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A recursive method for the dynamic analysis of a system of rigid bodies in plane motion

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In this study, a recursive method for generating the equations of motion of a system of rigid bodies with all common types of kinematic joints in plane motion is presented. The method rests upon the idea of replacing the rigid body by a dynamically equivalent system of particles with added geometric constraints that fix the distance between the particles. Some kinematic constraints due to common types of kinematic joints are automatically eliminated. The concepts of linear and angular momentums are used to generate the rigid body equations of motion without either introducing any rotational coordinates or distributing the external forces and moments over the particles. For the open loop case, the equations of motion are generated recursively along the open chains. For the closed loop case, the system is transformed to open loops by cutting suitable kinematic joints with the addition of cut-joints kinematic constraints. An example of a multi-branch closed-loop system is chosen to demonstrate the generality and simplicity of the proposed method.

1. Introduction

Many formulations have been used to carry out the dynamic analysis of planar mechanisms. Some formulations [3, 11, 8] use a large set of dependent coordinates. The location of each rigid body in the system is described in terms of a set of absolute coordinates; translational and rotational coordinates. The constraint equations are imposed to represent the kinematic joints that connect the rigid bodies. This formulation has the advantage that the constraint equations are easily introduced, however, it has the disadvantage of a large number of coordinates defined. Other formulations [2, 12] describe the configuration of the system in terms of relative coordinates. The location of each body is defined with respect to the adjacent body by means of an angle or a distance depending on the type of the kinematic pair joining the two bodies. Although this formulation yields the constraints as a minimal set of algebraic equations, it has the disadvantage that it does not directly determine the positions of the bodies and points of interest.

Other methods for generating the equations of motion use a two-step transformation. They group the advantages of the simplicity, generality and efficiency. One method [6, 7, 10] uses initially the absolute coordinate formulation. Then, the equations of motion are expressed in terms of the relative joint variables. Another method [1, 9] uses initially a dynamically equivalent constrained system of particles to replace the rigid bodies. The mass associated with each particle is determined as a function of the inertia characteristics of each body. The external forces and couples acting on the body are transformed to equivalent forces and redistributed over the system of particles. The equations of motion are derived using Newton's second law and the Lagrange multiplier technique which results in a large number of differential-algebraic equations. The simplicity and the absence of any rotational coordinates from the final form of the equations of motion are considered the main advantages of this formulation. For the purpose of computational efficiency, the equations of motion

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that are expressed in matrix form in terms of the Cartesian coordinates of the particles are rederived in terms of the relative joint variables. The main disadvantage of these two-step transformations is the necessity to transform at every time step from the joint variables to the original system (the absolute coordinates or the Cartesian coordinates of the particles). This transformation process which is known as the forward process [10,1] is very time consuming.

In the present paper, a recursive method for generating the equations of motion of a system of rigid bodies in plane motion that contains open and/or closed loop systems with the common types of kinematic joints revolute and prismatic is presented. The method rests upon the idea of replacing the rigid body by its dynamically equivalent constrained system of particles discussed in [1, 9] with essential modifications and improvements. Here, a reduced dynamically equivalent system of particles is chosen to improve the efficiency of the formulation without the need to introduce two kinds of particles primary and secondary. The concepts of the linear and angular momentums of the rigid body are used to formulate the rigid body dynamical equations. However, they are expressed in terms of the rectangular Cartesian coordinates of the equivalent system of particles. This groups the advantages of the automatic elimination of the unknown internal constraint forces as in Newton–Euler formulation while expresses the general motion of the rigid body in terms of a set of Cartesian coordinates without introducing any rotational coordinates. This process results in a reduced system of differential-algebraic equations and also eliminates the necessity of distributing the external forces and moments over the particles. For the open loop case, the equations of motion are generated recursively along the open chains instead of the matrix formulation derived in [1]. Geometric constraints that fix the distances between the particles are introduced while some kinematic constraints due to commom types of joints and the associated constraint forces are automatically eliminated by properly selecting the locations of the particles. For the closed loop case, the system is transformed to open loops by cutting suitable polar kinematic joints and introducing the cut-joint kinematic constraints. The special case of a system of rigid rods is also discussed. The dynamic analysis of a multi-branch closed-loop mechanism is carried out to demonstrate the generality and simplicity of the suggested method.

2. THE DYNAMIC MODEL

2.1. Construction of the equivalent system of particles

A system of three particles is chosen to replace the rigid body in plane motion as shown in Fig. 1. The rigid body and its dynamically equivalent system of particles should have the same mass, the same position of the centre of mass and the same polar moment of inertia about an axis perpendicular to the plane of motion. These conditions are expressed as

$$\begin{aligned} m &= \sum_{i=1}^{3} m_i , \\ m\mathbf{r}_G &= \sum_{i=1}^{3} m_i \mathbf{r}_i , \\ I_0 &= \sum_{i=1}^{3} m_i \bar{\mathbf{r}}_i^T \bar{\mathbf{r}}_i , \end{aligned} \tag{1}$$

where m is the mass of the rigid body, $\mathbf{r}_G$ is the position vector of the centre of mass of the body with respect to a body attached coordinate frame, I_0 is the polar moment of inertia of the rigid body about an axis perpendicular to the plane of motion, m_i is the mass of particle i and $\mathbf{r}_i$ is the position vector of particle i of the equivalent system with respect to the body attached coordinate frame. Equations (1) represent a system of 4 algebraic equations in 9 unknowns. Five unknowns may be chosen as free variables and then Eqs. (1) are solved for the rest of the unknowns. The mass

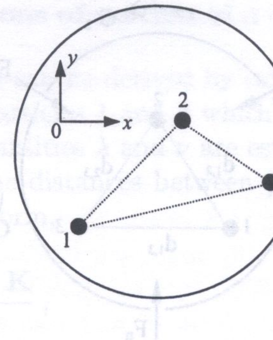

Fig. 1. The rigid body with its equivalent system of three particles

of particle 3 together with the Cartesian coordinates of particles 1 and 3 with respect to the system of axes shown in Fig. 1 are taken as free variables. Especially for $\bar{\mathbf{r}}_1 = \mathbf{0}$, masses m_1 and m_2 as well as the Cartesian coordinates of particle 2 can be estimated using Eqs. (1) in the following closed form,

$$m_2 = \frac{(m\mathbf{r}_G - m_3\bar{\mathbf{r}}_3)^T(m\mathbf{r}_G - m_3\bar{\mathbf{r}}_3)}{I_0 - m_3\bar{\mathbf{r}}_3^T\bar{\mathbf{r}}_3},$$

$$m_1 = m - m_2 - m_3,$$

$$\bar{\mathbf{r}}_2 = \frac{m\mathbf{r}_G - m_3\bar{\mathbf{r}}_3}{m_2}$$

The position vectors $\bar{\mathbf{r}}_1$ and $\bar{\mathbf{r}}_3$ may be chosen conveniently such that particles 1 and 3 are located at both ends of the rod in order to describe the position of the joints. The other remaining free variable m_3 should be chosen such that

$$m_3\bar{\mathbf{r}}_3 \neq m\mathbf{r}_G, \qquad m_3\bar{\mathbf{r}}_3^T\bar{\mathbf{r}}_3 \neq I_0.$$

In the case of a rigid rod of length l and mass m, particles 1 and 3 are located arbitrarely at both ends of the rod while particle 2 is located at the centre of the rod. The equality conditions for the mass, position vector of the centre of mass and moment of inertia can be solved to determine the unknown massses of the particles in the form

$$m_2 = \frac{4}{l^2}(mll_G - I_0),$$

$$m_3 = \frac{2}{l^2}\Big(I_0 - \frac{m}{2}ll_G\Big),$$

$$m_1 = m - m_2 - m_3,$$

where l_G is the location of the centre of mass of the rod with respect to the position of particle 1 and I_0 is the polar moment of inertia about an axis perpendicular to the rod and passing through its end associated with particle 1.

If the rigid body is connected to other bodies in a serial chain by revolute joints, then particles 1 and 3 can be conveniently located at the centers of these joints. Two adjacent rigid bodies contribute to the mass concentrated at the joint connecting them. This process reduces the total number of particles replacing the whole system and leads to the automatic elimination of the constraint forces associated with the revolute joints connecting the bodies.

2.2. Equations of motion of a single rigid body in plane motion

Consider the rigid body shown in Fig. 2 which is acted upon by external forces and force couples. The rigid body is replaced by an equivalent system of three particles. The distances between the

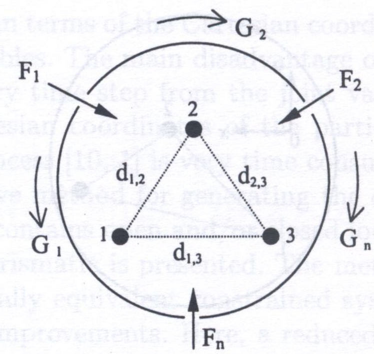

Fig. 2. The rigid body with its equivalent particles and the external forces and couples applied

three particles are invariant as a result of the internal constraint forces existing between them. The vector sum of these unknown internal forces and also the vector sum of their moments about any point equals zero by the law of action and reaction [5]. Then, the linear momentum equation for the whole system of particles yields

$$\mathbf{R} = \sum_{i=1}^{3} m_i \ddot{\mathbf{r}}_i \tag{2}$$

where $\mathbf{R}$ is the vector sum of the external forces acting on the rigid body and $\ddot{\mathbf{r}}_i$ is the acceleration vector of particle i with respect to the global coordinate frame. Also, the angular momentum equation for the whole system of particles with respect to particle 1 takes the form [5]

$$\mathbf{G}_1 = \sum_{i=2}^{3} \mathbf{r}_{i,1} \times m_i \ddot{\mathbf{r}}_i \tag{3}$$

where $\mathbf{G}_1$ is directed perpendicular to the plane of motion and represents the vector sum of the moments of the external forces and force couples acting on the body with respect to the location of particle 1 and $\mathbf{r}_{i,1}$ is the relative position vector between particles i and 1. The distance constraints between the three particles are given as

$$\begin{aligned} &\mathbf{r}_{2,1}^T \mathbf{r}_{2,1} - d_{2,1}^2 = 0, \\ &\mathbf{r}_{3,1}^T \mathbf{r}_{3,1} - d_{3,1}^2 = 0, \\ &\mathbf{r}_{3,2}^T \mathbf{r}_{3,2} - d_{3,2}^2 = 0. \end{aligned} \tag{4}$$

Differentiating Eqs. (4) with respect to time leads to the velocity constraints

$$\begin{aligned} &\mathbf{r}_{2,1}^T \dot{\mathbf{r}}_{2,1} = 0, \\ &\mathbf{r}_{3,1}^T \dot{\mathbf{r}}_{3,1} = 0, \\ &\mathbf{r}_{3,2}^T \dot{\mathbf{r}}_{3,2} = 0. \end{aligned} \tag{5}$$

Differentiating Eqs. (5) with respect to time leads to the acceleration constraints

$$\begin{aligned} &\mathbf{r}_{2,1}^T (\ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1) = -\dot{\mathbf{r}}_{2,1}^T \dot{\mathbf{r}}_{2,1}, \\ &\mathbf{r}_{3,1}^T (\ddot{\mathbf{r}}_3 - \ddot{\mathbf{r}}_1) = -\dot{\mathbf{r}}_{3,1}^T \dot{\mathbf{r}}_{3,1}, \\ &\mathbf{r}_{3,2}^T (\ddot{\mathbf{r}}_3 - \ddot{\mathbf{r}}_2) = -\dot{\mathbf{r}}_{3,2}^T \dot{\mathbf{r}}_{3,2}. \end{aligned} \tag{6}$$

The equations of motion (2), (3) and (6) represent a linear system of 6 scalar algebraic equations that can be solved to determine the unknown acceleration vectors $\ddot{\mathbf{r}}_i$, $i = 1, \ldots, 3$ of the particles at any instant of time.

2.3. The reduced form of equations of motion of a single rigid body

A reduced set of equations of motion can be derived by expressing the position vector of particle 2 in terms of the position vectors of particles 1 and 3 which are located at the centers of the joints. As shown in Fig. 3, two invarient quantities λ and ν are estimated with the aid of the two distance constraints (4a) and (4c) that fix the distances between particles 1 and 2 and particles 3 and 2, respectively, in the following closed form,

$$\bar{\nu} = \frac{\mathbf{r}_{2,1}^T \mathbf{r}_{3,1}}{|\mathbf{r}_{3,1}|}, \quad \bar{\lambda} = \frac{\mathbf{r}_{3,1} \times \mathbf{r}_{2,1} \cdot \mathbf{K}}{|\mathbf{r}_{3,1}|},$$

where $\mathbf{K}$ is the unit vector perpendicular to the plane of motion. Knowing the initial Cartesian coordinates of the particles, λ and ν can be determined using the above relations. In terms of these invarient quantities, the global position vector of particle 2 may be expressed as

$$\mathbf{r}_2 = \mathbf{r}_1 + \nu \mathbf{r}_{3,1} + \lambda \mathbf{K} \times \mathbf{r}_{3,1} \tag{7}$$

where

$$\nu = \frac{\bar{\nu}}{|\mathbf{r}_{3,1}|}, \quad \lambda = \frac{\bar{\lambda}}{|\mathbf{r}_{3,1}|}.$$

The corresponding velocity and acceleration vectors of particle 2 are estimated using time differentiation resulting in the following forms,

$$\dot{\mathbf{r}}_2 = \dot{\mathbf{r}}_1 + \nu \dot{\mathbf{r}}_{3,1} + \lambda \mathbf{K} \times \dot{\mathbf{r}}_{3,1}, \tag{8}$$

$$\ddot{\mathbf{r}}_2 = \ddot{\mathbf{r}}_1 + \nu \ddot{\mathbf{r}}_{3,1} + \lambda \mathbf{K} \times \ddot{\mathbf{r}}_{3,1} = \left((1-\nu) - \lambda \tilde{\mathbf{K}}\right) \ddot{\mathbf{r}}_1 + \left(\nu + \lambda \tilde{\mathbf{K}}\right) \ddot{\mathbf{r}}_3, \tag{9}$$

respectively. Substituting the estimated acceleration vector of particle 2 from Eqs. (9) in Eqs. (2) and (3), the differential equations of motion take the reduced form

$$\mathbf{R} = \left[(m_1 + (1-\nu) m_2)\mathbf{I} - m_2 \lambda \tilde{\mathbf{K}}\right] \ddot{\mathbf{r}}_1 + \left[(m_3 + \nu m_2) + m_2 \lambda \tilde{\mathbf{K}}\right] \ddot{\mathbf{r}}_3, \tag{10}$$

$$\mathbf{G}_1 = m_2 \left[\lambda \mathbf{r}_{2,1}^T + (1-\nu) \tilde{\mathbf{r}}_{2,1}^T\right] \ddot{\mathbf{r}}_1 + \left(m_3 \tilde{\mathbf{r}}_{3,1}^T + m_2 \nu \tilde{\mathbf{r}}_{2,1}^T + m_2 \lambda \mathbf{r}_{2,1}^T\right) \ddot{\mathbf{r}}_3, \tag{11}$$

$$\mathbf{r}_{3,1}^T (\ddot{\mathbf{r}}_3 - \ddot{\mathbf{r}}_1) = -\dot{\mathbf{r}}_{3,1}^T \dot{\mathbf{r}}_{3,1}, \tag{12}$$

where $\mathbf{I}$ is the 2×2 identity matrix and $\tilde{\mathbf{K}}\mathbf{A}$ denotes the cross product $\tilde{\mathbf{K}} \times \mathbf{A}$ and may simply written as $\tilde{\mathbf{A}}$ and where $\tilde{\mathbf{A}} = [-A_2, A_1]^T$ for the 2D vector $\mathbf{A} = [A_1, A_2]^T$. The above linear system of 4 scalar Eqs. (10), (11) and (12) represents the equations of motion of a single floating rigid body in plane motion. It can be solved at every time step to determine the unknown acceleration components of particles 1 and 3. Consequently, Eqs. (9) can be used to determine the acceleration components of

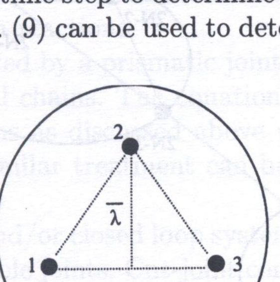

Fig. 3. The rigid body with its equivalent particles indicating the two invariant quantities $\bar{\lambda}$ and $\bar{\nu}$

particle 2. The acceleration components of the particles can be integrated numerically knowing their Cartesian coordinates and velocities at a certain time to determine the positions and velocities at the next time step. Gear's method [4] for the numerical integration of differential-algebraic equations is used to overcome the instability problem resulting during the modeling process of constraint mechanical systems. The rectilinear motion of the particles determines completely the translational and rotational motion of the rigid body. If the rigid body is rotating about a fixed axis, then particle 1 may be located at the intersection of the body with the axis of rotation. In this case, Eqs. (11) and (12) are used to solve for the unknown Cartesian accelerations of particle 3. Then Eq. (10) can be solved to determine the unknown reaction forces at the axis of rotation.

For the case of a rigid rod, $\nu = 0.5$, $\lambda = 0$, and correspondigly Eqs. (10) and (11) result respectively in the following simple forms,

$$\mathbf{R} = (m_1 + \bar{m}_2)\ddot{\mathbf{r}}_1 + (m_3 + \bar{m}_2)\ddot{\mathbf{r}}_3 ,$$

$$\mathbf{G}_1 = \bar{m}_2 \tilde{\mathbf{r}}_{2,1}^T \ddot{\mathbf{r}}_1 + (m_3 \tilde{\mathbf{r}}_{3,1}^T + \bar{m}_2 \tilde{\mathbf{r}}_{2,1}^T)\ddot{\mathbf{r}}_3 ,$$

where $\bar{m}_2 = m_2/2$.

2.4. Equations of motion of a serial chain of rigid bodies

Figure 4 shows a serial chain of N rigid bodies with the equivalent system of $(2N+1)$ particles where connected particles are unified from both bodies. For the last body "N" in the chain, the equations of motion are derived in a similar way as Eqs. (11) and (12) of a single rigid body. The angular momentum equation takes the form

$$\mathbf{G}_{N,2N-1} = m_{2N}\left[-\lambda_N \mathbf{r}_{2N,2N-1}^T + (1-\nu_N)\tilde{\mathbf{r}}_{2N,2N-1}^T\right]\ddot{\mathbf{r}}_{2N-1} + \left(m_{2N+1}\tilde{\mathbf{r}}_{2N+1,2N-1}^T + m_{2N}\nu_N \tilde{\mathbf{r}}_{2N,2N-1}^T + m_{2N}\lambda_N \mathbf{r}_{2N,2N-1}^T\right)\ddot{\mathbf{r}}_{2N+1}$$

where $\mathbf{G}_{N,2N-1}$ is the vector sum of the moments of the external forces and force couples acting on body N with respect to the location of particle $2N-1$. The distance constraint is given as

$$\mathbf{r}_{2N+1,2N-1}^T(\ddot{\mathbf{r}}_{2N+1} - \ddot{\mathbf{r}}_{2N-1}) = -\dot{\mathbf{r}}_{2N+1,2N-1}^T \dot{\mathbf{r}}_{2N+1,2N-1} .$$

Addition of one more body in the chain leads to the inclusion of an angular momentum equation that takes into consideration the contributions of all the ascending bodies in the chain together

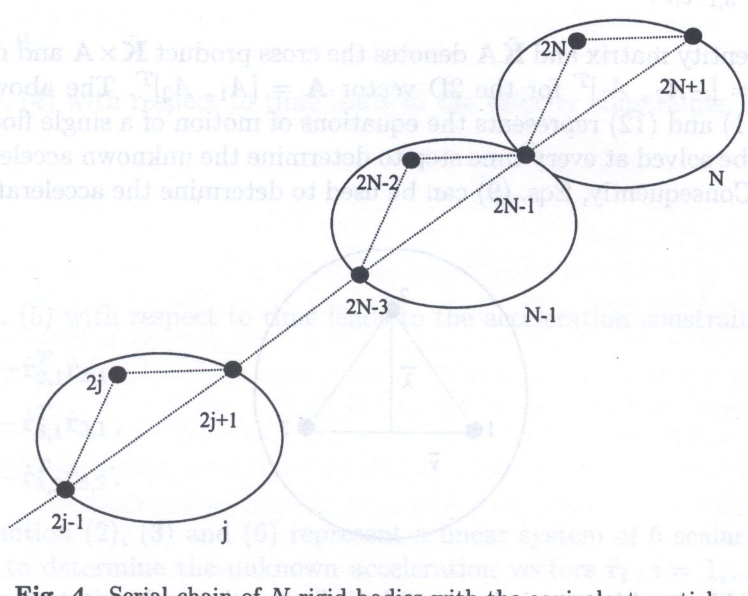

Fig. 4. Serial chain of N rigid bodies with the equivalent particles

with one distance constraint between the particles belonging to this body. These two equations are appended to the equations of motion derived for the leading bodies in the chain. For body j, the appended equations of motion take the form

$$\sum_{i=j}^{N} \mathbf{G}_{i,2j-1} = \sum_{i=j}^{N} (\mathbf{r}_{2i,2j-1} \times m_{2i}\ddot{\mathbf{r}}_{2i} + \mathbf{r}_{2i+1,2j-1} \times m_{2i+1}\ddot{\mathbf{r}}_{2i+1})$$

$$= \sum_{i=j}^{N} m_{2i}\left(-\lambda_i \mathbf{r}^T_{2i,2j-1} + (1-\nu_i)\tilde{\mathbf{r}}^T_{2i,2j-1}\right)\ddot{\mathbf{r}}_{2i-1}$$

$$+ \left(m_{2i+1}\tilde{\mathbf{r}}^T_{2i+1,2j-1} + m_{2i}\nu_i\tilde{\mathbf{r}}^T_{2i,2j-1} + m_{2i}\lambda_i\tilde{\mathbf{r}}^T_{2i,2j-1}\right)\ddot{\mathbf{r}}_{2i+1}\,,$$

$$\mathbf{r}^T_{2j-1,2j+1}\ddot{\mathbf{r}}_{2j-1} + \mathbf{r}^T_{2j+1,2j-1}\ddot{\mathbf{r}}_{2j+1} = -\dot{\mathbf{r}}^T_{2j+1,2j-1}\dot{\mathbf{r}}_{2j+1,2j-1}\,,$$

where

$$\ddot{\mathbf{r}}_{2i} = \left((1-\nu_i) - \lambda_i\tilde{\mathbf{K}}\right)\ddot{\mathbf{r}}_{2i-1} + \left(\nu_i + \lambda_i\tilde{\mathbf{K}}\right)\ddot{\mathbf{r}}_{2i+1}\,,$$

according to Eq. (9) has been used.

If body "j" is the floating base body in the chain then, two linear momentum equations, similar to Eqs. (10), are required to solve for the unknown acceleration components of particle 1. These linear momentum equations equate the sum of the external forces acting on all the bodies in the chain to the time rate of change of the linear momentums of all the equivalent particles that replace the chain and take the form

$$\sum_{i=j}^{N} \mathbf{R}_i = \sum_{i=j}^{N} (m_{2i-1}\ddot{\mathbf{r}}_{2i-1} + m_{2i}\ddot{\mathbf{r}}_{2i}) + m_{2N+1}\ddot{\mathbf{r}}_{2N+1}$$

$$= \sum_{i=j}^{N} \left[\left(m_{2i-1} + (1-\nu_i)m_{2i} - m_{2i}\lambda_i\tilde{\mathbf{K}}\right)\ddot{\mathbf{r}}_{2i-1} + m_{2i}\left(\nu_i + \lambda_i\tilde{\mathbf{K}}\right)\ddot{\mathbf{r}}_{2i+1}\right] + m_{2N+1}\ddot{\mathbf{r}}_{2N+1}\,.$$

In general, for a chain of N bodies, an equivalent system of $(2N+1)$ particles is constructed. By eliminating the coordinates of N particles, we are left with $N+1$ particles and consequently, $2N+2$ unknown acceleration components. To solve for these unknowns, N angular momentum equations can be generated recursively along the chain together with N distance constraints between the pair of particles located on each body. Finally, two linear momentum equations can be used to solve for the unknown acceleration components of particle 1 or for the unknown reaction forces if there is a fixation at point 1. If the chain is closed at its final end, a cut-joint at this end can be used to produce an open chain with the introduction of unknown reaction forces. The cut-joint constraint equations substitute for these unknown reactions.

If bodies "j" and "$j-1$" are connected by a prismatic joint, then the joint is cut and the original serial chain is separated into two serial chains. The equations of motion are generated recursively along each of the resulting serial chains as discussed above with the added kinematic constraints associated with the prismatic joint. Similar treatment can be used in dealing with all other kinds of lower or higher pair kinematic joints.

In the case of a multi-branch open and/or closed loop system, it can be transformed to a system of serial chains (branchs) by cutting suitable joints. Cut-joint constraints and the associated constraint reaction forces are introduced. For the multi-branch system shown in Fig. 5, the system is divided into 4 chains by cutting the connection joints at points 1, 2, 3 and 4. Equivalent particles are conveniently chosen to locate at the positions of the connection joints and in terms of their Cartesian coordinates the cut-joint constraint equations are easily formulated. These kinematic constraints substitute for the unknown constraint reaction forces that appear explicitly in the linear and angular

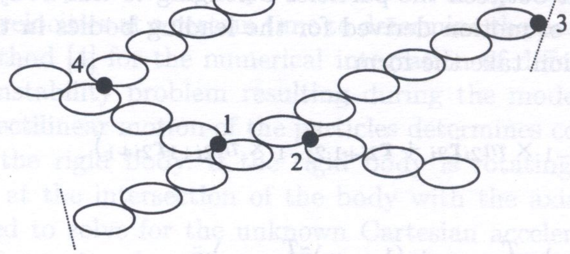

Fig. 5. A multi-branch system indicating four cut-joints

momentum equations. It is shown also in Fig. 5 that some bodies are connected with the others in many points. In such a case, though the number of particles that dynamically replace the rigid body is three which can be used to define two joints, more particles may be added to describe additional joints.

2.5. Dynamic analysis of a multi-branch closed-loop system

The planar three degree-of-freedom platform-type manipulator shown in Fig. 6 is chosen as an example of a multi-closed-loop system. The end-effector platform of the manipulator is constrained by three serial link trains each of which possesses three revolute joints. These three link trains form two independent closed loops. The mechanism is divided into two independent serial branches by cutting the joint at point 5, as indicated in Fig. 6. Each rigid body is replaced by an equivalent system of 3 particles. Two particles are conveniently located at the centers of the joints connecting the adjacent bodies in the chain, while the Cartesian coordinates of the third particle (not shown in Fig. 6) are expressed in terms of the coordinates of the other two particles with the aid of two distance constraints. As shown in Fig. 6, the platform is a multi-joint body, therefore, additional

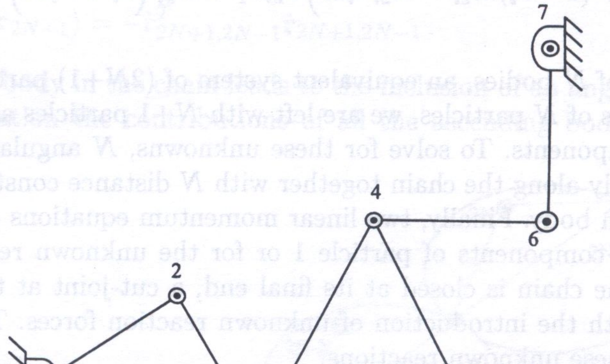

Fig. 6. The manipulator with the equivalent particles

particle 5 is located at the platform to describe the joint connecting the two separated branches. Locating the particles belonging to adjacent bodies together at the connection joints reduces the total number of particles replacing the whole system and leads to the automatic elimination of the kinematic constraints at these joints. An overall equivalent system of 10 particles is constructed. The equations of motion are generated recursively along each branch as discussed in Section 2.4 while additional cut-joint is needed at the grounded end of the first branch (the location of particle 7). The constraint equations due to cut-joints are introduced in the form,

$$\mathbf{r}_5 - \mathbf{r}_8 = \mathbf{0}, \qquad \mathbf{r}_7 - \mathbf{c}_1 = \mathbf{0},$$

where $\mathbf{c}_1$ is a constant known vector. A linear system of 20 algebraic equations can be solved at every time step to determine 16 unknown acceleration components of particles 2,. . . ,9 as well as 4 unknown reaction forces at the cut-joints. The motion is started from the rest position under the action of an external force of magnitude $10N$ applied at the centre of the end-effector in the x-direction as well as gravity forces. Figure 7 presents the trajectory of the platform in the plane of motion. Verification of the results is done by comparison with the absolute coordinate formulation.

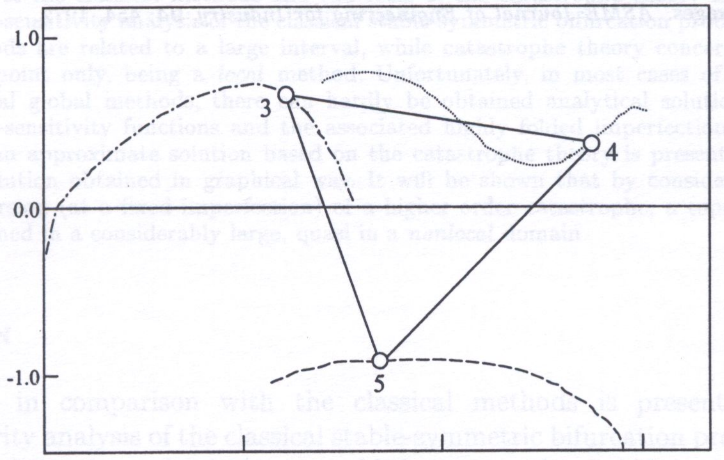

Fig. 7. The trajectory of the platform

3. Conclusions

In the present work, the concepts of linear and angular momentums are used in formulating the rigid body dynamical equations in plane motion. However, they are expressed in terms of the rectangular Cartesian coordinates of a dynamically equivalent constrained system of particles. This groups the advantages of the automatic elimination of the unknown internal constraint forces and describing the general motion of the rigid body in terms of a set of Cartesian coordinates without either introducing any rotational coordinates or distributing the external forces and force couples over the particles. The method results in a reduced system of differential-algebraic equations with the absence of the inconvenient rotational coordinates. The methodology is extended to a system of rigid bodies with all common types of kinematic joints, revolute or prismatic. The case of a system of rigid rods is also discussed.

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