11]. Such an approximation does not guarantee the convergence of the solution in the multi-domain formulation. Further more, many other approaches can be found in which the inter-domain continuity and boundary condition satisfaction is enforced in the weak (integral) sense (e.g. weighted residual, variational form, integral least squares [10]).

In the hybrid FEM formulations, the internal and boundary fields are chosen independently; the internal field variables are approximated by T-functions, and the boundary field enforces both the inter-domain continuity and the satisfaction of the boundary conditions in a weak sense [10, 13, 21, 24, 25]. In such formulations, the internal primary fields (displacements in hybrid-displacement FEM formulations) are incompatible between the elements, but usually the corresponding boundary fields are taken as representative of the solution.

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In this paper, FE approximation using T-functions is shown in the reciprocity based FEM formulations; these are well known from the BEM [1, 7, 8]. If non-singular T-functions [5] are used for the approximation of the variables' fields, difficulties encountered with numerical integration [1, 3, 18, 19] are not present. Numerical integration is a special procedure necessary for the integration of the singular integrals with weak, strong or hyper-singular kernels or the integrals with quasi-singularities. The complexity of the interpolation fields, however, increases with the complexity of the solved problem; therefore, the multi-domain solution yielding the T-polynomial reciprocity-based FEM (TPR FEM) was formulated [6, 16].

The element form of both hybrid and TPR FEM can be more general than that by using other FEM formulations because the integration is executed over the element boundaries only for linear problems. Consequently, large elements can also be used for the regions with complicated fields of variables (singularities, large gradients, local effects, elements with holes, etc.).

Finding the T-functions for a general problem is not possible for non-linear problems. The non-linearity yields volume integrals, with non-linear functions in the integrand. In solid statics, a suitable form for TPR FEM is the total Lagrangian formulation. In this formulation, the reference configuration does not change during the solution process and the integration is performed only once. The volume integrals containing the non-linear terms update the right side of the discretized form of the solution, leading to the initial stiffness formulation that does not converge by large strains. Improvement is achieved using an updated (tangential) stiffness matrix obtained from the nodal displacements of the previous iteration step.

The stress (and also strain) field is discontinuous between the elements in most FEM formulations, and its rate of convergence is lower than the rate of convergence of the displacements. If the stress is interpolated using T-polynomial interpolation functions with the moving least squares procedure from the nodal displacements (and known static conditions in the points near the boundaries), much better accuracy can be obtained with same rate of convergence for both displacements and stresses [15, 20].

Considering the examples of simple tension, pure bending, and tension of fully clamped rectangular plate (2D stress/strain problems) for large strain-large rotation problems, the use of the initial stiffness, the Newton–Raphson procedure, and the incremental Newton–Raphson procedure will be discussed.

2. Formulation for small strain, small rotation problems

When using the basic equations for time-independent small strain plasticity, the current state of deformation does not depend on only the current loading but also on the complete history of loading. At any loading time the total strain tensor can be split into the elastic and plastic parts,

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p} \,. \tag{1}$$

The strain is derived from displacements by the kinematic relation

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{2}$$

The stress–strain relation for isotropic material is

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{p}) = 2G\left[\left(\varepsilon_{ij} - \varepsilon_{ij}^{p}\right) + \frac{\nu}{1-\nu}\left(\varepsilon_{kk} - \varepsilon_{kk}^{p}\right)\delta_{ij}\right], \tag{3}$$

where C_{ijkl} is the elasticity tensor, G and ν are shear modulus and Poisson's ratio, respectively. Einstein's notation is used, in which the index after the comma denotes the partial derivative in the corresponding direction and with summation after repeated indices. The equilibrium equations in stresses

$$\sigma_{ji,j} = -b_i \tag{4}$$

are in displacements given by the Lame–Navier equations,

$$u_{i,jj} + \frac{1}{1-2\nu} u_{j,ji} = -\frac{b_i}{G} \tag{5}$$

with

$$b_i = \bar{b}_i - 2G\left[\varepsilon^p_{ij,j} + \frac{\nu}{1-2\nu}\varepsilon^p_{j,ji}\right], \tag{6}$$

where b_i is a pseudo-body force and the bar denotes prescribed value. The last relation contains the influence of the non-linear material behaviour.

The weak formulation of equilibrium (4) using the principle of weighted residuals can be written in the form

$$\int_\Omega (\sigma_{ji,j} + b_i)\, U_i \,\mathrm{d}\Omega = 0. \tag{7}$$

We can choose the weight function U_i to be the displacement field of a linear-elastic reference problem of the same body with body force absent. Capital letters will denote the quantities corresponding to this field.

The tractions corresponding to the stress field σ_{ij} are given by

$$t_i = \sigma_{ij} n_j \tag{8}$$

where n_i denotes the outer normal to the boundary.

Applying the integration by parts and the Gauss theorem to the first part of Eq. (7) we obtain

$$\int_\Gamma t_i U_i \,\mathrm{d}\Gamma - \int_\Omega \sigma_{ij} U_{j,i} \,\mathrm{d}\Omega + \int_\Omega b_i U_i \,\mathrm{d}\Omega = 0. \tag{9}$$

Using Hooke's law and identity

$$\Sigma_{ij}\varepsilon_{ij} = \Sigma_{ij} u_{j,i}\,, \tag{10}$$

we can write

$$\sigma_{ij} U_{j,i} = \Sigma_{ij} u_{j,i} - \Sigma_{ij}\varepsilon^p_{ij}\,, \tag{11}$$

where Σ_{ij} denotes the stress state corresponding to the displacement field U_i. T-polynomial displacements and tractions satisfy according to the definition of the following homogeneous equilibrium equations

$$U_{i,jj} + \frac{1}{1-2\nu} U_{j,ji} = 0, \tag{12}$$

$$\Sigma_{ji,j} = 0. \tag{13}$$

Then Eq. (9) can be written in the form

$$\int_\Omega \Sigma_{ij} u_{j,i} \,\mathrm{d}\Omega = \int_\Gamma t_i U_i \,\mathrm{d}\Gamma + \int_\Omega \Sigma_{ij}\varepsilon^p_{ij} \,\mathrm{d}\Omega + \int_\Omega b_i u_i \,\mathrm{d}\Omega. \tag{14}$$

Again, using integration by parts and Gauss theorem applied to the left side of the last equation, together with the equilibrium condition of the reference problem, the generalized form of Betti's theorem is obtained

$$\int_\Gamma T_i u_i \,\mathrm{d}\Gamma = \int_\Gamma t_i U_i \,\mathrm{d}\Gamma + \int_\Omega \Sigma_{ij}\varepsilon^p_{ij} \,\mathrm{d}\Omega + \int_\Omega b_i U_i \,\mathrm{d}\Omega. \tag{15}$$

This equation expresses the reciprocity of works done by two systems of forces: the one denoted by low letters which is looked for, and the other, reference state (for which all, displacements, strains, stresses and tractions are known inside and on the domain boundaries), denoted by capital letters.

In the BEM, the reference state defined by the Kelvin fundamental solution leads to the singular integral equation problem. Using Trefftz functions for the reference state, all integrals are regular; however, with complex problems many different reference states are needed for a numerical solution, and complex and inefficient high order T-polynomials are to be used. By using this formulation for sub-domains, a FEM is obtained with element matrices defined by the boundary integral equations (15) and their corresponding numerical resolution by the BEM. The displacements between the sub-domains will be compatible: the displacements on the element boundaries are common to the neighbour elements. However, the tractions will be incompatible between the elements; therefore, the inter-element equilibrium and natural boundary conditions will be satisfied in a weak (integral) sense as follows,

$$\int_{\Gamma_i} \delta u_i \,(t_i - \bar{t}_i)\, \mathrm{d}\Gamma + \int_{\Gamma_i} \delta u_i \,(t_i^A - t_i^B)\, \mathrm{d}\Gamma = \int_{\Gamma_e} \delta u_i t_i \,\mathrm{d}\Gamma - \int_{\Gamma_t} \delta u_i \bar{t}_i \,\mathrm{d}\Gamma = 0, \tag{16}$$

where Γ_i, Γ_t and Γ_e are the inter-element boundaries, the boundaries with prescribed tractions and element boundaries, respectively. The upper indices A and B denote the neighbouring elements.

For the discretization, the boundary displacements and tractions can be expressed by their values in nodal points (denoted by the dash) and shape functions, N,

$$u_i^e(\xi) = N_{Ju}(\xi)\hat{u}_{Ji}^e \quad \text{or} \quad \mathbf{u}^e = \mathbf{N}_u\hat{\mathbf{u}}^e, \tag{17}$$

$$t_i^e(\xi) = N_{Jt}(\xi)\hat{t}_{Ji}^e \quad \text{or} \quad \mathbf{t}^e = \mathbf{N}_t\hat{\mathbf{t}}^e, \tag{18}$$

where ξ is a local co-ordinate of an element boundary point and the capital letter index denotes the nodal point. The upper index e denotes the correspondence to the element.

Note that the tractions are discontinuous in the corner points, and thus a double node for tractions exists.

Then Eq. (15) leads to the system of equations

$$T_{iIJ}\,\hat{u}_{iJ}^e = U_{iIK}\,\hat{t}_{iK}^e + f_I^{ep} + f_I^{eb} \tag{19}$$

or, in the matrix form,

$$\mathbf{T}\mathbf{u}^e = \mathbf{U}\mathbf{t}^e + \mathbf{f}^p + \mathbf{f}^b, \tag{20}$$

where $\mathbf{u}^e$ and $\mathbf{t}^e$ are vectors of element nodal displacements and tractions, and

$$\begin{aligned}
T_{iIJ} &= \int_{\Gamma_e} T_{iI}(x(\xi))\, N_{Ju}(\xi)\,\mathrm{d}\Gamma = \sum_j \left(T_{iI}(x(\xi^{(j)}))\, N_{Ju}(\xi^{(j)})\, J(\xi^{(j)})\, w^{(j)}\right),\\
U_{iIK} &= \int_{\Gamma_e} U_{iI}(x(\xi))\, N_{Kt}(\xi)\,\mathrm{d}\Gamma = \sum_j \left(U_{iI}(x(\xi^{(j)}))\, N_{Kt}(\xi^{(j)})\, J(\xi^{(j)})\, w^{(j)}\right),\\
f_I^{ep} &= \int_{\Omega_e} \Sigma_{ijI}(x(\xi))\, \varepsilon_{ij}^p(x(\xi))\,\mathrm{d}\Omega = \sum_\alpha \left(\Sigma_{ijI}(x(\xi^{(\alpha)}))\, \varepsilon_{ij}^p(x(\xi^{(\alpha)}))\, J(\xi^{(\alpha)})\, w^{(\alpha)}\right),\\
f_I^{eb} &= \int_{\Omega_e} U_{iI}(x(\xi))\, b_i(x(\xi))\,\mathrm{d}\Omega = \sum_\alpha \left(U_{iI}(x(\xi^{(\alpha)}))\, b_i(x(\xi^{(\alpha)}))\, J(\xi^{(\alpha)})\, w^{(\alpha)}\right).
\end{aligned} \tag{21}$$

ξ_j and w_j are co-ordinates and weights in the Gauss quadrature formulas and J is Jacobian. Low indices in these expressions correspond to the field (vector or tensor) components, the lower index I corresponds to the I-th Trefftz function, and the lower index J corresponds to the nodal value. In order to distinguish the integration over the volume of the element from that over the element boundaries, we denote the volume Gauss points by α.

Equation (16) written in the discrete form is

$$\sum_e \sum_j \sum_L N_{Ku}(\xi^{(j)})\, N_{Lt}(\xi^{(j)})\, J(\xi^{(j)})\, w^{(j)} \hat{t}_{iL} = \sum_e \sum_j N_{Ku}(\xi^{(j)})\, \bar{t}_i(\xi^{(j)})\, J(\xi^{(j)})\, w^{(j)}, \tag{22}$$

or in the matrix form,

$$\sum_e \mathbf{M}\mathbf{t}^e = \sum_e \mathbf{p}^e, \tag{23}$$

with summation over all elements. The lower indices K and L in Eq. (22) correspond to the nodal displacements and tractions, respectively.

Setting for tractions from Eq. (20) into (23), the resulting system of discretized equations is obtained

$$\sum_e \mathbf{M}\mathbf{U}^{-1}\mathbf{T}\mathbf{u}^e = \sum_e \left(\mathbf{p}^e + \mathbf{M}\mathbf{U}^{-1}(\mathbf{f}^p + \mathbf{f}^b)\right) \tag{24}$$

or shortly

$$\mathbf{K}\mathbf{u} = \mathbf{p}. \tag{25}$$

This is a system of linear equations in which the non-linear term (containing the plastic strains) is the second term of the right hand side of Eq.(24).

Note that in order to obtain non-singular matrices, the vector of element nodal tractions must have as many, or more, independent components than the vector of element nodal displacements. Also, we must choose as many, or more, Trefftz functions for each element than we have element nodal tractions.

3. THE TOTAL LAGRANGIAN FORMULATION FOR FINITE DEFORMATION PROBLEMS

In this section, the application of the TPR FEM to geometrically and physically non-linear problems in the total Lagrangian approach will be illustrated. The basic equations refer to the undeformed configuration of the body.

Let X_i denote the coordinate of a material particle X in the undeformed body. After the deformation, the co-ordinate of this particle will be x_i. The Cartesian components f_{ij} of the deformation gradient are defined by

$$f_{ij}(X) = \frac{\partial x_i(X)}{\partial X_j}. \tag{26}$$

Using the displacement u_i of a material particle X,

$$u_i = x_i - X_i, \tag{27}$$

leads to the alternative expression for the deformation gradient,

$$f_{ij}(X) = \delta_{ij} + \frac{\partial u_i(X)}{\partial X_j} = \delta_{ij} + u_{i,j}(X). \tag{28}$$

Since the formulation used is presented in the undeformed configuration, partial derivatives denoted by $(\cdot)_{,i}$ are taken with respect to the undeformed co-ordinates X_i. The deformation gradient can be used to define the Green strain tensor,

$$e_{ij} = \frac{1}{2}\left(f_{ki} f_{kj} - \delta_{ij}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2} u_{k,i} u_{k,j}, \tag{29}$$

and the symmetric 2nd Piola–Kirchhoff stress tensor

$$s_{ij} = J f_{ik}^{-1} \sigma_{kl} f_{jl}^{-1} \quad \text{with} \quad J = \det(f_{ij}), \tag{30}$$

both referring to the undeformed configuration. σ_{ij} is the Cauchy stress tensor.

The equilibrium equation

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \tag{31}$$

can be transformed to the initial (undeformed) configuration,

$$(s_{jk} f_{ik})_{,j} + b_i^o = 0, \tag{32}$$

where b_i^0 and b_i denotes the body force with respect to the initial and deformed configuration, respectively. The relation between the tractions, t_i^0, which measure the force per unit undeformed area $\mathrm{d}A^0$, and the tractions, t_i, in the deformed configuration $\mathrm{d}A$ is given by

$$t_i^0 = f_{ij} s_{kj} n_k^0 = t_i \frac{\mathrm{d}A}{\mathrm{d}A^0} = \sigma_{ki} n_k \frac{\mathrm{d}A}{\mathrm{d}A^0}, \tag{33}$$

where n_i^0 and n_i is the outer surface normal in the initial and deformed configuration.

By using the derivation of the reciprocity relations, a similar procedure as in the case of infinitesimal displacements is followed, starting from the equilibrium equation of the deformed body relative to the initial configuration (32). Again, the T-polynomial displacements will be taken as the weight function in the weak formulation of the balance

$$\int_\Omega \left[(s_{kl} f_{il})_{,k} + b_i^0\right] U_i \, \mathrm{d}\Omega = 0. \tag{34}$$

The T-polynomials are taken in co-ordinates of the undeformed (initial) configuration. Due to the total Lagrangian approach, Ω is the domain of the undeformed body, we omit its upper index 0 (similarly we do for the surface Γ) and all derivatives are taken with respect to this configuration. Applying integration by parts and the Gauss' theorem to Eq. (34) we obtain

$$\int_\Gamma t_i^o U_i \, \mathrm{d}\Gamma - \int_\Omega (s_{kl} f_{il}) \, U_{i,k} \, \mathrm{d}\Omega + \int_\Omega b_i^0 U_i \, \mathrm{d}\Omega = 0. \tag{35}$$

Substituting the displacement gradients for the deformation gradient from Eq. (28) results in

$$\int_\Gamma t_i^0 U_i \, \mathrm{d}\Gamma + \int_\Omega b_i^0 U_i \, \mathrm{d}\Omega - \int_\Omega s_{ji} U_{i,j} \, \mathrm{d}\Omega - \int_\Omega (s_{jk} u_{i,k}) U_{i,j} \, \mathrm{d}\Omega = 0. \tag{36}$$

Again, the strain tensor can be split into the elastic and plastic parts

$$e_{ij} = e_{ij}^e + e_{ij}^p \tag{37}$$

and because of the linear dependence between the elastic part of the Green strain tensor and the 2nd Piola–Kirchhoff stress tensor, the reciprocity relation can be found in the form

$$s_{ij} U_{j,i} = u_{j,i} \Sigma_{ij} + \frac{1}{2} u_{k,i} u_{k,j} \Sigma_{ij} + e_{ij}^p \Sigma_{ij}. \tag{38}$$

Using this relation, the integration by parts, and the Gauss' theorem, Eq. (36) can be written in the form

$$\int_\Gamma t_i^0 U_i \, \mathrm{d}\Gamma + \int_\Omega b_i^0 U_i \, \mathrm{d}\Omega - \int_\Gamma u_i T_i \, \mathrm{d}\Gamma$$
$$- \int_\Omega \frac{1}{2} u_{k,i} u_{k,j} \Sigma_{ij} \, \mathrm{d}\Omega - \int_\Omega s_{jk} u_{i,k} U_{i,j} \, \mathrm{d}\Omega - \int_\Omega e_{ij}^p \Sigma_{ij} \, \mathrm{d}\Omega = 0. \tag{39}$$

If Eq. (39) is applied for the computation of the relation between the boundary displacements u and the tractions t^0 for each sub-domain (element), and the inter-domain traction continuity Eq. (16) is used to the weak satisfaction of equilibrium, the procedure described in the previous section can be used.

4. Linearization of resulting equations for large strain problems

For large strains we can linearize the expressions in the integrands. For this purpose the displacements inside the element can be approximated from its nodal values using the shape functions as it is by displacement FEM formulations [2, 27],

$$u_i = N_J\, u_{Ji}\,. \tag{40}$$

Similarly their derivatives are obtained from

$$u_{i,k} = N_{J,k}\, u_{Ji}\,. \tag{41}$$

The index denoted by capital letters will denote corresponding nodal displacement.

In the N-th iteration step the displacement will be given by

$$u_i^{(N)} = N_J\, u_{Ji}^{(N)}\,. \tag{42}$$

The resulting discretized equations (24) can be now written in the form

$$\left(\mathbf{K} + \mathbf{K}^{NL}\right)\mathbf{u}^{(N)} = \mathbf{p}^{(N-1)}, \tag{43}$$

where $\mathbf{K}$ corresponds to the linear part of Eq. (39) and $\mathbf{K}^{NL}$ to its non-linear part, which will be linearized for each iteration step and $\mathbf{p}^{(N-1)}$ denotes the configuration dependent load corresponding to the configuration of the previous iteration step. For this purpose we can write the integrand of the fourth integral (39) in the form

$$\frac{1}{2}\Sigma_{ij}(u_{k,i}u_{k,j})^L = \Sigma_{ij}\, u_{k,i}^{(N-1)}\, N_{J,j}\, u_{Jk}^{(N)} \tag{44}$$

or, if the integrand is written in the form

$$\frac{1}{2}\, u_{k,i}u_{k,j}\,\Sigma_{ij} = \frac{1}{2}\, C_{ijlm}\, u_{n,l}u_{n,m}\, U_{i,j} \tag{45}$$

and for the isotropic material, in which

$$C_{ijlm} = \lambda\delta_{ij}\delta_{lm} + \mu\left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right), \tag{46}$$

the linearized form is

$$\left(\frac{1}{2}\, C_{ijlm}\, u_{n,l}u_{n,m}\right)^L = \left[\mu\left(u_{n,i}^{(N-1)}N_{J,j} + u_{n,j}^{(N-1)}N_{J,i}\right) + \lambda\delta_{ij}u_{n,m}^{(N-1)}N_{J,m}\right] u_{Jn}^{(N)}. \tag{47}$$

The linearization of the fifth integral of Eq. (39) is realized using the 2-nd Piola–Kirchhoff stress in the form

$$s_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda\delta_{ij}(u_{k,k} + u_{k,l}u_{k,l}) + \mu\, u_{k,i}u_{k,j} \tag{48}$$

as follows

$$\begin{aligned} U_{j,i}(s_{ik}u_{j,k})^L = \Big[&U_{k,i}s_{ij}^{(N-1)}N_{J,j} + \mu\left(U_{j,k}u_{j,i}^{(N-1)} + U_{j,i}u_{j,k}^{(N-1)}\right)N_{J,i} \\ &+ \mu U_{j,i}u_{j,m}^{(N-1)}\left(u_{k,i}^{(N-1)}N_{J,m} + u_{k,m}^{(N-1)}N_{J,i}\right) \\ &+ \lambda U_{j,i}u_{j,i}^{(N-1)}\left(N_{J,k} + 2u_{k,m}^{(N-1)}N_{J,m}\right)\Big]\, u_{Jk}^{(N)}. \end{aligned} \tag{49}$$

Note that the field variables of the previous, $(N-1)$-th, iteration step are given by use of the nodal displacements computed in that step, whereas the field variables of the current, N-th, step are defined by the shape functions and the unknown nodal displacements of this step.

In the Newton–Raphson procedure, the increments are computed following Eq. (43) from

$$\left(\mathbf{K}+\mathbf{K}^{NL}\right)\Delta\mathbf{u}^{(N)}=\Delta\mathbf{p}^{(N)} \tag{50}$$

and the displacements in the N-th iteration step are

$$\mathbf{u}^{(N)}=\mathbf{u}^{(N-1)}+\Delta\mathbf{u}^{(N)}. \tag{51}$$

The iteration is stopped if the quadratic norm of the last displacement increment related to the quadratic norm of the displacements is less then specified value, i.e.

$$e \geq \left\|\Delta u^{(N)}\right\| \Big/ \left\|u^{(N)}\right\|. \tag{52}$$

5. Stress evaluation

Having obtained the nodal displacements, the tractions in nodal points in each element can be computed from Eq. (20), as

$$\mathbf{t}^{e}=\mathbf{U}^{-1}(\mathbf{T}\mathbf{u}^{e}-\mathbf{f}^{p}-\mathbf{f}^{b}). \tag{53}$$

The obtained tractions are discontinuous between the elements (according to the weak inter-element equilibrium continuity) and have the largest errors in the corner points [6, 14].

A continuous stress fields can be obtained using the Moving Least Square (MLS) techniques from displacements and tractions in the nodal points over some patches of nodes. Best results are achieved, if the interpolation polynomials satisfy the governing equation (T-polynomials). Similar ideas has been used in [4, 23]. We assume the displacement field (in a field point with the local co-ordinates $\mathbf{x}$ with the local origin in the point where the stresses are to be computed), $\mathbf{u}(\mathbf{x})$, given in the form

$$\mathbf{u}(\mathbf{x})=\mathbf{U}(\mathbf{x})\mathbf{c} \tag{54}$$

where $\mathbf{U}(\mathbf{x})$ is a matrix of T-displacement-functions and $\mathbf{c}$ is the vector of unknown coefficients. If T-polynomials are used for the T-functions, we can easy express strain and stress fields from displacements (54). The stress field can be expressed from

$$\boldsymbol{\sigma}(\mathbf{x})=\mathbf{S}(\mathbf{x})\mathbf{c} \tag{55}$$

where the matrix of T-stress-functions $\mathbf{S}(\mathbf{x})$ is derived from the matrix $\mathbf{U}(\mathbf{x})$. Similarly, we can express T-tractions

$$\mathbf{t}(\mathbf{x})=\mathbf{T}(\mathbf{x})\mathbf{c}. \tag{56}$$

In this approximation we use the full T-polynomials of the chosen order and the unknown coefficients $\mathbf{c}$ are computed by LS method by minimizing

$$\sum_i w_i^d(\mathbf{U}(\mathbf{x}_i)\mathbf{c}-\mathbf{u}_i)^2+\sum_i w_i^t(\mathbf{T}(\mathbf{x}_i)\mathbf{c}-\mathbf{t}_i)^2=\min \tag{57}$$

where $\mathbf{u}_i$ and $\mathbf{t}_i$ are the displacements and tractions in the nodal points, and w_i^d and w_i^t are corresponding weighting functions necessary for the dimensionality.

The accuracy of this method is high and its rate of convergence is the same as the rate of convergence of primary (displacement) field as it was shown in [16].

If also body forces are present in the solution then the T-polynomials have to be derived from corresponding non-homogeneous equilibrium equations. If the plasticity effect takes place, then the plastic part of strains is taken into account according to Eq. (3).

In the case of finite deformations, this effective procedure based on the T-polynomials cannot be used in the described form, as the T-polynomials are not available for the general deformation state. The methodology is under development. Currently we use classical polynomials for the local interpolation using same procedure as described in Eqs. (54)–(57).

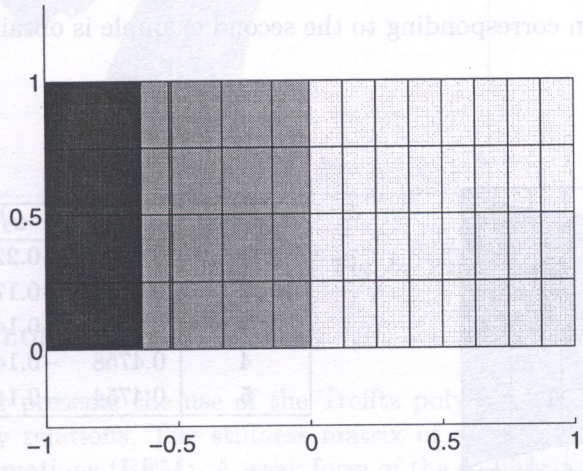
6. EXAMPLES

6.1. The first example

In the first example, a simple extension of a square domain of dimensions 1 by 1 by Young modulus E = 1 and plain stress conditions was examined. First, Poisson's ratio equal to zero was assumed. The relation between the tractions t and the stretching of the domain u is

$$t_1^0 = (u_1 + 1.5u_1^2 + 0.5u_1^3)\, E. \tag{58}$$

Numerically, the converged solution ($e = 0.00001$) was obtained for $t = 1.0$ in four iteration steps with the following results (Fig. 1).



iteration	u_1
1	0.63636
2	0.53039
3	0.52144
4	0.52138

Fig. 1. Simple tension with $\nu = 0$

The result in the last step agrees with the analytical result in all digits. Note that initial stiffness can be used for smaller deformations ($t < 0.19$) in this example.

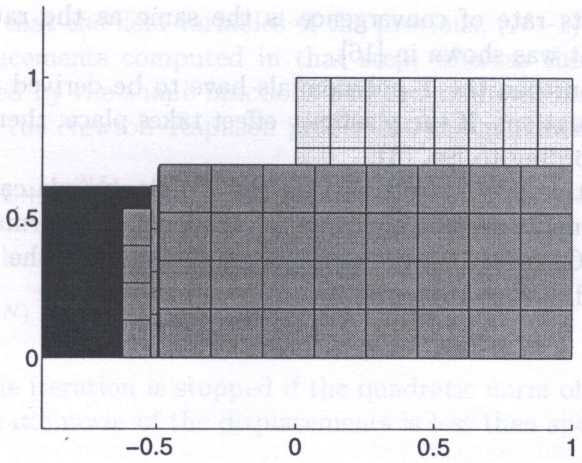
6.2. The second example

For real material with Poisson's ratio not equal to zero, the analytical solution is in the form

$$t_1^0 = \frac{E}{1-\nu^2}\,(1+u_1)\left[u_1 + 0.5u_1^2 + \nu(u_2 + 0.5u_2^2)\right] \tag{59}$$

where u_2 is the transverse contraction of the domain. The requirement for the transverse contraction to be zero also leads to t_1 defined by Eq. (58) in this case.

The numerical FE solution is obtained in four iteration steps (Fig. 2).



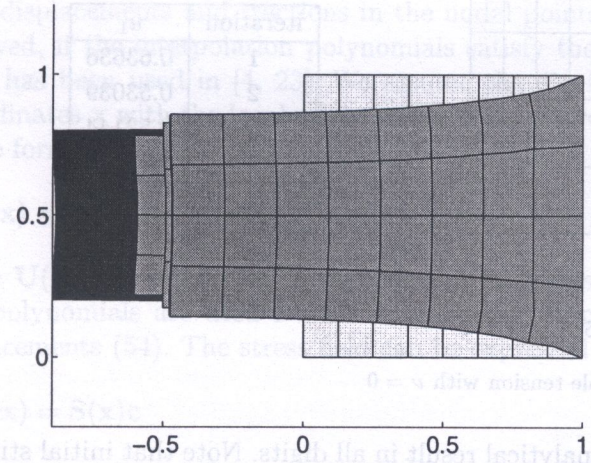
iteration	u_1	u_2
1	0.66364	−0.33581
2	0.54346	−0.24602
3	0.52292	−0.22460
4	0.52147	−0.22179

Fig. 2. Simple tension with $\nu = 0.3$

Note that we have to distinguish between true loading and dead loading [9]. If pressure is to be prescribed, it must be considered as a nominal traction (related to the deformed surface). The traction is changed in each iteration step, too, and the converged solution (0.80628, 0.43218) is reached in the 4-th iteration step.

6.3. The third example

If the domain is fully clamped, the solution corresponding to the second example is obtained in five iteration steps (Fig. 3).

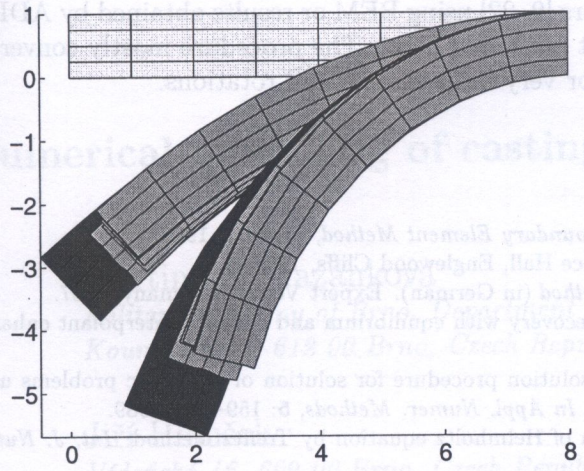


iteration	u_1	u_2
1	0.6044	−0.2206
2	0.5021	−0.1733
3	0.4796	−0.1470
4	0.4758	−0.1429
5	0.4754	−0.1423

Fig. 3. Simple tension with one side fully clamped

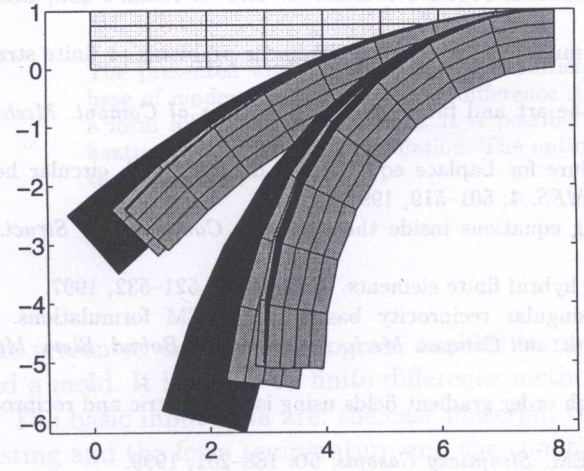
6.4. The fourth example

In the last example, the pure bending of the beam of dimensions 8 by 1 is demonstrated. The results with Poisson's ratio equal to 0.0 (bending moment of 0.023) and 0.3 (bending moment of 0.0195) are given in Fig. 4 and Fig. 5, respectively. Note that a larger moment is required to get the same bending if the transverse contraction is greater. In this case also, the error measures according to Eq. (52) are given. u_1 and u_2 denote the displacement components of the upper end point of the beam.



iteration	u_1	u_2	e
1	0.688	−6.877	7.93E−2
2	1.523	−5.810	1.84E−2
3	1.746	−5.333	4.13E−3
4	1.695	−5.067	2.00E−3
5	1.898	−5.221	4.69E−4
6	2.115	−5.405	9.91E−4
7	2.274	−5.526	4.78E−4
8	2.348	−5.580	1.06E−4
9	2.364	−5.591	4.91E−6

Fig. 4. Pure bending $\nu = 0$



iteration	u_1	u_2	e
1	0.907	−6.160	6.70E−2
2	1.842	−5.790	1.57E−2
3	2.240	−5.640	2.34E−3
4	2.341	−5.595	1.63E−4
5	2.375	−5.587	1.78E−5
6	2.377	−5.581	4.04E−6

Fig. 5. Pure bending $\nu = 0.3$

7. Conclusions

The paper presents the use of the Trefftz polynomials for the development of FEM based on the reciprocity relations. The stiffness matrix of an element is formulated by non-singular boundary integral equations (BEM). A weak form of the equilibrium is used for the inter-domain connectivity relations.

The formulation is shown for linear elasto-plastic problems and for the large strain problems using the total Lagrangian formulation. It leads to the boundary integrals over the original (related) domain computed only once during the whole iterative process and to the volume integrals which can be treated in different ways and thus obtaining the initial stiffness or one step or incremental Newton–Raphson formulations.

The Trefftz polynomials are used also in the post-processing phase of the FEM computations. In this way a smooth fields of the secondary variables (strains, stresses, etc.) can be found in the whole domain (if the material is homogeneous) using the Trefftz polynomials as a local interpolators and the MLS procedure. This approach increases the accuracy of the evaluated fields considerably and the same rate of convergence is obtained as the rate of the primary fields. Similar procedure was not developed yet for finite displacements and classical polynomials are used as interpolators.

The applications for linear problems are shown in the previous papers of the first author et. al. [15, 17, 20]. The applications for large strain, large rotation problems given in this paper show better

convergence than results obtained by other authors [9, 22] using BEM or results obtained by ADINA and MARC FEM programs (these results are not presented here). The procedure mostly converged in one increment with few iteration steps even for very large strains and rotations.

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