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Layout optimization of disks by the use of rigid-plastic element model

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In the paper, the layout optimization of rigid-plastic disks is presented. The method is based on a model where a disk is subdivided into rectangular elements interconnected by normal and shear forces along their edges. Using this model statically admissible stress fields are constructed and the static theorem of limit analysis is applied. Following the concept of porous materials the design variables are the unknown densities of the elements with variable yield stress expressed in terms of the densities. Two complementary optimum design problems are presented. The load intensity is maximized at given intensity of the load and the total amount of material is minimized at prescribed amount of material, respectively. Both problems are expressed in the forms of nonlinear mathematical programming. The application is illustrated by two examples.

1. Introduction

Recently, a number of topology optimization methods has been developed at which the layout problem is transferred into a material distribution problem (see e.g. [1, 2, 6, 9, 11–13, 15]). In some methods it is assumed that the elements of the discretized structures either have or have not material. This zero-one formulation leads, however, to combinatorial problem and the solution requires large computational work. To overcome this difficulty (see e.g. [8]) the concept of porous materials was proposed by Kohn and Strang [7] and applied among others by Yuge, Kikuchi [14] and Maute, Swartz, Ramm [10] to material topology optimization including nonlinear material properties.

In this paper the layout optimization of plastic disks will be presented. The method is based on the concept of porous materials and on the rigid-plastic element model of plane stress problems proposed by Kaliszky [3]. Using this model statically admissible stress fields can be constructed which form the basis of the application of the static theorem of limit analysis and the formulation of two optimum plastic design problems. At the first problem the load intensity is maximized at prescribed amount of material and at the second one the amount of material is minimized at given intensity of the load. The formulation of both problems leads to nonlinear mathematical programming. The solutions provide the optimal material distribution from which the optimal layout of the disk can be constructed.

2. Description of the problem

Consider a homogeneous rigid-perfectly plastic disk with constant thickness v, density ρ_0, mass m_0 and yield stress σ_{y0}, supported along the boundary S_u and loaded by a one-parameter load along the boundary S_P (Fig. 1). The mass forces are disregarded. This ground structure defines the available design domain Ω for the further investigations. To obtain a discrete system the disk is subdivided by perpendicular straight lines into rectangular elements. It is assumed that between the elements along their edges normal and shear forces arise, as it is shown for the element (i,j) in Fig. 2. Following the concept of porous materials the elements have different unknown densities and yield stresses defined by the relations

$$\rho^{(ij)} = x_{ij}\rho_0 , \qquad \sigma_y^{(ij)} = x_{ij}^{\beta}\sigma_{y0} . \tag{1}$$

Here x_{ij} are design variables that characterize the material distribution and β is an appropriately chosen constant.

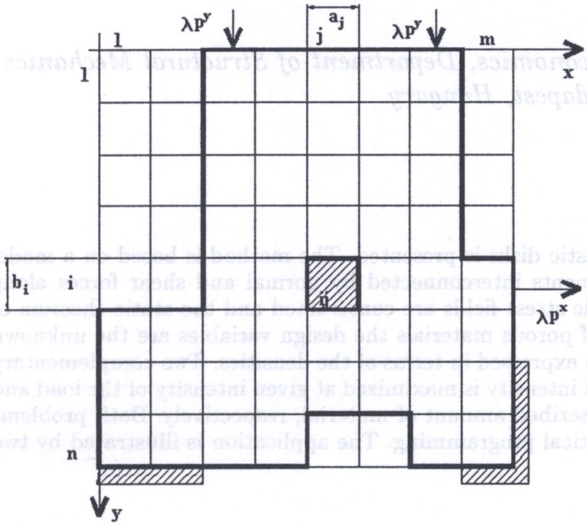

Fig. 1. Available design domain

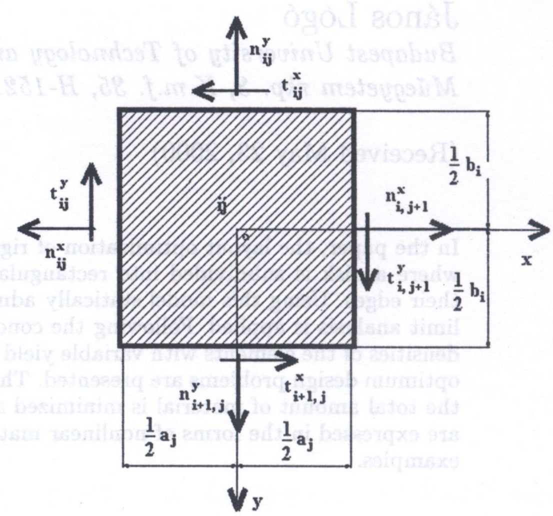

Fig. 2. Rigid-plastic element

By the use of a sixth order Airy stress function statically admissible stress fields can be derived for each element [3]. The stresses σ_x^{ij}, σ_y^{ij}, τ_{yx}^{ij} of the elements are expressed in terms of the forces acting at the edges. Making use of these relationships and assembling the elements fourth degree statically admissible stress fields can be constructed for the entire disk, in which the lines of the mesh are discontinuity lines across which the interior normal stresses are discontinuous. It is to be noted that the thicknesses of the elements can not be considered design variables because then the thicknesses and consequently also the exterior stresses would be discontinuous along the discontinuity lines which would violate the equilibrium conditions [3, 4]. The statically admissible stress fields form the basis for the application of the static theorem of limit analysis.

Using the proposed model the following two material optimization problems will be presented:

Problem 1. At given mass $m = \alpha m_0$, $\alpha \leq 1$ available for the design the material distribution of the disk characterized by the densities ρ_{ij} of the elements is to be determined such that the statically admissible load multiplier λ be maximum.

Problem 2. At given load multiplier $\lambda = \lambda_0$, the material distribution of the disk characterized by the densities ρ_{ij} of the elements is to be determined such that the mass m of the disk be minimum.

In both cases the goal of the design is the determination of the optimal material distribution of the disk with prescribed amount of material and with given intensity of the load, respectively.

3. Equilibrium conditions

Collecting the forces acting along the edges of the elements (i,j), $(i+1,j)$, and $(i,j+1)$ in the vectors

$$\mathbf{q}^*_{i,j} = \begin{bmatrix} n^x_{ij} \\ t^x_{ij} \\ n^y_{ij} \\ t^y_{ij} \end{bmatrix}, \quad \mathbf{q}^*_{i+1,j} = \begin{bmatrix} n^x_{i+1,j} \\ t^x_{i+1,j} \\ n^y_{i+1,j} \\ t^y_{i+1,j} \end{bmatrix}, \quad \mathbf{q}^*_{i,j+1} = \begin{bmatrix} n^x_{i,j+1} \\ t^x_{i,j+1} \\ n^y_{i,j+1} \\ t^y_{i,j+1} \end{bmatrix}, \tag{2}$$

the equilibrium equation of the (i,j) element can be expressed in the following form,

$$\mathbf{N}_{ij}\bar{\mathbf{q}}_{ij} = \mathbf{0}. \tag{3}$$

Here,

$$\mathbf{N}_{ij} = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & a_j & 0 & -b_i & 0 & 0 & 0 & -b_i & 0 & a_j & 0 & 0 \end{bmatrix} \quad \bar{\mathbf{q}}_{ij} = \begin{bmatrix} \mathbf{q}^*_{i,j} \\ \mathbf{q}^*_{i+1,j} \\ \mathbf{q}^*_{i,j+1} \end{bmatrix}. \tag{4}$$

Using these equations the equilibrium equation of the entire disk and the statical boundary conditions can be written,

$$\mathbf{Nq} = \mathbf{0} \qquad \text{in } \Omega, \tag{5}$$

$$\mathbf{q}_0 = \lambda\mathbf{p} \qquad \text{on } S_p\,. \tag{6}$$

Here $\mathbf{q}_0$ contains the element forces acting on the free edges of elements along the boundary.

4. Yield conditions

In case of discrete models the yield conditions can be satisfied only at certain points of the elements. In the following we present the simplest approach when the yield conditions the stresses $\bar{\sigma}^{ij}_x$ $\bar{\sigma}^{ij}_y$ and $\bar{\tau}^{ij}_{xy}$, acting in the middle points of the elements are taken into consideration. Using the stress fields derived in [3] these stresses can be expressed in terms of the element forces as below,

$$\bar{\sigma}^{ij}_x = \frac{1}{2vb_i}\left(n^x_{ij} + n^x_{i,j+1}\right), \tag{7}$$

$$\bar{\sigma}^{ij}_y = \frac{1}{2va_j}\left(n^y_{ij} + n^y_{i+1,j}\right), \tag{8}$$

$$\bar{\tau}^{ij}_{xy} = \frac{3}{16va_j}\left[\left(t^y_{ij} + t^y_{i,j+1}\right) + \frac{a_j}{b_i}\left(t^x_{ij} + t^x_{i,j+1}\right)\right] = \frac{3}{8vb_i}\left(t^x_{ij} + t^x_{i,j+1}\right). \tag{9}$$

In Eq. (9) the last equation of Eq. (3) was substituted.

Using the Huber–Mises–Hencky yield condition,

$$f_{ij} = \left(\bar{\sigma}^{ij}_x\right)^2 + \left(\bar{\sigma}^{ij}_y\right)^2 - \left(\bar{\sigma}^{ij}_x\right)\left(\bar{\sigma}^{ij}_y\right) + 3\left(\bar{\tau}^{ij}_{xy}\right)^2 - \left(\bar{\sigma}^{ij}_y\right)^2 \le 0, \tag{10}$$

and substituting Eqs. (1) and (7)–(9), we get the yield condition of the element (i,j),

$$f_{ij} = \frac{1}{b_i^2}\left(n^x_{ij} + n^x_{i,j+1}\right)^2 + \frac{1}{a_j^2}\left(n^y_{ij} + n^y_{i+1,j}\right)^2 - \frac{1}{a_j b_i}\left(n^x_{ij} + n^x_{i,j+1}\right)\left(n^y_{ij} + n^y_{i+1,j}\right)$$
$$+ \frac{27}{16b_i^2}\left(t^x_{ij} + t^x_{i,j+1}\right)^2 - \left(2v\sigma_{y0}x^\beta_{ij}\right)^2 \le 0. \tag{11}$$

Writing the above yield condition for all elements the yield conditions of the entire disk becomes

$$\mathbf{f}(\mathbf{q}, x_{ij}) \le \mathbf{0}. \tag{12}$$

More accurate solution can be obtained if the yield conditions in the Gaussian points of the elements are satisfied. Here the details are not described, however, at the solution of the examples presented in Section 6. this approach was used.

5. Formulation of the Problem

Using Eqs. (5)–(6) and (12) and the expression of the mass of the disk

$$m = v\rho_0 \sum_{i=1}^{n} \sum_{j=1}^{z} b_i a_j x_{ij}\,, \tag{13}$$

the optimal design problems described in Section 2 can be formulated as below.

Problem 1. Determine the maximum value of the load multiplier λ and the corresponding material distribution of the disk characterized by the design variables x_{ij} at prescribed amount of mass $m = \alpha m_0$, $\alpha \le 1$:

$$\max \lambda \tag{14}$$

$$\mathbf{N}\mathbf{q} = \mathbf{0} \qquad \text{in } \Omega, \tag{15}$$

$$\mathbf{q}_0 = \lambda \mathbf{p} \qquad \text{on } S_p, \tag{16}$$

$$\mathbf{f}(\mathbf{q}, x_{ij}) \le \mathbf{0} \qquad \text{in } \Omega, \tag{17}$$

$$x_{\min} \le x_{ij} \le x_{\max} \qquad \text{in } \Omega, \tag{18}$$

$$v\rho_0 \sum_{i=1}^{n} \sum_{j=1}^{z} b_i a_j x_{ij} - \alpha m_0 \le 0. \tag{19}$$

Problem 2. Determine the minimum mass m and the corresponding material distribution of the disk characterized by the design variables x_{ij} at given load multiplier λ_0:

$$\min m = v\rho_0 \sum_{i=1}^{n} \sum_{j=1}^{z} b_i a_j x_{ij}\,, \tag{20}$$

$$\mathbf{N}\mathbf{q} = \mathbf{0} \qquad \text{in } \Omega, \tag{21}$$

$$\mathbf{q}_0 = \lambda \mathbf{p} \qquad \text{on } S_p, \tag{22}$$

$$\mathbf{f}(\mathbf{q}, x_{ij}) \le \mathbf{0} \qquad \text{in } \Omega, \tag{23}$$

$$x_{\min} \le x_{ij} \le x_{\max} \qquad \text{in } \Omega, \tag{24}$$

$$\lambda - \lambda_0 \le 0. \tag{25}$$

In Eqs. (18) and (24) $x_{\min}$ and $x_{\max}$ denote the prescribed minimum and maximum values of the design variables. It is seen that both problems lead to nonlinear mathematical programming. Through the optimality conditions it is easy to prove that the two problems lead to the same optimal solution.

For the numerical solution a computer program system was elaborated. A standard finite element program was applied for the calculation of the internal forces of the elements with the application of simple force method. A sequential quadratic programming algorithm was used for the optimization. The convergence of the iteration is good. The only limitation is the capacity of the optimizer program. Determining the design variables the optimal layout of the material and the optimal shape of the disks can be constructed.

6. Numerical example

Example 1. Consider a rectangular disk with a hole loaded by concentrated forces $P = 80$ kN and supported at its corners (Fig. 3). The material constants are $E_0 = 200\,000$ MPa and $\sigma_{y0} = 600$ Mpa and the thickness of the disk is $v = 3$ cm. The disk was subdivided into 22 equal elements and at the solution $\beta = 1$ was applied. The problem is to minimize the mass of the disk. Figure 4 shows the optimal material distribution of the disk. At the illustration the standard MS EXCEL representation was used at which the results are presented by surface. That is the reason that in the figure the densities of some elements are not constant. In case of applying fine mesh this result can come directly from the calculation. The symmetry of the Problem 1. is not used the decrease the size of the computational task, because the symmetry of the final solution, what has obtained rather preciously, is one control point of the method.

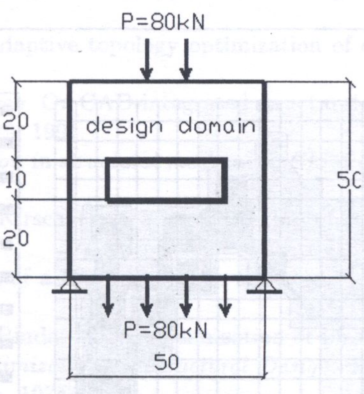

Fig. 3. First example. Available design domain

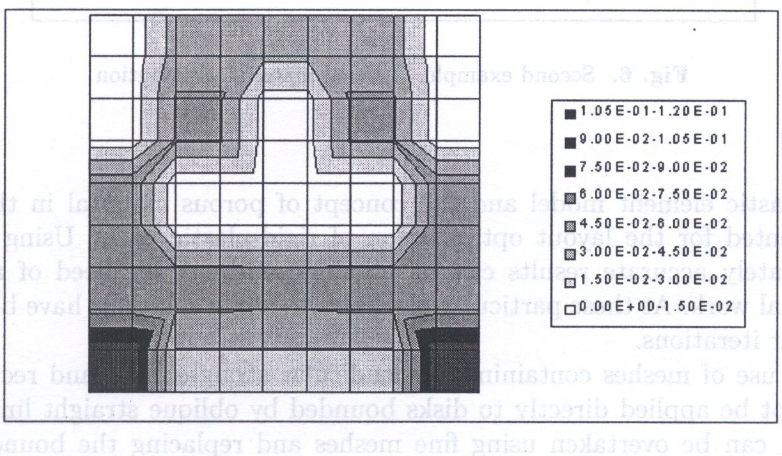

Fig. 4. First example. Optimal material distribution

Example 2. Consider a disk with irregular shape and with a hole is loaded by concentrated forces and supported at its corners (Fig. 5). The minimum mass problem was considered with $\lambda = 12.5$. The material constants are the same as in the first example ($E_0 = 200\,000$ MPa and $\sigma_{y0} = 600$ Mpa) and the thickness of the disk is $v = 3$ cm. The design domain was subdivided into 81 equal elements and $\beta = 1$ was applied. Figure 6 illustrates the optimal material distribution of the disk. Here also the standard MS EXCEL representation was used.

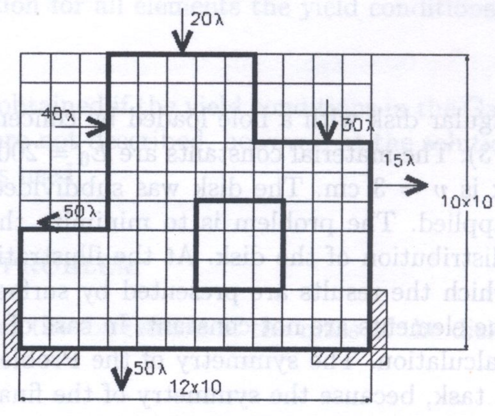

Fig. 5. Second example. Available design domain

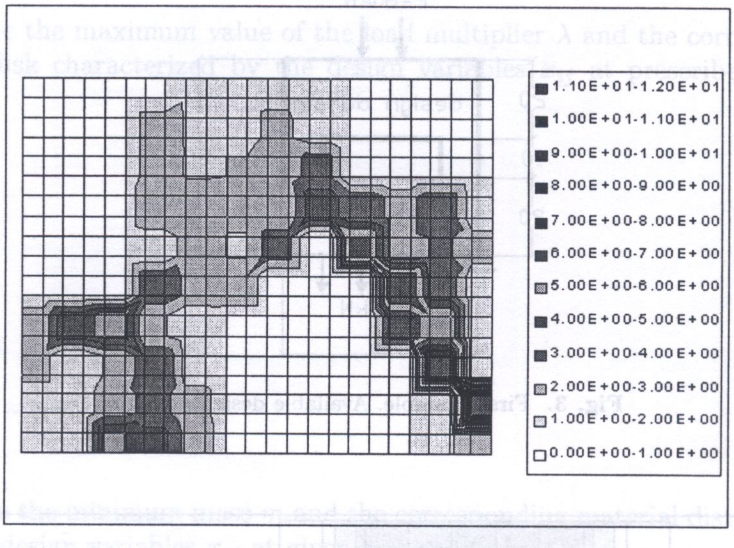

Fig. 6. Second example. Optimal material distribution

7. Conclusion

Using the rigid-plastic element model and the concept of porous material in the paper a simple method was presented for the layout optimization of rigid-plastic disks. Using larger number of elements appropriately accurate results can be obtained without the need of application of too great computational work. At these particular examples the final solutions have been obtained after two hundred major iterations.

Because of the use of meshes containing perpendicular straight lines and rectangular elements the method can not be applied directly to disks bounded by oblique straight lines or curves. This difficulty however, can be overtaken using fine meshes and replacing the boundaries by stepped lines.

Using the finite element method the presented method has been extended to the layout optimization of elasto-plastic disks subjected to multiparameter loading. In the solution bounds on plastic deformation and residual displacement have also been taken into consideration [5].

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