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Nordsieck form of multirate integration method for flexible multibody dynamic analysis[1]

Sung-Soo Kim
Department of Mechatronics Engineering, Chungnam National University
220 Kung-dong, Yusong-ku, Taejon, 305-764, Korea

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A Nordsieck form of multirate integration scheme has been proposed for flexible multibody dynamic systems of which motions are represented by large gross motion coupled with small vibration. Based on the conventional flexible multibody dynamics formulation, vibrational modal coordinates with floating reference frame and relative joint coordinates are employed to describe the motion in this research. In the multirate integration, the fast variables of the flexible multibody system are integrated with smaller stepsize, whereas the slow variables are integrated with larger stepsize. It is assumed that vibrational modal coordinates are treated as fast variables, whereas the relative joint coordinates are treated as slow variables to apply multirate integration method. A method that decomposes the equations of motion for flexible multibody systems into a fast system with flexible coordinates and a slow system with joint relative coordinates has been also proposed. The proposed multirate integration method is based on the Adams–Bashforth–Moulton predictor–corrector method and implemented in the Nordsieck vector form. The Nordsieck form of multirate integration method provides effective step-size control and at the same time, inherits the efficiency from the Adams integration method. Simulations of a flexible gun and turret system of a military tank have been carried out to show the effectiveness and efficiency of the proposed method.

1. Introduction

In flexible multibody systems, such as a space satellite system with flexible antenna or lightweight high-speed mechanisms, a flexible body is allowed to have large motion coupled with small vibration. Thus, the equations of motion are expressed in terms of two sets of coordinates; one for describing large translational and rotational motion and the other for expressing vibrational motion. Although they are coupled through inertia matrices in equations of motion, they are varied with different frequency rate. In most cases, the coordinate set associated with deformation can be treated as faster variables. In numerical integration for such systems, it is difficult to solve equations of motion efficiently, since smaller stepsize must be used to meet the accuracy requirement for the solutions of such fast variables.

A multirate integration method is known to be one of the efficient methods, if the system has two or more distinctive different frequencies. In the multirate integration, the fast variables are integrated with smaller stepsize, whereas the slow variables are integrated with larger stepsize. Thus, the multirate integration may provide an efficient means to analyze flexible multibody systems.

Hofer [4] introduced a method with the multistage one-step type that combines implicit and explicit formulas. Gear [3] investigated dual rate methods for ODE systems. Srinivasin [11] applied a multirate integration method to planar flexible multibody systems. Solis [10] applied conventional multirate integration method to a mechanical system with high stiffness elements such as hydraulic subsystems. This kind of system is known to be a force-coupled subsystem. In the structural dynamics area, Daniel [1] investigated subcycling algorithms which apply different integration time steps

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to different sizes of finite elements in a structure. Kim [5] proposed a multirate integration method for multibody dynamic analysis with a decomposed subsystem method. System equations of motion can be decomposed into several small sizes of the equations of motion. Different rate integrator has been then applied to solve each subsystem.

In this paper, a constant stepsize multirate integration scheme is developed for flexible multibody system analysis, based on the Nordsieck form of Adams–Bashforth–Moulton integration method. The system equations of motion for a flexible body are derived using a recursive formulation. A method to decompose flexible multibody system equations of motion is introduced to make a suitable form for the multirate integration. Section 2 explains derivation of equations of motion for flexible multibody systems. Section 3 introduces the constant stepsize multirate integration method that is implemented in the Nordsieck form of Adams–Bashforth–Moulton integrator. The proposed method is applied to a flexible gun–turret system of a military tank to investigate the efficiency in Section 4. Finally, the conclusions are made in Section 5.

2. FLEXIBLE MULTIBODY SYSTEM EQUATIONS OF MOTION

2.1. Equations of motion for a flexible body based on modal coordinates

To describe the configuration of a flexible body, it is necessary to define a set of coordinates that define the position of the every point in a flexible body. Figure 1 shows a flexible body in the deformed state. The X–Y–Z frame is the inertial reference frame. The x_i–y_i–z_i is a body reference frame chosen to define the position and orientation of a flexible body in the undeformed state, relative to the inertial reference frame. The position of a typical point P on a flexible body can be represented in the deformed state as

$$\mathbf{r}^p = \mathbf{r}_i + \mathbf{A}_i \rho'^p = \mathbf{r}_i + \mathbf{A}_i(\mathbf{s}'^p + \mathbf{u}'^p) \tag{1}$$

where $\mathbf{r}_i$ is the position vector of the origin of the body reference frame, $\mathbf{A}_i$ is the orientation matrix of the reference frame, $\mathbf{s}'^p$ is the position vector from the origin of the body reference frame to the point P in the undeformed state, and $\mathbf{u}'^p$ is the deformation displacement of the point P with respect to the body reference frame. In order to make practical analysis, a Ritz approximation is employed to represent the displacement field that can be expressed as a linear combination of the deformation modes, i.e.

$$\mathbf{u}'^p = \Psi_t^p \mathbf{a}_i\,, \tag{2}$$

where Ψ_t^p is a modal matrix whose columns consist of linearly independent deformation modes. Deformation modes represent displacement field that is expressed in terms of nodal coordinates

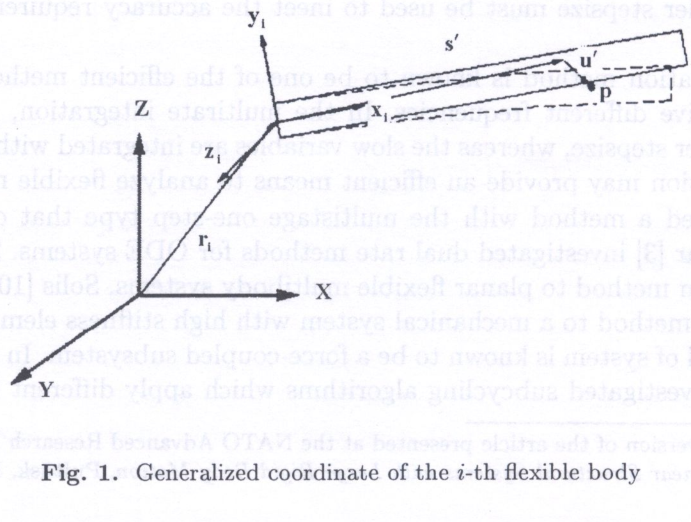

Fig. 1. Generalized coordinate of the i-th flexible body

associated with the point P and $\mathbf{a}_i$ is the modal coordinate vector of the flexible body i. Thus, the position vector of the point P in the flexible body i can be written as

$$\mathbf{r}^p = \mathbf{r}_i + \mathbf{A}_i(\mathbf{s}'^p + \mathbf{\Psi}_t^p \mathbf{a}_i). \tag{3}$$

The variational equation of motion of a flexible body is written as [12]

$$-\int_\Omega \mu \delta\, \mathbf{r}^{PT}\, \ddot{\mathbf{r}}^P \, \mathrm{d}\Omega + \int_\Omega \delta\, \mathbf{r}^{PT}\, \mathbf{f}^P \, \mathrm{d}\Omega + \int_\sigma \delta\, \mathbf{r}^{PT}\, \mathbf{T}^P \, \mathrm{d}\sigma = \int_\Omega \delta\, \boldsymbol{\varepsilon}^{PT}\, \boldsymbol{\tau}^P \, \mathrm{d}\Omega = \delta W_i \tag{4}$$

where μ is material density, $\ddot{\mathbf{r}}^P$ is the acceleration of the point P, $\mathbf{f}^P$ is body force density at the point P, $\mathbf{T}^P$ is surface traction at the point P, $\delta\mathbf{r}^P$ is a virtual displacement of the point P, $\boldsymbol{\tau}^P$ is stress vector, and $\delta\boldsymbol{\varepsilon}^P$ is strain variation vector. The acceleration of the point P can be obtained by differentiating Eq. (1) twice. The virtual displacement of the point P is procured by taking a variation of Eq. (1). Substituting those acceleration and virtual displacement terms into Eq. (4) yields

$$-\left[\, \delta\mathbf{r}_i^T\,,\ \delta\boldsymbol{\pi}_i^T\,,\ \delta\mathbf{a}_i^T \,\right] \left\{ \mathbf{M}_i \begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}_i \\ \ddot{\mathbf{a}}_i \end{bmatrix} + \mathbf{S}_i - \mathbf{Q}_i \right\} = \delta W_i \tag{5}$$

where $\delta\mathbf{r}_i$ is the virtual displacement of the origin of the x_i–y_i–z_i frame, $\delta\boldsymbol{\pi}_i$ is the virtual rotation of the flexible body reference frame, and $\delta\mathbf{a}_i$ is the variation of modal coordinates of the flexible body i. The detail expressions of generalized mass matrix $\mathbf{M}_i$, velocity coupling vector $\mathbf{S}_i$, and generalized force vector $\mathbf{Q}_i$ can be found in [12].

The internal virtual work term in Eq. (5) can be represented in terms of modal coordinate variation and corresponding restoration force vector, using the linear strain-displacement relationship and the linear stress and strain relationship. This derivation is based on small deformation assumption. Thus, the right hand side of Eq. (5) can be written as [12]

$$\delta W_i = \int_\Omega \delta\, \boldsymbol{\varepsilon}^{PT}\, \boldsymbol{\tau}^P = \left[\, \delta\mathbf{r}_i^T\,,\ \delta\boldsymbol{\pi}_i^T\ \ \delta\mathbf{a}_i^T \,\right] \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{K}_i\mathbf{a}_i \end{bmatrix} = \left[\, \delta\mathbf{r}_i^T\,,\ \delta\boldsymbol{\pi}_i^T\,,\ \delta\mathbf{a}_i^T \,\right] \mathbf{U}_i(\mathbf{a}_i) \tag{6}$$

where $\mathbf{K}_i$ is modal stiffness matrix.

Therefore, the following variational form of equations of motion for a flexible body is obtained by substituting Eq. (6) into Eq. (5) as

$$\left[\, \delta\mathbf{r}_i^T,\ \delta\boldsymbol{\pi}_i^T,\ \delta\mathbf{a}_i^T \,\right] \left\{ \mathbf{M}_i \begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}_i \\ \ddot{\mathbf{a}}_i \end{bmatrix} + \mathbf{S}_i - \mathbf{Q}_i + \mathbf{U}_i(\mathbf{a}_i) \right\} = \mathbf{0}. \tag{7}$$

2.2. Flexible multibody system equations of motion using recursive formulation

A flexible multibody system is composed of rigid and flexible bodies that are interconnected by kinematic joints. To derive system equations of motion, the recursive formulation [8] can be used. Figure 2 shows a pair of contiguous flexible bodies. To explain transformation between the reference frames, we denote frame name as follows: The frame F represents the inertial reference frame. The frame F' denotes body reference frame before deformation occurs. The frame F'' is the joint reference frame that is attached to body at the joint definition point. The frame F''' denotes alternate joint reference frame that is fixed to the joint definition points. The frame F''' is also parallel to the frame F' in the undeformed state. This frame is used to define joint rotation due to deformation.

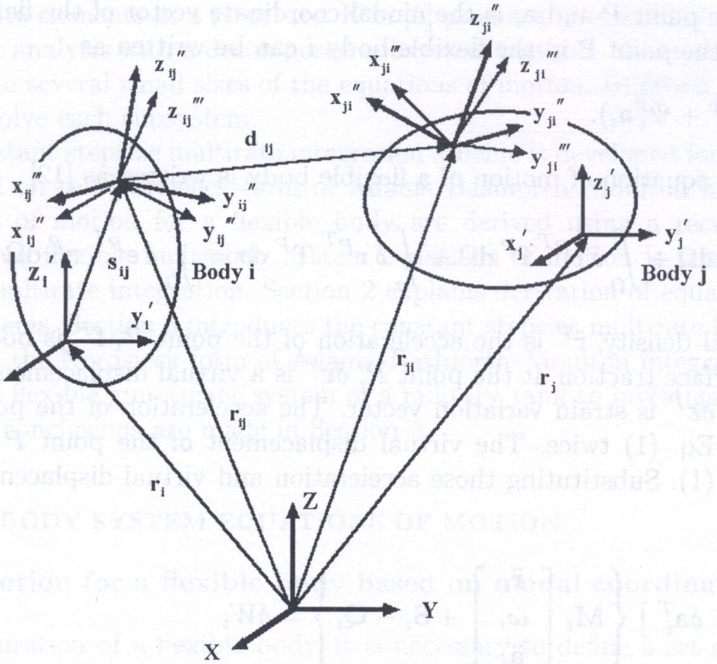

Fig. 2. Adjacent flexible bodies

Using these definitions of the frames, the orientation matrix of body j reference frame can be obtained by the following sequence of transformation matrices,

$$\mathbf{A}_j = \mathbf{A}_i\,\mathbf{B}_{ij}(\mathbf{a}_i)\,\mathbf{C}_{ij}\,\mathbf{D}_{ij}(q_{ij})\,\mathbf{C}_{ji}^T\mathbf{B}_{ji}^T(\mathbf{a}_j)\,, \tag{8}$$

where $\mathbf{A}_i$ is the orientation matrix with respect to the frame F, the matrix $\mathbf{B}_{ij}$ is the transformation matrix from the frame F''' to the frame F' and the matrix $\mathbf{C}_{ij}$ is the transformation matrix from the frame F'' to the frame F'''. $\mathbf{D}_{ij}$ is the transformation matrix from the frame F''_{ji} to F''_{ij} and is also function of joint relative coordinate q_{ij}.

The position vector of the origin of body j reference frame can be written as a vector summation

$$\mathbf{r}_j = \mathbf{r}_i + \mathbf{A}_i(\mathbf{s}'_{ij} + \mathbf{u}'_{ij}) + \mathbf{d}_{ij}(\mathbf{A}_{ij}, q_{ij}) - \mathbf{A}_j(\mathbf{s}'_{ji} + \mathbf{u}'_{ji}) \tag{9}$$

where the vector $\mathbf{d}_{ij}$ is the position vector from the joint definition point P_{ij} to the point P_{ji}, and also it is the function of joint coordinate q_{ij}.

In the recursive formulation [8], a state vector of velocity is introduced to have more efficient expression of velocity relationship between the inboard body i and the outboard body j. If we define the composite velocity vector of the body i as $\mathbf{Y}_i = [\,\dot{\mathbf{r}}_i \;\; \omega_i\,]^T$, then the state vector can be obtained from the following transformation as

$$\hat{\mathbf{Y}}_i = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{r}}_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \omega_i \end{bmatrix} \equiv \mathbf{T}_i\mathbf{Y}_i\,. \tag{10}$$

Thus, the state vector relationship between the body i and the body j can be obtained by differentiating Eq. (9), and augmenting angular velocity relationship between the body i and the body j,

$$\hat{\mathbf{Y}}_j = \hat{\mathbf{Y}}_i + \mathbf{\Lambda}_{ij}\dot{\mathbf{a}}_i + \bar{\mathbf{B}}_{ij}\dot{q}_{ij} - \mathbf{\Lambda}_{ji}\dot{\mathbf{a}}_j\,, \tag{11}$$

where

$$\bar{\mathbf{B}}_{ij} = \begin{bmatrix} \tilde{\mathbf{r}}_{ji}\mathbf{H}_{ij} + \dfrac{\partial \mathbf{d}_{ij}}{\partial q_{ij}} \\ \mathbf{H}_{ij} \end{bmatrix}, \qquad \mathbf{\Lambda}_{ij} = \begin{bmatrix} \mathbf{A}_i\mathbf{\Psi}_t^{ij} + \tilde{\mathbf{r}}_{ij}\mathbf{A}_i\mathbf{\Psi}_r^{ij} \\ \mathbf{A}_i\mathbf{\Psi}_r^{ij} \end{bmatrix},$$

and $\mathbf{H}_{ij}$ is a matrix of which columns consists of unit vectors for joint axis. $\mathbf{\Psi}_t^{ij}$ and $\mathbf{\Psi}_r^{ij}$ are modal matrices of which columns consist of translational and rotational nodal displacements associated with the point P_{ij} on the body i, respectively.

The state variation relationship between the body i and the body j can be expressed in the same manner as the state vector relationships,

$$\delta\hat{\mathbf{Z}}_j = \delta\hat{\mathbf{Z}}_i + \mathbf{\Lambda}_{ij}\delta\mathbf{a}_i + \bar{\mathbf{B}}_{ij}\,\delta\mathbf{q}_{ij} - \mathbf{\Lambda}_{ji}\,\delta\mathbf{a}_j\,. \tag{12}$$

This state variation can be also obtained by following transformation of the virtual displacement and rotation in Cartesian coordinate as

$$\delta\hat{\mathbf{Z}}_i = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{r}}_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\boldsymbol{\pi}_i \end{bmatrix} \equiv \mathbf{T}_i\delta\mathbf{Z}_i\,. \tag{13}$$

Differentiating Eq. (11) yields following acceleration state relationship between the body i and the body j,

$$\dot{\hat{\mathbf{Y}}}_j = \dot{\hat{\mathbf{Y}}}_i + \mathbf{\Xi}_{ij} + \mathbf{\Lambda}_{ij}\,\ddot{\mathbf{a}}_i + \mathbf{D}_{ij} + \mathbf{B}_{ij}\,\ddot{\mathbf{q}}_{ij} - \mathbf{\Xi}_{ji} - \mathbf{\Lambda}_{ij}\,\ddot{\mathbf{a}}_j\,, \tag{14}$$

where $\mathbf{D}_{ij} = \dot{\mathbf{B}}_{ij}\dot{\mathbf{q}}_{ij}$, $\mathbf{\Xi}_{ij} = \dot{\mathbf{\Lambda}}_{ji}\dot{\mathbf{a}}_j$.

The relationship between acceleration state vector and Cartesian acceleration vector is also obtained by differentiating Eq. (10).

$$\dot{\mathbf{Y}}_j = \mathbf{T}_i^{-1}\dot{\hat{\mathbf{Y}}}_i + \dot{\mathbf{T}}_i^{-1}\hat{\mathbf{Y}}_i\,. \tag{15}$$

Once the kinematic relationships between the inboard body i and the outboard body j are obtained as shown in Eqs. (8), (9), (11), and (14), the position, velocity, and acceleration state of each body in the open chain multibody system can be computed recursively from the base body to the tree-end body [8].

The equations of motion for a flexible body shown in Eq. (7) can be now transformed into the one in terms of state variation and acceleration state vector, using Eqs. (13) and (15),

$$\begin{bmatrix} \delta\hat{\mathbf{Z}}_i^T & \delta\mathbf{a}_i^T \end{bmatrix} \left\{ \bar{\mathbf{M}}_i \begin{bmatrix} \dot{\hat{\mathbf{Y}}}_i \\ \ddot{\mathbf{a}}_i \end{bmatrix} - \bar{\mathbf{Q}}_i \right\} = \mathbf{0}, \tag{16}$$

where

$$\bar{\mathbf{M}}_i = \begin{bmatrix} \mathbf{T}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \mathbf{M}_i \begin{bmatrix} \mathbf{T}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{M}}_i^{mm} & \bar{\mathbf{M}}_i^{ma} \\ \bar{\mathbf{M}}_i^{am} & \bar{\mathbf{M}}_i^{aa} \end{bmatrix},$$

$$\bar{\mathbf{Q}}_i = \begin{bmatrix} \mathbf{T}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \left(\mathbf{M}_i \begin{bmatrix} \mathbf{R}_i \\ \mathbf{0} \end{bmatrix} - \mathbf{S}_i + \mathbf{U}_i + \mathbf{Q}_i \right) = \begin{bmatrix} \bar{\mathbf{Q}}_i^z \\ \bar{\mathbf{Q}}_i^a \end{bmatrix}.$$

For the flexible multibody chain system that consists of n+1 number of flexible bodies interconnected by kinematic joints, the system equations of motion in the variation form can be expressed as [8]

$$\sum_{i=0}^{n} \left\{ \delta\hat{\mathbf{Z}}_i^T \left(\bar{\mathbf{M}}_i^{mm}\,\dot{\hat{\mathbf{Y}}}_i + \bar{\mathbf{M}}_i^{ma}\,\ddot{\mathbf{a}}_i - \bar{\mathbf{Q}}_i^z \right) + \delta\mathbf{a}_i^T \left(\bar{\mathbf{M}}_i^{am}\,\dot{\hat{\mathbf{Y}}}_i + \bar{\mathbf{M}}_i^{aa}\,\ddot{\mathbf{a}}_i - \bar{\mathbf{Q}}_i^a \right) \right\} = 0. \tag{17}$$

Once Eq. (12) and Eq. (14) are substituted into Eq. (17) with proper indices from the tree end body to the base body, recursively, then following system equations of motion are obtained in terms of the base body, joint, and deformation modal coordinates [8],

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{P}, \tag{18}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}^{aa} & \mathbf{M}^{aq} & \mathbf{M}^{ay} \\ & \mathbf{M}^{qq} & \mathbf{M}^{qy} \\ symm & & \mathbf{M}^{yy} \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} \mathbf{P}^{a} \\ \mathbf{P}^{q} \\ \mathbf{P}^{y} \end{bmatrix}, \quad \ddot{\mathbf{q}} = \begin{bmatrix} \ddot{\mathbf{a}} \\ \ddot{\mathbf{q}} \\ \hat{\dot{\mathbf{Y}}}_0 \end{bmatrix},$$

the superscripts a, q, and y, denote terms that are related to modal, joint, and base body coordinate, respectively.

3. Nordsieck form of Adams multirate integration

3.1. Nordsieck form of constant stepsize Adams predictor–corrector method

The Adams family of integration methods is known to be a multistep integration method for solving nonlinear differential equations such as equations of motion. Although numerical methods of solving differential equations are explained in the several literatures [2, 9], to explain the constant stepsize Nordsieck form of Adams method, basic formula of the Adams method is briefly mentioned in this paper.

To solve the following first order differential equation with time variable t and state variable x, and an initial value,

$$\dot{x} = f(t, x) \qquad \text{with} \qquad x(t_0) = x_0 \,. \tag{19}$$

The conventional form of the Adams predictor–corrector integration method [2] with constant stepsize can be given by following formulation:

3rd order Adams–Bashforth predictor:

$$\mathbf{x}_{n,(0)} = \mathbf{B}\mathbf{x}_{n-1}\,, \tag{20}$$

4th order Adams–Moulton corrector:

$$\mathbf{x}_{n,(m+1)} = \mathbf{x}_{n,(m)} + \mathbf{c}G(\mathbf{x}_{n,(m)}), \qquad m \geq 0, \tag{21}$$

where

$$\mathbf{x}_{n,(0)} \equiv \begin{bmatrix} x_{n,(0)} & h\,\dot{x}_{n,(0)} & h\,\dot{x}_{n-1} & h\,\dot{x}_{n-2} \end{bmatrix}^T \tag{22}$$

$$\mathbf{x}_{n-1} \equiv \begin{bmatrix} x_{n-1} & h\,\dot{x}_{n-1} & h\,\dot{x}_{n-2} & h\,\dot{x}_{n-3} \end{bmatrix}^T, \tag{23}$$

$$\mathbf{B} = \begin{bmatrix} 1 & \frac{23}{12} & -\frac{16}{12} & \frac{5}{6} \\ 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} \frac{3}{8} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{24}$$

$$G(\mathbf{x}_{n,(m)}) \equiv h\,f(x_{n,(m)}\,,\,t_n) - (3h\,\dot{x}_{n-1} - 3h\,\dot{x}_{n-2} + h\,\dot{x}_{n-3}), \tag{25}$$

and the subscript n represents the n-th time grid point in the descretized solution curve and the subscript m in the prenteses ( ) denotes iteration numbers of the fixed point iteration in the corrector stage, and h is a stepsize in the Adams integration method.

The polynomials of the conventional Adams method, which approximate solution curves, are represented as a linear combination of basis polynomials, so called cardinal functions, based on the Lagrange polynomial formulation. However, Nordsieck made use of Newton's divided difference formula for the interpolated polynomials, instead of the Lagrange formulation. In this formulation, the value of the dependent variables at the current time step is approximated by a combination of the dependent variables and their higher order derivatives at the previous time step.

The constant stepsize Nordsieck form 3rd order predictor–4th order corrector formula can be obtained from the conventional Adams formula given in Eqs. (20) and (21) by the following transformation,

$$\mathbf{a}_{n,(0)} = \mathbf{T}\,\mathbf{x}_{n,(0)}\,, \tag{26}$$

where

$$\mathbf{a}_{n,(0)} \equiv \left[\, x_{n,(0)}\,,\;\; h\,\dot{x}_{n,(0)}\,,\;\; \frac{h^2}{2!}\,\ddot{x}_n\,,\;\; \frac{h^3}{3!}\,\dddot{x}_n \,\right]^T, \tag{27}$$

and

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{3}{4} & -1 & \frac{1}{4} \\ 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix} \tag{28}$$

Thus, the 3rd order Nordsieck form predictor is given as

$$\mathbf{a}_{n,(0)} = \mathbf{A}\,\mathbf{a}_{n-1} \tag{29}$$

which is followed by the 4th order corrector given by

$$\mathbf{a}_{n,(m+1)} = \mathbf{a}_{n,(m)} + \mathbf{l}\,F(\mathbf{a}_{n,(m)}), \qquad m \geq 0. \tag{30}$$

Substituting Eq. (26) into Eqs. (20) and (21) yields the coefficients of the method that can be expressed as

$$\mathbf{A} = \mathbf{T}\mathbf{B}\mathbf{T}^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{31}$$

$$\mathbf{l} = \mathbf{T}\mathbf{c} = \left[\, \frac{3}{8}\,,\;\; 1,\;\; \frac{3}{4}\,,\;\; \frac{1}{6} \,\right]^T \tag{32}$$

$$F(\mathbf{a}_{n,(m)}) = G(\mathbf{T}^{-1}\mathbf{x}_{n,(m)}) = h\,f(x_{n,(m)}, t_n) - h\left(\dot{x}_{n-1} + h\ddot{x}_{n-1} + \frac{h^2}{2!}\dddot{x}_{n-1}\right). \tag{33}$$

There are two important features of the Nordsieck form of Adams integration method to apply for multirate scheme, compared with the conventional Adams method. First, the conventional constant stepsize Adams predictor–corrector method requires a dependent variable and its first order derivatives at a number of previous time steps, whereas the Nordsieck form of Adams predictor–corrector requires a dependent variable and the higher order time derivatives of just one previous time step. Second, as shown in Eq. (27), the Nordsieck vector, obtained using the predictor–corrector formula in Eqs. (29) and (30), contains higher order derivative information at the current step. Since, in the multirate integration, some variables are integrated with smaller stepsizes and some are integrated with larger stepsizes, variable dependency of just one previous time step enables the stepsize to be adjusted more straightforwardly. In the multirate integration, which will be explained in the next subsection, higher order derivative information is also required. This information is readily available using the Nordsieck vector.

3.2. Nordsieck form of multirate integration with decoupled equations of motion

Multirate integration methods are usually used when a system has largely different frequency contents. To apply multirate integration, a system must be decomposed into the one associated with high frequency contents and the others with low frequency ones. The fast variables related to high frequencies are integrated with smaller stepsize, whereas, the slow variables related to low frequencies are integrated with relatively larger stepsize.

In order to explain a method to decouple the flexible multibody equations of motion for the multirate integration method, typical equations of motion are represented symbolically as follows,

$$\begin{bmatrix} m_{ss} & m_{sf} \\ m_{fs} & m_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\bar{q}}_s \\ \ddot{\bar{q}}_f \end{bmatrix} = \begin{bmatrix} \underline{g}^s(\bar{q}_s, \bar{q}_f, \dot{\bar{q}}_s, \dot{\bar{q}}_f, t) \\ \underline{g}^f(\bar{q}_s, \bar{q}_f, \dot{\bar{q}}_s, \dot{\bar{q}}_f, t) \end{bmatrix}, \tag{34}$$

where the left hand side matrix represents a generalized mass matrix and the right hand side vector represents a generalized force vector, variables $\bar{q}$, $\dot{\bar{q}}$, and $\ddot{\bar{q}}$ represent position, velocity, and acceleration, respectively, and subscripts s and f denote slow and fast variables, respectively.

In Eq. (34), the slow and fast acceleration variables are coupled through inertia matrix. To apply Nordsieck form of multirate integration, taking inertia force term to the right hand side and rearranging Eq. (34) yield

$$\begin{bmatrix} m_{ss} & 0 \\ 0 & m_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\bar{q}}_s \\ \ddot{\bar{q}}_f \end{bmatrix} = \begin{bmatrix} \underline{g}^s(\bar{q}_s, \bar{q}_f, \dot{\bar{q}}_s, \dot{\bar{q}}_f, t) - m_{sf}\ddot{\bar{q}}_f \\ \underline{g}^f(\bar{q}_s, \bar{q}_f, \dot{\bar{q}}_s, \dot{\bar{q}}_f, t) - m_{fs}\ddot{\bar{q}}_s \end{bmatrix} \tag{35}$$

Since Eq. (35) is now decoupled, the first and seconnd equations can be solved separately, if each of the inertia force term in the right hand side vector is known. However, high frequency acceleration $\ddot{\bar{q}}_f$ in the right hand side of Eq. (35) is unknown, so the value of $\ddot{\bar{q}}_f$ must be estimated using the approximation method. Since it is high frequency acceleration, approximated value of right hand side term may cause overall solution unstable. In order to overcome this difficulty, first we solve for the second equation of Eq. (35) to obtain expression of $\ddot{\bar{q}}_f$ in terms of $\ddot{\bar{q}}_s$ explicitly, and then substitute this expression into the first equation to eliminate the dependency of $\ddot{\bar{q}}_f$ in the right hand side of Eq. (35). Then the following decoupled equations of motion can be obtained,

$$\begin{bmatrix} m_e & 0 \\ 0 & m_f \end{bmatrix} \begin{bmatrix} \ddot{\bar{q}}_s \\ \ddot{\bar{q}}_f \end{bmatrix} = \begin{bmatrix} \underline{g}^s(\bar{q}_s, \bar{q}_f, \dot{\bar{q}}_s, \dot{\bar{q}}_f, t) - \frac{m_{sf}}{m_{ff}}\underline{g}^f \\ \underline{g}^f(\bar{q}_s, \bar{q}_f, \dot{\bar{q}}_s, \dot{\bar{q}}_f, t) - m_{sf}\ddot{\bar{q}}_s \end{bmatrix}, \tag{36}$$

where $m_e = m_{ss} - \frac{m_{sf}m_{fs}}{m_{ff}}$. Now, the right hand side of the first equation of Eq. (36) does not depend on the acceleration $\ddot{\bar{q}}_f$ anymore. To apply the Nordsieck form of Adams-integrator, Eq. (36) can be changed to the first order form of differential equations as

$$\begin{bmatrix} \dot{\mathbf{x}}_s \\ \dot{\mathbf{x}}_f \end{bmatrix} = \begin{bmatrix} \mathbf{F}^s(\mathbf{x}_s, \mathbf{x}_f, \dot{\mathbf{x}}_f, t) \\ \mathbf{F}^f(\mathbf{x}_s, \mathbf{x}_f, \dot{\mathbf{x}}_s, t) \end{bmatrix} \tag{37}$$

where $\mathbf{x}_s = [\,\bar{q}_s\,,\ \dot{\bar{q}}_s\,]^T$ and $\mathbf{x}_f = [\,\bar{q}_f\,,\ \dot{\bar{q}}_f\,]^T$.

If the Nordsieck form of multirate integration is applied to the above equations, with larger stepsize for the slow variables, and smaller stepsize for the fast variables, the second equation must be evaluated with extrapolated values $\mathbf{x}_s^*$ and $\dot{\mathbf{x}}_s^*$ for slow variables $\mathbf{x}_s$ and their derivatives $\dot{\mathbf{x}}_s$ as follows,

$$\dot{\mathbf{x}}_s = \mathbf{F}^S(\mathbf{x}_s, \mathbf{x}_f, \dot{\mathbf{x}}_f, t), \tag{38}$$

$$\dot{\mathbf{x}}_f = \mathbf{F}^f(\mathbf{x}_s^*, \mathbf{x}_f, \dot{\mathbf{x}}_s^*, t). \tag{39}$$

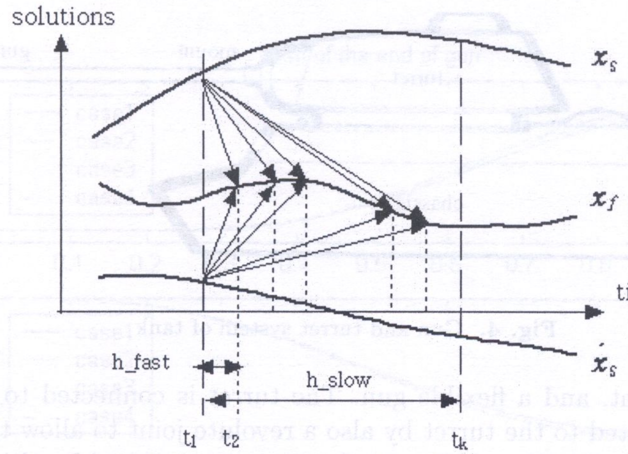

Fig. 3. Schematic diagram of multirate integration

Figure 3 shows schematic diagram of the Nordsieck form multirate integration. Approximated values of slow variables and their derivatives at t_2, t_3, ... t_{k-1} are extrapolated using information at t_1 by the following equations,

$$\mathbf{a}_n^*|_{t_i} = \mathbf{AC}(\alpha)\,\mathbf{a}_n|_{t_i} \qquad t_i = t_1 + (i-1) \times h_{fast}, \qquad i = 2, \ldots, k-1, \tag{40}$$

where

$$\mathbf{C}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & \alpha^3 \end{bmatrix} \quad \text{and} \quad \alpha = \frac{(i-1) \times h_{fast}}{h_{slow}}.$$

Thus, $\mathbf{x}_s^*$ and $\dot{\mathbf{x}}_s^*$ can be easily obtained from only 1st and 2nd equations of Eq. (40) as

$$\mathbf{x}_s^* = \mathbf{x}_s + \alpha h_{slow}\left(\dot{\mathbf{x}}_s + \frac{\alpha h_{slow}}{2}\left(\ddot{\mathbf{x}}_s + \frac{\alpha h_{slow}}{3}\dddot{\mathbf{x}}_s\right)\right) \tag{41}$$

$$\dot{\mathbf{x}}_s^* = \dot{\mathbf{x}}_s + \alpha h_{slow}\left(\ddot{\mathbf{x}}_s + \frac{\alpha h_{slow}}{2}\dddot{\mathbf{x}}_s\right). \tag{42}$$

Fast variables are then integrated using the predictor–corrector method with extrapolated values $\mathbf{x}_s^*$ and $\dot{\mathbf{x}}_s^*$ at the time points t_2, t_3, ... t_{k-1}. Slow and fast variables are then integrated together with the Nordsieck predictor–corrector method at t_k.

In the conventional multirate integration for multibody dynamics [10], one must solve the equations of Eq. (34) simultaneously for the accelerations. After then one must form the first order form differential equations and must extrapolate only slow variables using information from several previous steps. Whereas, in the proposed Nordsieck multirate integration method, Eq. (34) is converted into the decoupled equations of motion as shown in Eq. (36) and it is solved and evaluated separately for slow variable acceleration and for the fast variable acceleration. The first order form differential equations can be then separately formed for fast and slow variables. In this multirate integration, not only slow variables but also their derivatives are efficiently extrapolated using Eqs. (41) and (42), since the Nordsieck vector provides higher order derivatives.

4. A FLEXIBLE GUN–TURRET SYSTEM ANALYSIS WITH THE MULTIRATE INTEGRATION

In order to investigate the efficiency of the proposed multirate integration, the method is applied to a flexible gun–turret system as shown in Figure 4. The flexible gun–turret system [6] consists of

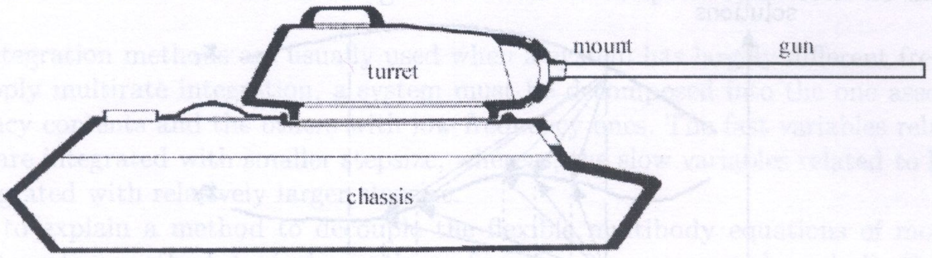

Fig. 4. Gun and turret system of tank

a chassis, a turret, a mount, and a flexible gun. The turret is connected to chassis by a revolute joint. The mount is connected to the turret by also a revolute joint to allow the gun elevation. The flexible gun is fixed to the mount, since the recoil motion is not considered in the simulation. The chassis is assumed to be attached to ground.

For deformation modes of the flexible gun, the first and the second bending modes in vertical plane and in horizontal plane, respectively, are employed. To excite vibration of the gun, the turret and the mount are activated with following torques, simultaneously,

$$T_1(t) = -45500\sin(2\pi t), \tag{43}$$

$$T_2(t) = 17600\sin(2\pi t) - 33451. \tag{44}$$

The torque in Eq. (43) is imposed to the turret to rotate the gun and the torque in Eq. (44) is for elevating the gun.

The resulting flexible gun–turret system equations of motion using the recursive formulation are written as

$$\begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sf} \\ \mathbf{M}_{fs} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_s \\ \ddot{\mathbf{q}}_f \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_s \\ \mathbf{Q}_f \end{bmatrix} \tag{45}$$

where $\ddot{\mathbf{q}}_s$ is the joint acceleration vector for gun rotation and elevation and $\ddot{\mathbf{q}}_f$ is the deformation modal acceleration vector for the flexible gun. As explained earlier in Section 3.2, Eq. (45) can be decoupled to apply the Nordsieck form of multirate integration as

$$\begin{bmatrix} \mathbf{M}_e & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_s \\ \ddot{\mathbf{q}}_f \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_e \\ \mathbf{Q}_f - \mathbf{M}_{fs}\ddot{\mathbf{q}}_s \end{bmatrix} \tag{46}$$

where $\mathbf{M}_e = \mathbf{M}_{ss} - \mathbf{M}_{sf}\mathbf{M}_{ff}^{-1}\mathbf{M}_{fs}$ and $\mathbf{Q}_e = \mathbf{Q}_s - \mathbf{M}_{sf}\mathbf{M}_{ff}^{-1}\mathbf{Q}_f$. Although there is inverse matrix computation, it is not necessary to evaluate inverse matrix, since modal mass matrix with the Eigen modes is identity, i.e., $\mathbf{M}_{ff} = \mathbf{I}$. The right hand side term $\ddot{\mathbf{q}}_s$ in the second equation of Eq. (46) must be extrapolated using Eq. (40) for fast variable acceleration calculation.

In the equations of motion, 4 modal coordinates are assumed to be fast variables, whereas, the rest of variables(i.e., 2 joint variables) in the system equation as slow variables. Four cases of simulations are carried out, according to the stepsize ratio of the slow to the fast variables. In the first case, the same ratio between the fast and the slow variables is used and the stepsize is 0.0001 sec. With the fast variable stepsize being fixed to 0.0001 sec., the slow variable stepsize is increased twice, five-times, and ten-times for the 2nd, 3rd, and 4th simulations, respectively. Figure 5 shows the muzzle displacements of the flexible gun for four cases of simulations. The muzzle velocities of the flexible gun are also shown in Fig. 6. Essentially the same results are obtained from four simulations.

Figure 7 shows the modal coordinate histories for the four cases of the simulations. Comparing with the simulation results for the four cases, the maximum values of modal coordinates for the first bending mode in the horizontal plane are different, although overall trend is the same. The

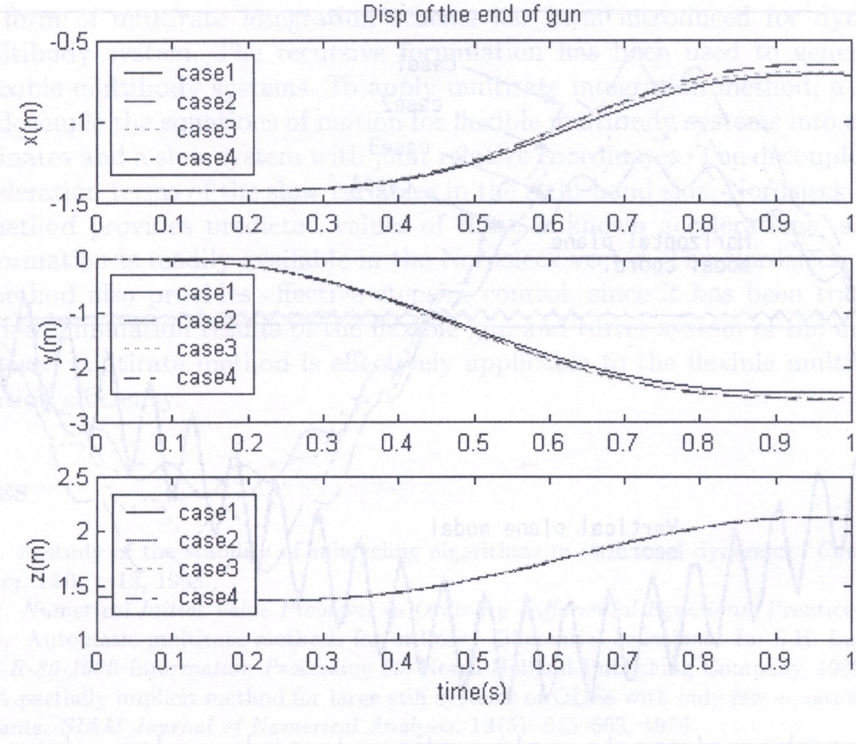

Fig. 5. Muzzle position of the gun

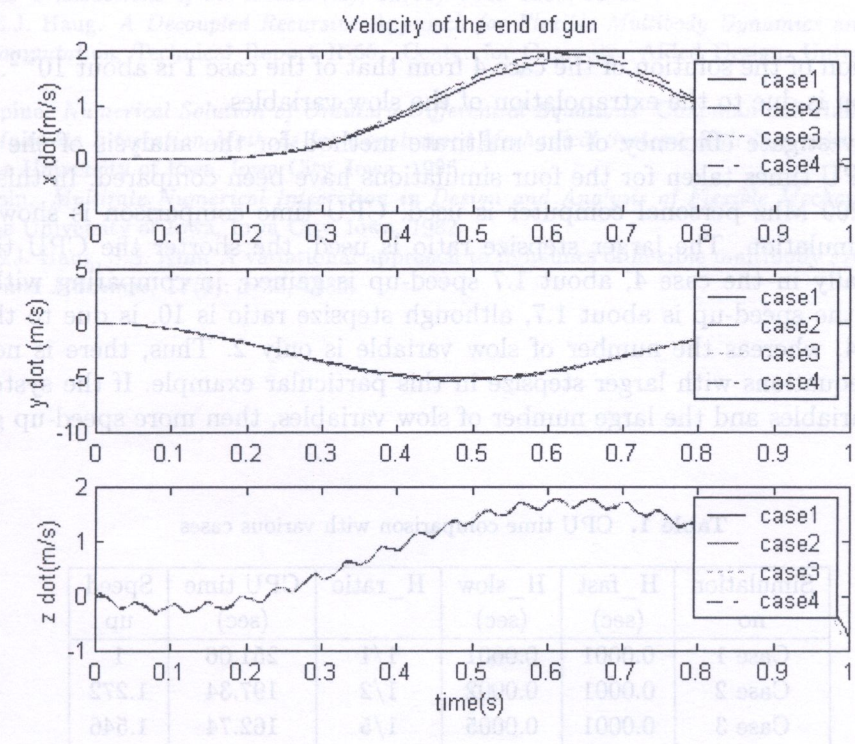

Fig. 6. Muzzle velocity of the gun

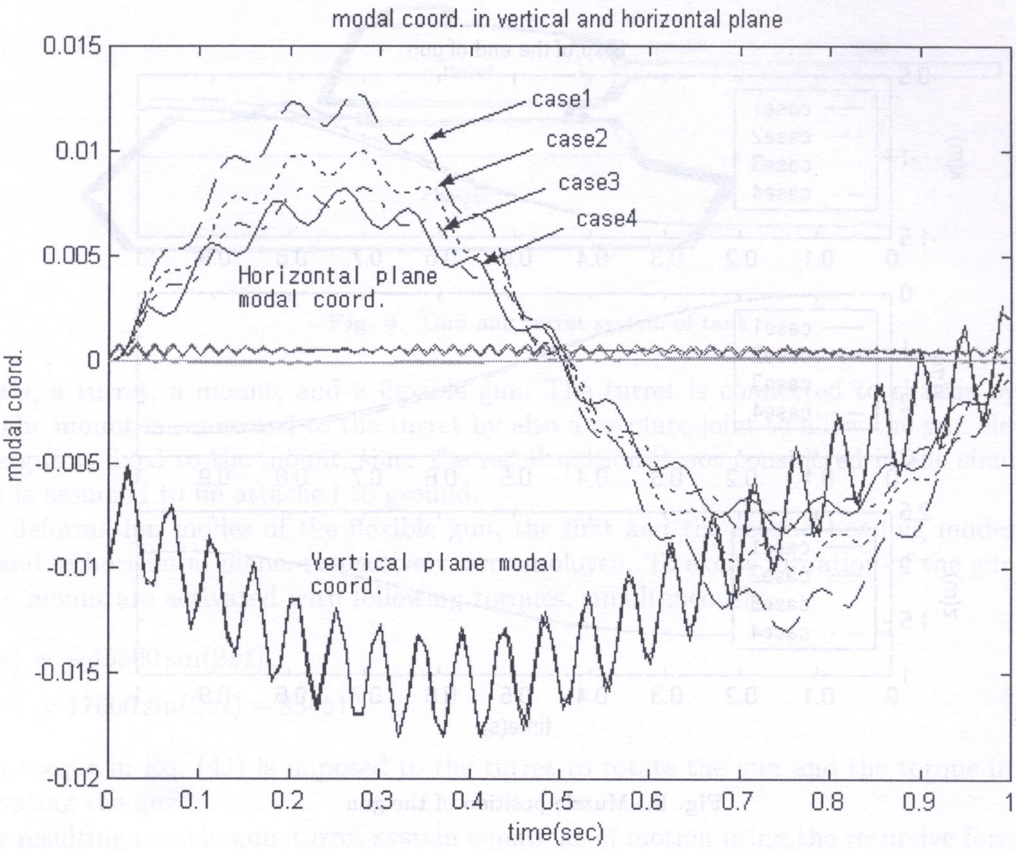

Fig. 7. Time history of modal coordinate

maximum deviation of the solution of the case 4 from that of the case 1 is about 10^{-2}. It is thought that this deviation is due to the extrapolation of the slow variables.

In order to investigate efficiency of the multirate method for the analysis of the flexible gun-turret system, CPU times taken for the four simulations have been compared. In this comparison, Pentium MMX 200 Mhz personal computer is used. CPU time comparison is shown in Table 1 for one-second simulation. The larger stepsize ratio is used, the shorter the CPU time takes, as expected. Especially in the case 4, about 1.7 speed-up is gained, in comparing with the case 1. The reason that the speed-up is about 1.7, although stepsize ratio is 10, is due to the number of fast variables is 4, whereas the number of slow variable is only 2. Thus, there is not much gain to solve smaller equations with larger stepsize in this particular example. If the system has a few number of fast variables and the large number of slow variables, then more speed-up gain could be obtained.

Table 1. CPU time comparison with various cases

Simulation no	H_fast (sec)	H_slow (sec)	H_ratio	CPU time (sec)	Speed up
Case 1	0.0001	0.0001	1/1	251.06	1
Case 2	0.0001	0.0002	1/2	197.34	1.272
Case 3	0.0001	0.0005	1/5	162.74	1.546
Case 4	0.0001	0.001	1/10	148.90	1.696

5. Conclusions

A Nordsieck form of multirate integration scheme has been introduced for dynamic analysis of a flexible multibody system. The recursive formulation has been used to generate equations of motion for flexible multibody systems. To apply multirate integration method, a method has been developed to decouple the equations of motion for flexible multibody systems into a fast system with flexible coordinates and a slow system with joint relative coordinates. The decoupled equations have unknown acceleration terms of the slow variables in the right hand side. Nordsieck form of multirate integration method provides predicted values of those unknown accelerations, since higher order derivative information is readily available in the Nordsieck vector. The Nordsieck form of multirate integration method also provides effective stepsize control, since it has been transformed into an one-step method. Simulation results of the flexible gun and turret system of the military tank show that the proposed multirate method is effectively applicable to the flexible multibody system and improves solution efficiency.

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