The natural boundary conditions as a variational basis for finite element methods

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Variational formulations that can be employed in the approximation of boundary value problems involving essential and natural boundary conditions are presented in this paper. They are based on trial functions so chosen as to satisfy a priori the governing differential equations of the problem. The essential boundary conditions are used to construct the displacement approximation basis at finite element level. The natural boundary conditions are enforced on average and their integral forms constitute the variational expression of the finite element approach. The shape functions contain both homogeneous and particular terms, which are related through the interpolation technique used. The application in the framework of the finite element method of the approach proposed here is not trouble free, particularly in what concerns the inter-element continuity condition. The Gauss divergence theorem is used to enforce the essential boundary conditions and the continuity conditions at the element boundary. An alternative but equivalent boundary technique developed for the same purpose is presented also. It is shown that the variational statement of the Trefftz approach is recovered when the Trefftz trial functions are so chosen as to satisfy the essential boundary conditions of the problem.

1. INTRODUCTION

The number of trial functions used to construct conforming displacement elements is usually identified with the number of degrees of freedom of the nodes of the element. This option leads, in general, to an indirect and discrete relationship between the internal displacement functions and the element loading. In the hybrid stress approach to the finite element method, the assumed force functions at element level are directly related to the loading, as the homogeneous and particular terms of the stress functions are established separately. In both cases, the relationship between the displacements and the loading is established when the elementary algebraic systems of equations are assembled for the finite element mesh, and after eliminating the free terms present in the hybrid stress models.

There is, however, a direct relation between the internal displacement functions, or the conjugated stress functions, and the load functions at finite level, expressed by the governing differential equations of the problem under analysis. This relationship is considered explicitly in the alternative Trefftz approaches to the finite element method, which, in general preserve the independence of homogeneous and non-homogeneous solution terms.

A survey of Trefftz-type finite element formulations can be found in [12]. In all formulations reported there, the internal displacement field contains a homogeneous part weighted by a vector of undetermined coefficients and a particular term associated with the integral form of the non-homogeneous governing equation. The basic idea supporting most research works is to find a conventional-type finite element formulation described by such displacement fields rather than the assessment of the essential boundary and inter-element continuity conditions [7–9]. The contributions of several authors to the application of the Trefftz method [26], differ from each other mainly in the technique used to enforce the boundary and inter-element continuity conditions [10, 11, 20, 21, 23].

The use of Trefftz displacement trial functions had already been reported in [1] in the derivation of equilibrating stress fields. It is shown there how hybrid stress and hybrid-mixed elements suggested by T.H.H. Pian and other authors [17, 18, 25, 27], can be developed to model plate bending and plane stress elasticity problems, as well as geometrically linear problems on folded structures [2, 13–16].

Some applications of the hybrid stress and hybrid-mixed approaches are reported in [1], using equilibrating stress fields and compatible displacement fields independently assumed on the element boundary. In the subsequent research reported in [3–6], the essential boundary conditions at finite element level are used in an interpolation technique to derive the internal displacement field. The homogeneous term of this field satisfies the homogeneous Lagrange equation and the added particular term contains shape functions that satisfy the non-homogeneous Lagrange-equation. These particular shape functions depend on the geometry and on the loading of the element and are explicitly linked to the homogeneous shape functions.

Beam and plate bending elements based on the application of such internal displacement fields in the context of the conventional finite element displacement are presented in [3, 4]. Unfortunately, this plate bending element does not satisfy the inter-element continuity condition [3, 5]. Therefore, a more general formulation should be developed for such cases in association with a variational statement on the natural boundary conditions, in order to ensure the enforcement of the inter-element and boundary conditions.

To enforce inter-element continuity, the strategy suggested by J. Jirousek is followed here. It consists in introducing at the element boundary an independent conforming displacement 'frame function' field. Conformity of the internal displacement function with the frame function on the element boundary is enforced using the Gauss divergence theorem for the stress variation and the essential boundary conditions at finite element level.

For the convenience of the reader, the interpolation technique introduced in [3] is first recalled. It is shown that a simple modification in the use of the interpolation theory enables the modelling of the element loading at the finite element level and establishes a direct relationship between the internal displacement functions and the load functions at element level. It relates, also, the homogeneous shape functions with the non-homogeneous shape functions.

2. DISPLACEMENT APPROXIMATION BASIS

In the following, Latin indices in brackets range over the nodal points, where Greek indices identify the natural co-ordinates. Consider the variational problem governed by differential equation (1) and boundary conditions (2) and (3),

$$\Delta^{\alpha\beta} u_\alpha = \bar{q}^\beta, \tag{1}$$

$$u_\alpha = \bar{u}_\alpha \quad \text{on } s_u, \tag{2}$$

$$\sigma^{\alpha\beta} n_\beta = \bar{T}^\alpha \quad \text{on } s_\sigma. \tag{3}$$

In Eq. (1), $\Delta^{\alpha\beta}$ is a matrix of differential operators, u_α is the displacement function, and $\bar{q}^\beta$ is the load function. In the boundary conditions (2) and (3), s_u is the surface of the domain on which the displacements are prescribed and s_σ is the surface of the domain on which the forces are prescribed.

After dividing the domain into finite elements, in addition to the requirements on continuity, linearly independence and completeness of the assumed displacement functions, the essential boundary conditions of the element require further that

$$[u_\alpha]_{\theta^\alpha = \theta^\alpha_{(k)}} = u_{\alpha(k)} \tag{4}$$

where $\theta^\alpha(k)$ are the co-ordinates of the element nodes and $u_{\alpha(k)}$ are their displacements.

To describe the internal displacement field, the usual parameter form (5) can be used, where the terms of matrix M_α^ρ are functions of co-ordinates θ^α ($\alpha = 1, 2, 3$), and c_ρ represents the undetermined parameters,

$$u_\alpha = M_\alpha^\rho c_\rho . \tag{5}$$

The number of parameters c_ρ is now chosen to be larger than the usual (nodal) number, in order to identify the load functions. Substitution of Eq. (5) in the differential equation (1) yields the relationship between free parameters and the load functions,

$$\Delta^{\alpha\beta} M_\alpha^\rho c_\rho = \bar{q}^\beta . \tag{6}$$

Equation (6) shows that a subset of free parameters c_ρ can be expressed in terms of the element loading, $\bar{q}^\beta$, as stated by Eq. (7), where $N q_{(p)}$ are loading shape functions and $\bar{q}^{\beta(p)}$ the corresponding nodal values,

$$\Delta^{\alpha\beta} M_\alpha^\rho c_\rho = \bar{q}^\beta = N q_{(p)} \bar{q}^{\beta(p)} . \tag{7}$$

A suitable solution of Eq. (6) or (7) enables the separation of the trial functions present in Eq. (5) into a homogeneous term, with the same dimension as the number of degrees of freedom of the element, and a particular term dependent on the element loading,

$$u_\alpha = M_\alpha^{\eta(m)} c_{\eta(m)} + \bar{M}_{\alpha(p)} \bar{q}^{(p)} , \tag{8a}$$

or

$$u_\gamma = M_\gamma^{\zeta(n)} c_{\zeta(n)} + \bar{M}_{\gamma(p)} \bar{q}^{(p)} . \tag{8b}$$

The free parameters $c_{\zeta(n)}$ can now be eliminated using property (4) at the element nodes, to yield,

$$u_{\alpha(k)} = A_{\alpha(k)}^{\eta(m)} c_{\eta(m)} + \bar{A}_{\alpha(k)(p)} \bar{q}^{(p)} , \tag{9a}$$

$$c_{\zeta(n)} = B_{\zeta(n)}^{\alpha(k)} \left(u_{\alpha(k)} - \bar{A}_{\alpha(k)(p)} \bar{q}^{(p)} \right) , \tag{9b}$$

where $B_{\zeta(n)}^{\alpha(k)}$ is the inverse matrix of $A_{\alpha(k)}^{\eta(m)}$, $A_{\alpha(k)}^{\eta(m)}$ and $\bar{A}_{\alpha(k)(p)}$ are matrices derived from $M_\alpha^{\eta(m)}$ and $\bar{M}_{\alpha(p)}$, respectively, by substituting the co-ordinates of the element nodes, and index n varies as m and k.

Substituting in Eq. (9) the free parameters defined by Eq. (8), the following relationship between the internal displacements u_γ, the nodal displacements $u_{\alpha(k)}$ and the loading $\bar{q}^\beta$, is obtained,

$$u_\gamma = M_\gamma^{\zeta(n)} B_{\zeta(n)}^{\alpha(k)} \left(u_{\alpha(k)} - \bar{A}_{\alpha(k)(p)} \bar{q}^{(p)} \right) + \bar{M}_{\gamma(p)} \bar{q}^{(p)} , \tag{10a}$$

$$u_\gamma = N_\gamma^{\alpha(k)} u_{\alpha(k)} + \bar{N}_{\gamma(p)} \bar{q}^{(p)} , \tag{10b}$$

where the following definitions apply,

$$N_\gamma^{\alpha(k)} = M_\gamma^{\zeta(n)} B_{\zeta(n)}^{\alpha(k)} , \tag{11a}$$

$$\bar{N}_{\gamma(p)} = -M_\gamma^{\zeta(n)} B_{\zeta(n)}^{\alpha(k)} \bar{A}_{\alpha(k)(p)} + \bar{M}_{\gamma(p)} = -N_\gamma^{\alpha(k)} \bar{A}_{\alpha(k)(p)} + \bar{M}_{\gamma(p)} . \tag{11b}$$

In the equations above, $N_\gamma^{\alpha(k)}$ are the homogeneous shape functions, weighted by the nodal displacements the homogeneous term of u_γ, and $\bar{N}_{\gamma(p)}$ are the non-homogeneous shape functions, weighted by the non-homogeneous part of u_γ associated with the element nodal loading.

The displacement approximation basis (10), constructed as suggested above, can be used in the application of the displacement version of the finite element method. As the displacement trial functions defined by Eq. (8) satisfy the non-homogeneous differential equation that governs the problem under analysis, they satisfy also the non-homogeneous equilibrium equations. Therefore, they can be used to derive the force functions needed to implement hybrid stress version of the finite element method and they can, also, be used directly in the application of the Trefftz-type approach.

3. VARIATIONAL APPROXIMATION BASIS

Let us consider a linear elasticity problem described by the domain equilibrium, compatibility and constitutive Eqs. (12)–(14) and the boundary conditions (15) and (16), where $\bar{u}_\alpha$ are the prescribed displacements, $\bar{T}^\alpha$ are prescribed boundary tractions and n_β denotes the unit outward normal vector,

$$\sigma^{\alpha\beta}|_\beta + \bar{f}^\alpha = 0 \qquad \text{in } V, \tag{12}$$

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha|\beta} + u_{\beta|\alpha}\right) \qquad \text{in } V, \tag{13}$$

$$\sigma^{\alpha\beta} = c^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta} \qquad \text{in } V, \tag{14}$$

$$u_\alpha = \bar{u}_\alpha \qquad \text{on } s_u \subseteq s, \tag{15}$$

$$\sigma^{\alpha\beta}n_\beta = \bar{T}^\alpha \qquad \text{on } s_\sigma = s/s_u. \tag{16}$$

The governing system of differential equation (1) is obtained combining Eqs. (12)–(14) to eliminate the stress and strain components as independent variables.

Assume that the virtual variation δu_α of the actual displacement field u_α satisfies the governing differential equation and the kinematic boundary conditions, so that the variational expression leads to an approximation of the mechanical boundary conditions. This variational expression can be stated as the following global enforcement of the mechanical boundary condition,

$$\int_{s_\sigma} \left(\sigma^{\alpha\beta}n_\beta - \bar{T}^\alpha\right)\delta u_\alpha \, \mathrm{d}s = 0. \tag{17}$$

The Trefftz approximation given in [19, 26] reduces to form (17) if the Trefftz trial functions are so chosen as to satisfy the kinematic boundary conditions. As Eq. (17) is simply a global enforcement of the mechanical boundary conditions it can be applied, as in the Trefftz method, to problems for which a variational principle may not be available. It can also be applied as a boundary method in its current form or as a domain method after appropriate transformations.

For this purpose, we introduce the first surface integral in Eq. (17),

$$\int_{s_\sigma} \sigma^{\alpha\beta}n_\beta \, \delta u_\alpha \, \mathrm{d}s = \int_s \sigma^{\alpha\beta}n_\beta \, \delta u_\alpha \, \mathrm{d}s - \int_{s_u} \sigma^{\alpha\beta}n_\beta \, \delta u_\alpha \, \mathrm{d}s, \tag{18}$$

to obtain

$$\delta I = \int_s \sigma^{\alpha\beta}n_\beta \, \delta u_\alpha \, \mathrm{d}s - \int_{s_u} \sigma^{\alpha\beta}n_\beta \, \delta u_\alpha \, \mathrm{d}s - \int_{s_\sigma} \bar{T}^\alpha \, \delta u_\alpha \, \mathrm{d}s. \tag{19}$$

Now, the integral over boundary s can be replaced by its domain equivalent in different manners. Firstly, we can replace it directly using the Gauss integral theorem for the displacement variation,

$$\int_s \sigma^{\alpha\beta}n_\beta \, \delta u_\alpha \, \mathrm{d}s = \int_V \sigma^{\alpha\beta}|_\beta \, \delta u_\alpha \, \mathrm{d}V - \int_V \sigma^{\alpha\beta} \, \delta u_{\alpha|\beta} \, \mathrm{d}V, \tag{20}$$

to obtain the following relation, according to result (12),

$$\int_V \sigma^{\alpha\beta} \, \delta u_{\alpha|\beta} \, \mathrm{d}V - \int_V \bar{f}^\alpha \, \delta u_\alpha \, \mathrm{d}V - \int_{s_u} \sigma^{\alpha\beta}n_\beta \, \delta u_\alpha \, \mathrm{d}s - \int_{s_\sigma} \bar{T}^\alpha \, \delta u_\alpha \, \mathrm{d}s = 0. \tag{21}$$

The variational expression (22) is obtained by introducing the kinematic field equation and enforcing the kinematic boundary condition in Eq. (21),

$$\int_V \sigma^{\alpha\beta} \, \delta\varepsilon_{\alpha|\beta} \, \mathrm{d}V - \int_V \bar{f}^\alpha \, \delta u_\alpha \, \mathrm{d}V - \int_{s_\sigma} \bar{T}^\alpha \, \delta u_\alpha \, \mathrm{d}s = 0. \tag{22}$$

It is mathematically similar to the principle of virtual work but differs in what concerns the subsidiary condition of the variation. In Eq. (22), δu_α must be kinematically admissible and the conjugate $\delta\sigma^{\alpha\beta}$ of δu_α must satisfy the static field equation in the entire domain.

A second possible transformation of Eq. (19) can be obtained by introducing the following modification,

$$\int_s \sigma^{\alpha\beta} n_\alpha \, \delta u_\alpha \, \mathrm{d}s = \delta \int_s \sigma^{\alpha\beta} n_\beta u_\alpha \, \mathrm{d}s - \int_s \left(\delta\sigma^{\alpha\beta} n_\beta \right) u_\alpha \, \mathrm{d}s, \tag{23}$$

and using next the Gauss integral theorem for the stress variation,

$$\int_s \left(\delta\sigma^{\alpha\beta} n_\beta \right) u_\alpha \, \mathrm{d}s = \int_V u_\alpha \, \delta\sigma^{\alpha\beta}\big|_\beta \, \mathrm{d}V + \int_V u_{\alpha|\beta} \, \delta\sigma^{\alpha\beta} \, \mathrm{d}V, \tag{24}$$

as well as the variation of Eq. (12), $\delta\sigma^{\alpha\beta}\big|_\beta + \delta\bar{f}^\alpha = 0$, to obtain

$$-\int_V u_{\alpha|\beta} \, \delta\sigma^{\alpha\beta} \, \mathrm{d}V + \delta \int_s \left(\sigma^{\alpha\beta} n_\beta \right) u_\alpha \, \mathrm{d}s - \int_{s_\sigma} \bar{T}^\alpha \, \delta u_\alpha \, \mathrm{d}s - \int_{s_u} \sigma^{\alpha\beta} n_\beta \, \delta u_\alpha \, \mathrm{d}s = 0. \tag{25}$$

The following variational expression, which is mathematically similar to the modified principle of complementary virtual work but distinct in what concerns the subsidiary conditions of the variation, is obtained introducing the kinematic field equation and the kinematic boundary conditions,

$$-\int_V \varepsilon_{\alpha\beta} \, \delta\sigma^{\alpha\beta} \, \mathrm{d}V + \int_{s_u} \left(\delta\sigma^{\alpha\beta} n_\beta \right) \bar{u}_\alpha \, \mathrm{d}s - \int_{s_\sigma} \delta \left[\left(\sigma^{\alpha\beta} n_\beta - \bar{T}^\alpha \right) u_\alpha \right] \mathrm{d}s = 0. \tag{26}$$

In Eq. (26), $\delta\sigma^{\alpha\beta}$ must satisfy the static field equation and the δu_α associated with $\delta\sigma^{\alpha\beta}$ must be kinematically admissible. Moreover, u_α on s_σ can not be selected independently.

The following functionals can be obtained from Eqs. (22) and (26) for systems for which a potential exists,

$$\delta \left\{ \frac{1}{2} \int_V \varepsilon_{\alpha\beta} c^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} \, \mathrm{d}V - \int_V \bar{f}^\alpha u_\alpha \, \mathrm{d}V - \int_{s_\sigma} \bar{T}^\alpha u_\alpha \, \mathrm{d}s \right\} = 0, \tag{27}$$

$$\delta \left\{ -\frac{1}{2} \int_V \sigma^{\alpha\beta} \left(c_{\alpha\beta\gamma\delta} \right) \sigma^{\gamma\delta} \, \mathrm{d}V - \int_{s_u} \sigma^{\alpha\beta} n_\beta \bar{u}_\alpha \, \mathrm{d}s - \int_{s_\sigma} \left(\sigma^{\alpha\beta} n_\beta - \bar{T}^\alpha \right) u_\alpha \, \mathrm{d}s \right\} = 0. \tag{28}$$

Equations (27) and (28) are mathematically similar to the principle of minimum total potential energy and to the modified principle of complementary energy, respectively.

It can be seen that the approach suggested here can be applied as a boundary method in form (17) or as a domain and mixed method in forms (27) and (28), following a computational technique similar to those used in the implementation of the conventional displacement and hybrid finite element approaches, respectively. It is expected, also, that the different computational techniques will yield the same result provided that all the conditions assumed in the variation are strictly observed.

4. GENERAL APPROXIMATION OF THE ESSENTIAL BOUNDARY CONDITIONS

The geometrical interpolation technique described above may not comply with the boundary conditions and the continuity conditions. In this case, a hybrid technique or a boundary technique at the finite element level can be used to enforce the essential boundary conditions.

As we use displacement field functions that satisfy the governing differential equation and, consequently, the equilibrium equation, the Gauss divergence theorem for the stress variation reduces to the following form, where the integral over surface s is uncoupled in the sum of two integrals on the static and kinematic boundaries,

$$\int_V u_{\alpha|\beta} \, \delta\sigma^{\alpha\beta} \, \mathrm{d}V = \int_s \left(\delta\sigma^{\alpha\beta} n_\beta \right) u_\alpha \, \mathrm{d}s = \int_{s_\sigma} \left(\delta\sigma^{\alpha\beta} n_\beta \right) u_\alpha \, \mathrm{d}s + \int_{s_u} \left(\delta\sigma^{\alpha\beta} n_\beta \right) u_\alpha \, \mathrm{d}s. \tag{29}$$

The following equation is obtained introducing the essential boundary condition (15),

$$\int_V u_{\alpha|\beta}\,\delta\sigma^{\alpha\beta}\,dV = \int_s \left(\delta\sigma^{\alpha\beta}n_\beta\right)u_\alpha\,ds = \int_{s_\sigma}\left(\delta\sigma^{\alpha\beta}n_\beta\right)u_\alpha\,ds + \int_{s_u}\left(\delta\sigma^{\alpha\beta}n_\beta\right)\bar{u}_\alpha\,ds. \tag{30}$$

Instead of result (9), the equation above can be used to calculate in two different ways the free parameters of the trial functions, $c_{\eta(m)}$, that depend on $u_{\alpha(k)}$.

The first option is hybrid and yields the following equation,

$$\int_V u_{\alpha|\beta}\,\delta\sigma^{\alpha\beta}\,dV = \int_{s_\sigma}\left(\delta\sigma^{\alpha\beta}n_\beta\right)u_\alpha\,ds + \int_{s_u}\left(\delta\sigma^{\alpha\beta}n_\beta\right)\bar{u}_\alpha\,ds. \tag{31}$$

The second optional form is encoded by an equivalent boundary description technique and results from the following identity,

$$\int_s \left(\delta\sigma^{\alpha\beta}n_\beta\right)u_\alpha\,ds = \int_{s_\sigma}\left(\delta\sigma^{\alpha\beta}n_\beta\right)u_\alpha\,ds + \int_{s_u}\left(\delta\sigma^{\alpha\beta}n_\beta\right)\bar{u}_\alpha\,ds. \tag{32}$$

As Eqs. (31) and (32) are valid both for the whole domain and for every sub-domain, they can be used also at finite element level to enforce inter-element continuity using the so-called 'frame function' concept in the selection of an independent inter-element displacement field for $\bar{u}_\alpha$.

The application of the trial functions (8) in association with the variational statement of the natural boundary conditions (17) and the equivalence (32) enables the encodement of the method using boundary integral expressions only. The undetermined parameters may then be eliminated from Eq. (32) and the degrees of freedom of the system can be calculated from the variational statement of the natural boundary conditions (17) after collocation at the nodal points.

Another possibility for enforcing the essential boundary conditions consists in the application of the internal displacement functions in form (10) in association with modified version of Eqs. (19) or (27) using the Lagrange-multiplier method. To do so, the difference between the displacement functions on boundary s_u, resulting from the geometrical interpolation in the element and the pre-scribed term, $\bar{u}_\alpha$, can be added to the variational statement using the Lagrange multiplier method.

Different element have been developed for plate bending using the hybrid and boundary techniques described above in addition to the conventional displacement technique. These elements are described in the following section and the results they produce are compared next with the solutions obtained with Trefftz-type hybrid and boundary elements. They are compared also with analytical solutions for different boundary and loading conditions.

The displacement functions used in the implementation of the plate bending elements satisfy the Lagrange equation and the displacement (but not the slope) continuity condition. Therefore, hybrid and boundary techniques are used at the finite element level to enforce conformity and inter-element continuity.

It is noted that the application of the interpolation techniques suggested here to the solution of beam bending problems produces shape functions that satisfy the Lagrange equation and the inter-element continuity. The same results are recovered by applying such functions with the any of the variational statements described above, provided that the 'actual' essential boundary conditions are strictly observed [2, 4, 6].

5. RECTANGULAR PLATE BENDING ELEMENTS

The elements presented here have been developed by the author to solve the Kirchhoff plate bending problem [3–5]. The application is implemented in natural co-ordinates.

In the governing differential equation (33) for the Kirchhoff plate, $u_{\theta3}^0$ is the plate deflection, D is the plate stiffness, $g_i^\alpha g_j^\beta g_k^\gamma g_l^\delta$ represents the product of the components of the contravariant base vectors and δ^{jk} is the Kronecker delta,

$$g_i^\alpha g_j^\beta g_k^\gamma g_l^\delta\,u_{\theta3}^0{}_{|\alpha\beta\gamma\delta}\,\delta^{jk}\delta^{il} = \frac{\bar{q}^{\theta3}}{D}. \tag{33}$$

5.1. Displacement element (ZDE)

This four-node element, represented in Fig. 1, is subjected to a distributed load $\bar{q}^{\theta^3}(\theta^1, \theta^2)$ with nodal values $\bar{q}^{\theta^3(i)}$.
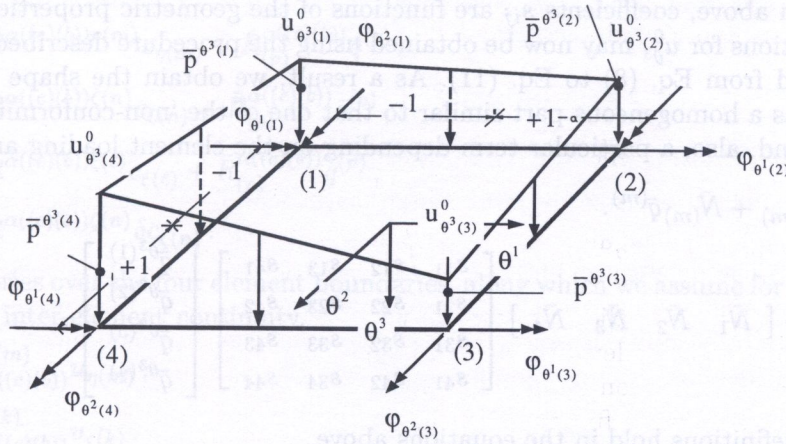

Fig. 1. Co-ordinates and loading for plate bending element

The conventional finite element displacement approach is based on twelve polynomial fields and undetermined parameters. The essential boundary conditions for the deflection and slopes at the four element nodes are sufficient to determine the free parameters. In the present approach a symmetric sixteen-term polynomial is selected in order to accommodate the load function,

$$u_{\theta3}^0(\theta^1, \theta^2) = M^\rho c_\rho,$$ (34a)

$$M^\rho = \begin{bmatrix} 1 & \theta^1 & \theta^2 & (\theta^1)^2 & \theta^1\theta^2 & (\theta^2)^2 & (\theta^1)^3 & (\theta^1)^2\theta^2 & \theta^1(\theta^2)^2 & (\theta^2)^3 \\ & & (\theta^1)^3\theta^2 & \theta^1(\theta^2)^3 & (\theta^1)^2(\theta^2)^2 & (\theta^1)^4\theta^2 & \theta^1(\theta^2)^4 & (\theta^1)^3(\theta^2)^3 \end{bmatrix}.$$ (34b)

The parameters are chosen using the conditions obtained from Eq. (34) as constrained by condition (33). There are, of course, other choices available, for example, the selection of a 22-term polynomial containing all sixth-order terms in the Pascal triangle (21 terms) and term $(\theta^1)^3(\theta^2)^3$. However, in such cases the differential equation does not provide directly the sufficient conditions required for determining the additional parameters, as there are other possible solutions of Eq. (33).

For the parameters implied by definition (34) the loading function $\bar{q}^{\theta^3}(\theta^1, \theta^2)$ is approximated by the following four-term polynomial,

$$\bar{q}^{\theta3}(\theta^1, \theta^2) = \begin{bmatrix} 1 & \theta^1 & \theta^2 & \theta^1\theta^2 \end{bmatrix} \begin{bmatrix} c_{17} \\ c_{18} \\ c_{19} \\ c_{20} \end{bmatrix}.$$ (35)

Using Eqs. (34) and (33) and comparing the coefficients of the trial functions with those of the loading functions, the following decomposition of Trefftz trial functions in homogenous and particular terms is obtained, after eliminating the parameters of the loading functions depending on their load values,

$$u_{\theta3}^0(\theta^1, \theta^2) = M^{\eta(m)} c_{\eta(m)} + \bar{M}_{(p)} \bar{q}^{(p)},$$ (36a)

$$M^{\eta(m)} = \begin{bmatrix} 1 & \theta^1 & \theta^2 & (\theta^1)^2 & \theta^1\theta^2 & (\theta^2)^2 & (\theta^1)^3 & (\theta^1)^2\theta^2 & \theta^1(\theta^2)^2 & (\theta^2)^3 & (\theta^1)^3\theta^2 & \theta^1(\theta^2)^3 \end{bmatrix},$$ (36b)

$$\bar{M}_{(p)} = \frac{1}{4D} \begin{bmatrix} (\theta^1)^2(\theta^2)^2 & (\theta^1)^4\theta^2 & \theta^1(\theta^2)^4 & (\theta^1)^3(\theta^2)^3 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{41} \\ s_{21} & s_{22} & s_{23} & s_{42} \\ s_{31} & s_{32} & s_{33} & s_{43} \\ s_{41} & s_{42} & s_{34} & s_{44} \end{bmatrix}. \tag{36c}$$

In the definition above, coefficients s_{ij} are functions of the geometric properties of the element.
The shape functions for $u^0_{\theta3}$ may now be obtained using the procedure described above, following
the steps described from Eq. (8) to Eq. (11). As a result, we obtain the shape functions for the
displacement $u^0_{\theta3}$ as a homogeneous part similar to that one of the 'non-conforming' 12 DOF plate
bending element and, also, a particular term depending on the element loading and geometry,

$$u^0_{\theta3} = N^{\eta(m)} u_{\eta(m)} + \bar{N}_{(m)} \bar{q}^{(m)}. \tag{37a}$$

$$\bar{N}_{(m)} \bar{q}^{(m)} = \frac{1}{4D} \begin{bmatrix} \bar{N}_1 & \bar{N}_2 & \bar{N}_3 & \bar{N}_4 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{41} \\ s_{21} & s_{22} & s_{23} & s_{42} \\ s_{31} & s_{32} & s_{33} & s_{43} \\ s_{41} & s_{42} & s_{34} & s_{44} \end{bmatrix} \begin{bmatrix} \bar{q}^{\theta3(1)} \\ \bar{q}^{\theta3(2)} \\ \bar{q}^{\theta3(3)} \\ \bar{q}^{\theta3(4)} \end{bmatrix}. \tag{37b}$$

The following definitions hold in the equations above,

$$\begin{aligned}
\bar{N}_1 &= \left(1 - (\theta^1)^2\right)\left(1 - (\theta^2)^2\right) \\
\bar{N}_2 &= \theta^2 \left(1 - (\theta^1)^2\right)^2 \\
\bar{N}_3 &= \theta^1 \left(1 - (\theta^2)^2\right)^2 \\
\bar{N}_4 &= \theta^1\theta^2 \left(1 - (\theta^1)^2\right)\left(1 - (\theta^2)^2\right).
\end{aligned} \tag{38}$$

We obtain from Eqs. (37) and (38) the particular shape functions for the uniformly distributed
load case, $\bar{q}^{\theta3(i)} = \bar{q}$. The particular term is defined by

$$\bar{N}_{(m)} \bar{q}^{(m)} = \frac{\bar{q}}{8D} \left(1 - (\theta^1)^2\right)\left(1 - (\theta^2)^2\right). \tag{39}$$

The derivation of the element matrices is based on the application of the variational state-
ment (27), which leads to the following FEM relations for rectangular elements,

$$\delta \left\{ \frac{1}{2} u_{\eta(m)} k^{\eta(m)\zeta(n)} u_{\zeta(n)} + \frac{1}{2} u_{\eta(m)} \bar{f}1^{\eta(m)} + \frac{1}{2} u_{\zeta(n)} \bar{f}1^{\zeta(n)} + \frac{1}{2}c - u_{\eta(m)} \bar{f}2^{\eta(m)} \right\} = 0. \tag{40}$$

In the equation above, the element stiffness matrix is defined by Eq. (41) and the term of the
equivalent nodal forces associated with the particular term of the displacement function is defined
by Eq. (42),

$$k^{\eta(m)\zeta(n)} = \int_A N^{\eta(m)}_{,\alpha\beta} E^{\alpha\beta\gamma\delta} N^{\zeta(n)}_{,\gamma\delta} \, dA, \tag{41}$$

$$\bar{f}1^{\eta(m)} = \int_A N^{\eta(m)}_{,\alpha\beta} E^{\alpha\beta\gamma\delta} \bar{N}_{(p),\gamma\delta} \bar{q}^{(p)} \, dA. \tag{42}$$

In Eq. (40), c is a constant term and $f2$ is the equivalent nodal force resulting from the application
of the conventional displacement finite element model,

$$c = \int_A \bar{q}^{(p)} \bar{N}_{(p),\alpha\beta} E^{\alpha\beta\gamma\delta} \bar{N}_{(q),\gamma\delta} \bar{q}^{(p)} \, dA, \tag{43}$$

$$\bar{f}2^{\eta(m)} = \int_A N^{\eta(m)} \bar{N}_{(p)} \bar{q}^{(p)} \, dA. \tag{44}$$

5.2. Hybrid element (TFE)

This element derives from the variational basis (19) and condition (31). The internal displacement field is approximated by trial function (36). At the element boundary, the conjugate vector of boundary tractions is determined from Eq. (36) and can be written using different index notations, as follows,

$$T^{\alpha((e)(b))} = R^{\alpha((e)(b))\eta(m)} c_{\eta(m)} + \bar{R}_{(p)}^{\alpha((e)(b))} \bar{q}^{(p)}, \tag{45a}$$

$$T^{\alpha((e)(b))} = R^{\alpha((e)(b))\zeta(n)} c_{\zeta(n)} + \bar{R}_{(p)}^{\alpha((e)(b))} \bar{q}^{(p)}, \tag{45b}$$

$$T^{\alpha((e)(b))} = R^{\alpha((e)(b))\xi(\ell)} c_{\xi(\ell)} + \bar{R}_{(p)}^{\alpha((e)(b))} \bar{q}^{(p)}, \tag{45c}$$

$$\delta T^{\alpha((e)(b))} = R^{\alpha((e)(b))\zeta(n)} \delta c_{\zeta(n)}. \tag{45d}$$

Index $((e)(b))$ varies over the four element boundaries, along which we assume for $\bar{u}_\alpha$ a displacement field that ensures inter-element continuity,

$$\bar{u}_{\alpha((e)(b))} = L_{\alpha((e)(b))}^{\eta(m)} u_{\eta(m)}, \tag{46a}$$

$$\bar{u}_{\alpha((e)(b))} = L_{\alpha((e)(b))}^{\zeta(k)} u_{\zeta(k)}. \tag{46b}$$

Matrix $L_{\alpha((e)(b))}^{\eta(m)}$ contains shape functions on the element boundaries. It is built here using Hermite polynomials.

In some Trefftz-type formulations the conformity between the internal displacement field (36) and the boundary displacement (46) is enforced in a weak weighted residual sense [7, 8, 11], and in others in a least square sense [10, 20]. Here, the undetermined parameters $c_{\eta(m)}$, or $c_{\xi(\ell)}$, may now be eliminated using Eq. (31) and hybrid-type technique that leads straightforwardly to the following relation, where $H_{\xi(\ell)\zeta(n)}$ is the inverse of matrix (48),

$$c_{\xi(\ell)} = H_{\xi(\ell)\zeta(n)} \left(T^{\eta(m)\zeta(n)} u_{\eta(m)} - \bar{H}_{(p)}^{\zeta(n)} \bar{q}^{(p)} \right), \tag{47}$$

$$H^{\eta(m)\zeta(n)} = \int_A M_{,\alpha\beta}^{\eta(m)} E^{\alpha\beta\gamma\delta} M_{,\gamma\delta}^{\zeta(n)} \, dA. \tag{48}$$

The remaining terms in Eq. (47) are defined as follows,

$$T^{\eta(m)\zeta(n)} = \int_s R^{\alpha((e)(b))\zeta(n)} L_{\alpha((e)(b))}^{\eta(m)} \, ds, \tag{49}$$

$$\bar{H}_{(p)}^{\zeta(n)} = \int_A \bar{M}_{(p),\alpha\beta} E^{\alpha\beta\gamma\delta} M_{,\gamma\delta}^{\zeta(n)} \, dA. \tag{50}$$

Definitions (45) and (47) for the boundary traction vector and for parameters $c_{\xi(\ell)}$, respectively, can be used to derive the following 'force-displacement' finite element relationship using the variational statement of (19),

$$k^{\varsigma(k)\eta(m)} u_{\eta(m)} + \bar{r}^{0\varsigma(k)} = \bar{r}^{1\varsigma(k)}. \tag{51}$$

In this equation, the symmetric finite element stiffness matrix and the equivalent nodal force vectors are defined by the equations below,

$$k^{\varsigma(k)\eta(m)} = T^{\varsigma(k)\xi(\ell)} H_{\xi(\ell)\zeta(n)} T^{\zeta(n)\eta(m)}, \tag{52}$$

$$\bar{r}^{0\varsigma(k)} = -T^{\varsigma(k)\xi(\ell)} H_{\xi(\ell)\zeta(n)} \bar{H}_{(m)}^{\zeta(n)} \bar{q}^{(m)} + \bar{T}_{(m)}^{\varsigma(k)} \bar{q}^{(m)}, \tag{53}$$

$$\bar{r}^{1\varsigma(k)} = \int_{s_\sigma} \bar{T}^{\alpha((e)(b))} L_{\alpha((e)(b))}^{\varsigma(k)} \, ds, \tag{54}$$

$$\bar{T}^{\varsigma(k)} = \int_{s_\sigma} \bar{R}_{(m)}^{\alpha((e)(b))} L_{\alpha((e)(b))}^{\varsigma(k)} \, ds. \tag{55}$$

5.3. Boundary element (JFE)

This element has the same computational basis as element TFE. It differs only in what concerns the elimination of the undetermined parameters $c_{\eta(m)}$, or $c_{\xi(\ell)}$, using Eq. (32) in a hybrid boundary technique equivalent to the one described above.

To apply this technique, we derive first from $u_{\theta 3}^0$, Eq. (36), the corresponding displacement functions $u_{\alpha((e)(b))}$ on the element boundary,

$$u_{\alpha((e)(b))} = M_{\alpha((e)(b))}^{\eta(m)} c_{\eta(m)} + \bar{M}_{\alpha((e)(b))(p)} \bar{q}^{(p)}. \tag{56}$$

Using these functions and the boundary tractions (45) we can evaluate the boundary integrals present in Eq. (32), which leads to the equivalently hybrid $H^{\eta(m)\zeta(n)}$ and $\bar{H}_{(p)}^{\zeta(n)}$ matrices,

$$H^{\eta(m)\zeta(n)} = \int_s M_{\alpha((e)(b))}^{\eta(m)} R^{\alpha((e)(b))\zeta(n)} \, ds, \tag{57}$$

$$\bar{H}_{(p)}^{\zeta(n)} = \int_s \bar{M}_{\alpha((e)(b))(p)} R^{\alpha((e)(b))\zeta(n)} \, ds. \tag{58}$$

The elimination of the undetermined parameters $c_{\eta(m)}$, or $c_{\xi(\ell)}$, using Eq. (32) recovers relation (47), so that the application can be carried out in an analogous way.

6. Numerical results

The first set of examples address the analysis of a square plate with various boundary conditions and loads. Besides comparing the finite element results with the analytical solutions available, attention is paid also to the convergence obtained with different finite element meshes. The second and third sets of tests are chosen to compare the solutions obtained with the procedures described here with Trefftz-type finite element solutions.

Three testing problems defined on a square plate are represented in Fig. 2: a) a simply supported plate subject to an hydrostatic pressure; b) a plate with two simply supported edges and two clamped edges subject to a sinusoidal load, and; c) a simply supported plate subject to pyramid-type load.

The following normalised values for the transverse displacement, bending and torsion moments and shear force are used below,

$$w(x^1, x^2) = u_{x3}^0(x^1, x^2) D / (\bar{q}^0 a^4) \tag{59}$$

$$m^{ij}(x^1, x^2) = M^{x^i x^j}(x^1, x^2) / (\bar{q}^0 a^2) \tag{60}$$

$$q^j(x^1, x^2) = Q^{x^i}(x^1, x^2) / (\bar{q}^0 a) \tag{61}$$

The results obtained for the plate tests defined in Fig. 2 with different elements and meshes are summarised in Tables 1–3.

The approach presented here is assessed also using the results obtained with the hybrid-Trefftz elements HTQI1 and HTQI2 due to J. Jirousek [8]. These quadrilateral elements have mid-side and corner nodes, for a total of 24 degrees of freedom and 21 trial functions. The frame functions are polynomials of degree five for the transverse displacement and of the second degree for the slopes. This relative assessment is based on the number of degrees of freedom.

Table 4 shows the results obtained for the displacement and the bending moment at the centre of the square plate, simply supported and subject to a uniform load, $\bar{q}^0$. It shows also the drill-moment at the plate corner and the shear force at the mid-side boundary point, see Fig. 3.

The percentage errors in the same displacement and force components measured now in the clamped square plate, subjected also to a uniform load $\bar{q}^0$, shown in the same figure, are presented in Table 5. The error is computed for the solutions found in [8]. Bearing in mind the simplicity of the elements presented here, their numerical performance seems to be quite satisfactory.
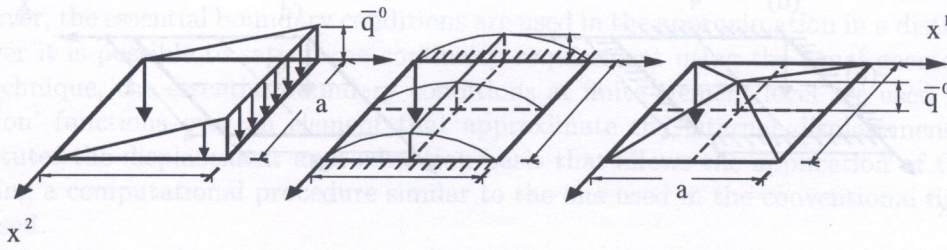
Fig. 2. Square plate with different loadings and boundary conditions

Table 1. Square plate subject to hydrostatic pressure

Mesh and element	Normalised displacement $w(a/2, a/2)$			Normalised moment $m^{11}(a/2, a/2)$		
	ZDE	TFE	JFE	ZDE	TFE	JFE
2×2	0.002023	0.001953	0.001935	0.030834	0.023698	0.023700
4×4	0.002038	0.002026	0.002026	0.025063	0.023923	0.023966
6×6	0.002036	0.002031	0.002031	0.024409	0.023961	0.023963
8×8	0.002034	0.002031	0.002031	0.024187	0.023946	0.023941
10×10	0.002033	0.002031	0.002032	0.024121	0.023951	0.023973
12×12	0.002032	0.002031	0.002032	0.024023	0.023925	0.023941
14×14	0.002032	0.002031	0.002031	0.024016	0.023934	0.023945
Ref. [22]	0.00203			0.0235		

Table 2. Square plate subject to sinusoidal load

Mesh and element	Normalised displacement $w(a/2, a/2)$			Normalised moment $m^{22}(a/2, a/2)$		
	ZDE	TFE	JFE	ZDE	TFE	JFE
2×2	0.001490	0.000868	0.000936	0.040118	0.025190	0.022615
4×4	0.001550	0.001353	0.001359	0.029862	0.024854	0.024652
6×6	0.001549	0.001460	0.001462	0.028341	0.026130	0.026121
8×8	0.001546	0.001496	0.001497	0.027889	0.026609	0.026601
10×10	0.001544	0.001513	0.001513	0.027626	0.026857	0.026820
12×12	0.001543	0.001522	0.001522	0.027516	0.026954	0.026938
14×14	0.001543	0.001527	0.001528	0.027396	0.027011	0.027057
Ref. [22]	0.00154			0.0268		

Table 3. Square plate subject to pyramid-type load

Mesh and element	Normalised displacement $w(a/2, a/2)$			Normalised moment $m^{22}(a/2, a/2)$		
	ZDE	TFE	JFE	ZDE	TFE	JFE
4×4	0.001979	0.001776	0.001776	0.0266	0.0228	0.0228
6×6	0.002069	0.001975	0.001975	0.0273	0.0254	0.0254
8×8	0.002057	0.002003	0.002004	0.0271	0.0259	0.0260
10×10	0.002062	0.002027	0.002027	0.0271	0.0263	0.0263
Ref. [22]	0.002083			0.0271		
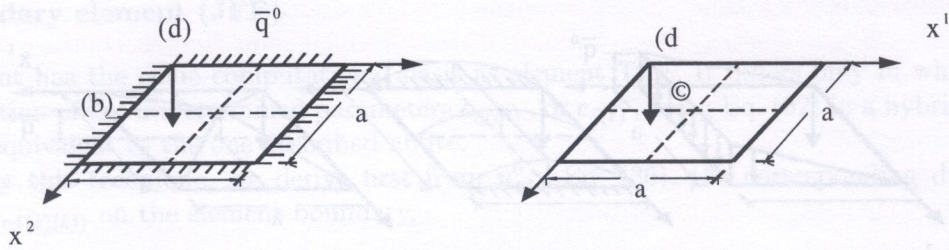
Fig. 3. Square plate subject to a uniform load with different boundary conditions

Table 4. Simply supported square plate subject to uniform load

Elements and nodes	$w(a/2,a/2)$	$m^{11}(a/2,a/2)$	$-m^{12}(0,0)$	$q^1(0,a/2)$
ZDE (45)	0.004071	0.048829	0.033685	0.30845
TFE (45)	0.004061	0.047933	0.036033	0.29890
JFE (45)	0.004061	0.047937	0.036064	0.29931
HTQI1 (40)	0.00406	0.0479	0.0348	0.341
Analytical	0.00406	0.0479	0.0325	0.338

Table 5. Clamped square plate subject to uniform load (percent errors)

Elements	$w(a/2,a/2)$	$m^{11}(a/2,a/2)$	$-m^{11}(0,a/2)$	$q^1(0,a/2)$
ZDE	−1.9	+3.1	−1.1	+9.9
TFE	+0.2	+0.08	−0.5	+8.3
JFE	+0.2	+0.13	−0.7	+9.2
CIQ	−4.4	+4.9	−8.2	−
HTQI1	−0.0	+0.3	−0.6	−1.7
HTQI2	−0.0	+0.3	−0.7	−1.3
Analytical	0.001265	0.02291	0.0513	0.441

Table 6. Simply supported rectangular plate subject to a unit uniform load

Solution	ZDE	TFE	JFE	Ref. [21]	Ref. [24]
$u^0_{x^3}$	2672.88	2696.10	2694.66	2698.70	2697.60
$M^{x^1x^1}$	1.3567	1.2950	1.2916	1.2984	1.2992
$M^{x^2x^2}$	0.8061	0.7737	0.7770	0.7974	0.7984

The last test is on a 4×6 rectangular plate with thickness $h = 0.2$, unit modulus of elasticity and Poisson ratio $\nu = 0.3$. The plate is simply supported and solved for a unit uniform load. The results for the centre transverse displacement and bending moments obtained with a 4×4 element mesh (symmetry is not used) and eight boundary elements are presented in Table 6. Shown in the same table are the solutions obtained with Trefftz elements [21], and the analytical solutions presented in [24]. The quality of the results obtained with the elements suggested here compare favourably with those obtained with the relatively more complex approach proposed by Piltner [21].

7. CONCLUDING REMARKS

The approach proposed here exploits the basic concept of the Trefftz method in the use of assumed internal displacement fields that solve the Lagrange equation. What distinguishes this approach is the technique used to approximate the essential boundary conditions.

The basic concept consists in operating on the natural boundary conditions enforced in integral form. However, the essential boundary conditions are used in the approximation in a distinct manner.

Whenever it is possible to satisfy the continuity requirement using the usual geometrical interpolation technique, the essential boundary conditions at finite element level are used to construct 'interpolation' functions over an element that approximate the internal displacement field. This field constitutes the displacement approximation basis that allows the application of the proposed method using a computational procedure similar to the one used in the conventional finite element method based.

Two alternative procedures are followed to enforce conformity and inter-element continuity when the interpolation cannot ensure the continuity requirement. The 'frame function' concept is applied in both cases. The first procedure is hybrid and consists simply in applying the Gauss divergence theorem for the stress variation at the finite element level, after imposing the essential boundary conditions of the element and enforcing the equation of equilibrium, in order to eliminate the undetermined parameters. The second procedure is a boundary equivalent. It results from the application of condition (32) at finite element level in order to relate the undetermined parameters with the nodal degrees of freedom.

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