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Numerical analysis of dielectrics in powerful electrical fields

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The limiting analysis problem for dielectrics in nonhomogeneous powerful electrical fields is considered. In the framework of this problem the external charges for which the appropriate electrostatical variational problem has no solution are calculated, that solution is treated as a beginning of the electrical puncture of dielectric. From the mathematical point of view the limiting analysis problem is non-correct and needs a relaxation. This is achieved using a partial relaxation based on a special discontinuous finite-element approximation.

1. Introduction

Investigation of the electrostatical boundary-value problem (BVP) for dielectrics in powerful electrical fields is of particular interest in both theory and practice. It is stimulated by significance and practical interests in electrical engineering and microelectronics.

The electrical state of a medium in a given domain is characterized by the bulk and surface density of charges and by the vectors of electrical tensity E, electrical induction D and electrical current density J. Vector D is introduced by the rule $D = \varepsilon_0 E + P$, where $\varepsilon_0 \approx 8.85 \cdot 10^{-12}$ is the electrical constant and P is the vector of polarization [9, 12]. For the electrical tensity the scalar potential u is used such that $E = -\nabla u$.

In weak electrical fields the currents of conductivity in dielectrical media are practically absent, i.e. $J \approx 0$, and the simplest linear constitutive relation $E \mapsto D$ is used [12]. As a result, for the solution of the appropriate linear BVPs, various effective analytical and numerical methods have been worked out [9].

In powerful electrical fields the essentially nonlinear phenomena of polarization saturation ($|P| \le P_* < +\infty$) and ionization ($J \neq 0$) must be taken into account [11, 12]. As a result, the integral model of bounded electrical induction is used, where $|D| \le \lambda < +\infty$ and $\lambda > 0$ is the complementary physical parameter of dielectrical medium (*the parameter of saturation*) which is easily calculated [4].

In the framework of the model of bounded electrical induction, the existence of the limiting electrostatical load (such external charges with no solution of BVP) has been proved by the author recently [4]. From the physical point of view this effect is treated as the beginning of the electrical puncture of dielectric.

For calculation of the limiting electrostatical load the original variational problem was formulated in [4]. From the mathematical point of view this problem is non-correct because its solution belongs to the space BV of scalar functions with bounded variations, having the generalized gradient as the bounded Radon's measure [8]. Using the simplest example, it is demonstrated that the limiting analysis problem has discontinuous solutions with breaks of the first type. As a result, this problem needs a relaxation [13]. We use a partial relaxation based on the special discontinuous finite-element approximation (FEA) [3, 10].

After this approximation the limiting analysis problem is transformed into the non-linear system of algebraic equations which can be ill conditioned. Therefore, the decomposition method of adaptive block relaxation is used for the numerical solution because it practically disregards the condition number of the global stiffness matrix [2].

The numerical results show that the proposed technique has qualitative advantages over the standard continuous finite-element approximations when applied to the determination of the limiting electrostatical load.

2. THE LIMITING ANALYSIS PROBLEM IN ELECTROSTATICS

In the general case the polarization properties of a dielectrical medium are discribed by the vector-function $D = D(x, E)$. The constitutive relation $D_i = \varepsilon_{ij}(x, E)E_j$, where $\{\varepsilon_{ij}\}$ is the symmetrical tensor of real dielectrical penetration [9, 12] is used in practice. Here and in what follows the addition over repeating indeces is assumed. For an isotropic medium $\varepsilon_{ij} = \varepsilon\delta_{ij}$, where $\varepsilon = \varepsilon(x, |E|)$ is the scalar function and δ_{ij} is the Kronecker symbol. For a homogeneous medium $\{\varepsilon_{ij}\} = \mathrm{const}(x)$.

Let a dielectrical medium occupy a domain $\Omega \subset \mathbb{R}^3$. We consider the following boundary-value problem. The quasi-static electrical influences acting on the dielectric are: a bulk charge with density ρ in Ω, a surface charge with density g on a portion Γ^2 of the boundary, and a potential u^0 on a portion Γ^1 of the boundary is also given. Here $\Gamma^1 \cup \Gamma^2 = \partial\Omega$, $\Gamma^1 \cap \Gamma^2 = \emptyset$ and $\mathrm{area}(\Gamma^1) > 0$.

From the classical Thomson's principle [9, 12] it follows that the free energy of the electrical field in dielectric has the global minimum on the real potential, i.e. the potential u is the solution of the following variational problem,

$$u_* = \arg(\inf\{\, I(u) : \ u \in V\,\}), \tag{1}$$

where

$$I(u) = \int_\Omega \Phi(x, \nabla u(x))\, \mathrm{d}x - A(u), \qquad A(u) = \int_\Omega \rho u\, \mathrm{d}x + \int_{\Gamma^2} g u\, \mathrm{d}\gamma.$$

Here $V = \{u : \overline{\Omega} \to \mathbb{R};\ u(x) = u^0(x),\ x \in \Gamma^1\}$ is the set of admissible electrical potentials, $A(u)$ is the work of the electrical field on the external charges and Φ is the specific full energy of the electrical field such that $D_i(x, E) = \partial\Phi(x, E)/\partial E_i$ for every $E \in \mathbb{R}^3$ and almost every $x \in \Omega$. In the general case the function $\Phi(x, E)$ has the following form [7],

$$\Phi(x, E) = E_i \int_0^1 D_i(x, tE)\, \mathrm{d}t.$$

Concerning the constitutive relation of a medium we make the following hypotheses:

(H1) The Caratheodory vector-function $D(x, E)$ is continuous and strongly monotonuous in vector argument [7], i.e. for any vectors $(E^1 \neq E^2) \in \mathbb{R}^3$ and almost every $x \in \Omega$ the following estimation is true,

$$\big(D_i(x, E^1) - D_i(x, E^2)\big)\,(E_i^1 - E_i^2) > 0.$$

(H2) Vector-function $D(x, E)$ is coercive and has polynomial growth in modulus of vector argument, i.e. there exist constants $a_0 \geq a_1 > 0$, $a_2 \in \mathbb{R}$, $p > 1$ and a function $b \in L^q(\Omega)$ with $q = p/(p-1)$ such that for every vector $E \in \mathbb{R}^3$ and almost every $x \in \Omega$ the following estimations are true,

$$a_2 + a_1|E|^{p-1} \leq |D(x, E)| \leq a_0\,|E|^{p-1} + b(x).$$

From hypothesis (H2), the set of admissible electrical potentials is defined in the following way,

$$V^p = \{ u \in W^{1,p}(\Omega) : u(x) = u^0(x),\ x \in \Gamma^1 \} . \tag{2}$$

Theorem 1. *In the framework of hypotheses (H1) and (H2) and of the standard hypotheses for domain Ω and external influences (u^0, ρ, g) [1, 7], the electrostatical variational problem (1) has a single solution in V^p.*

In powerful electrical fields the current of conductivity must be taken into account. In the general case the ionization properties of a dielectrical medium are discribed by the vector-function $J = J(x, E)$ [9, 12]. In practice the Ohm law $J_i = \sigma_{ij}(x, E)E_j$ is used, where $\{\sigma_{ij}\}$ is the symmetric tensor of conductivity. For an isotropic medium $\sigma_{ij} = \sigma\delta_{ij}$, where $\sigma = \sigma(x, |E|)$ is the scalar function, and for a homogeneous medium $\{\sigma_{ij}\} = \mathrm{const}(x)$.

A part of the full energy of the electrical field is lost on the work of the electrical current. As a result, in the problem (1) function $\Phi(x, E)$ is replaced by function $\Phi(x, E) - \Psi(x, E)$, where

$$\Psi(x, E) = E_i \int_0^1 J_i(x, tE)\, \mathrm{d}t$$

is the specific work of the electrical current (the generalized Joule–Lenz law) [9, 12].

In Fig. 1 the characteristic relations $|E| \mapsto |D|$ and $|E| \mapsto |J|$ for real isotropic dielectrical media are presented (lines 1 and 2, respectively). It is easily seen that there exists always a parameter $\lambda(x) > 0$ such that $\Phi(x, E) - \Psi(x, E) \le \lambda(x)|E|$.

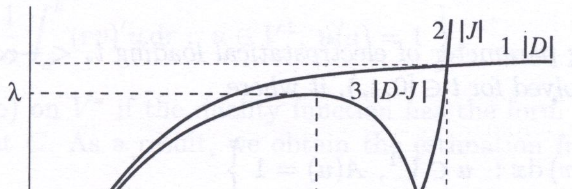

Fig. 1

The appropriate model of bounded electrical induction (line 3 in Fig. 1) describes integrally both nonlinear phenomena of dielectrical media in powerful electrical fields: the polarization saturation and ionization. In the general case, the parameter of saturation is calculated as the solution of the following problem for almost every $x \in \Omega$

$$\lambda(x) = \sup \left\{ \frac{\Phi(x, E) - \Psi(x, E)}{|E|} : E \in \mathbb{R}^3 \right\} , \tag{3}$$

it being clear that for homogeneous dielectrical media $\lambda = \mathrm{const}(x)$.

For the model of bounded electrical induction, parameter $p \le 1$ in (H2) because after point $E_0 = |E^*|$, where $E^* \in \mathbb{R}^3$ is the maximizer of the function $[\Phi(x, E) - \Psi(x, E)]/|E|$, the effective energy of the electrical field $I(u)$ has a less than linear growth in $\|u\|$. However the work of the electrical field on the external charges $A(u)$ has always a linear growth in $\|u\|$. As a result, the appropriate variational problem is non-correct [6, 7, 13]. The existence of the limiting electrostatical load (such external charges with no solution of BVP) has been proved by the author recently [4]. From a physical point of view, this effect is treated as the beginning of the electrical puncture of dielectric.

We remind the definition of the limiting electrostatical load [4]. For this reason we introduce the set of admissible external charges for which functional $I(u)$ is bounded from below on V^1 and, therefore, a solution of the problem (1) exists:

$$B = \{\, (\rho, g) \in L^\infty(\Omega) \times L^\infty(\Gamma^2) : \inf\{ I(u) : u \in V^1 \} > -\infty \,\}.$$

This set is non-empty because for small external influences the problem (1) is transformed into the classical variational problem of linear electrostatics which always has a solution. It follows from the Theorem 1 because $p = 2$ for linear dielectrics.

Definition 2. For external charges $(\rho, g) \in B$ we examine the sequence of charges which are proportional to the real parameter $t \geq 0$. The number $t_* \geq 0$ is called *the limiting parameter of electrostatical loading* and $(t_*\rho, t_*g)$ is called *the limiting electrostatical load,* if $(t\rho, tg) \in B$ for $0 \leq t \leq t_*$ and $(t\rho, tg) \notin B$ for $t > t_*$.

The limiting analysis problem is the investigation of the set of positive parameters t, for which functional,

$$I_t(u) = \int_\Omega \Phi_\lambda(x, u(x))\, \mathrm{d}x - t\, A(u)$$

is bounded from below on the set of admissible electrical potentials V^1 where the integrand Φ_λ corresponds to the model of bounded electrical induction.

Theorem 3. *The limiting parameter of electrostatical loading $t_* < +\infty$ exists such that the limiting analysis problem is solved for $t \in [0, t_*)$, it where,*

$$t_* = \inf\left\{ \int_\Omega |\nabla u(x)| \lambda(x)\, \mathrm{d}x : \ u \in V^1, \ A(u) = 1 \right\} \tag{4}$$

and λ is the parameter of saturation from (3).

From the Definition 2 it follows that for $t_* < 1$ the variational problem (1) for the model of bounded electrical induction has no solution. If this phenomenon is treated as the beginning of the electrical puncture of a dielectric then the limiting analysis problem (3) is the main problem for estimation of puncture conditions for dielectrics of complex shape in nonhomogeneous electrical fields. As a result, we can formulate the following shape optimization problem for the dielectric of maximum electrical strength,

$$\Omega_* = \arg\left(\sup\{\, t_*(\Omega) : \ \Omega \in C^{0,1}, \ \mathrm{vol}(\Omega) = \omega \,\}\right), \tag{5}$$

where $t_*(\Omega)$ is the solution of the limiting analysis problem (4) on a domain Ω, ω is the prescribed dielectric's bulk and $C^{0,1}$ is the set of domains with Lipschitz boundary [7].

3. Existence of discontinuous fields

Problem (4) is a variational problem with a multiple integral functional of linear growth. The appropriate set of admissible fields V^1 is the subset of the non-reflexive Sobolev's space $W^{1,1}(\Omega)$ [7, 13]. As a result, the variational problem (4) is non-correct because its solution really belongs to the space of scalar functions with bounded variations $BV(\Omega) \supset W^{1,1}(\Omega)$, having the generalized gradient as the bounded Radon's measure [8]. The space $BV(\Omega)$ contains both continuous and discontinuous fields with breaks of the first type.

Example

We consider the simplest problem on the long cylindrical condenser formed by two co-axial conducting cylinders [9, 12]. The inside cylinder of radius a and unit length has charge $Q = 2\pi ag$ and the outside cylinder of radius b is grounded. The space between cylinders is filled by a homogeneous and isotropic dielectric with parameter of saturation λ.

In this case the limiting analysis problem (4) takes the form,

$$g_* = \lambda \inf \left\{ \frac{1}{a} \int_a^b |u'| r dr : u \in V^1, u(a) = 1 \right\}, \tag{6}$$

where $V^1 = \{ u \in W^{1,1}(a,b) : u(b) = 0 \}$.

Using the simplest equality $|a| = \sup\{ab : |b| \leq 1\}$ we can rewrite this problem in the following way,

$$g_* = \lambda \inf \left(\sup \left\{ \frac{1}{a} \int_a^b u' v \, r dr : v \in V^* \right\} : u \in V^1, u(a) = 1 \right),$$

where $V^* = \{ v \in W^{1,\infty}(a,b) : |v| \leq 1 \}$.

From the classical inequality $\inf\limits_u \sup\limits_v \{L(u,v)\} \geq \sup\limits_v \inf\limits_u \{L(u,v)\}$ [6] the estimation follows,

$$g_* \geq \lambda \sup\{ K(v) : v \in V^* \},$$

where the duality functional,

$$K(v) = -v(a) - \sup \left\{ \frac{1}{a} \int_a^b (rv)' u \, dr : u \in V^1, u(a) = 1 \right\},$$

is proper (i.e. $K(v) \not\equiv -\infty$) on V^* if the duality function has the form $v(r) = C\,a/r$ for $r \in [a,b]$ with an indefinite constant C. As a result, we obtain the estimation from below for the limiting electrostatical load,

$$g_* \geq \lambda \sup\{ -C : |C| \leq 1 \} = \lambda.$$

It is easily seen that for a minimizing sequence of the simplest form $u_m(r) = (b-r)^m/(b-a)^m$ ($m \in \mathbb{N}$) the minimum value λ of the functional in the problem (6) is reached on the function with a break of the first type because $u_m(r) \to 1 - H(r-a)$ as $m \to \infty$ for almost every $r \in (a,b)$, where H is the Heaviside function of bounded variation [8]. As a result, we have $g_* = \lambda$.

The concept of a generalized solution and its mathematical and physical justification for the variational problem with the integral functional of linear growth are presented in many publications. For example, we can refer here the variational problem of the non-parametric minimal surface [8] and of ideal elasto-plasticity [13].

4. Discontinuous FEA and the partial relaxation

From the previous section it follows that the limiting analysis problem (4) needs a relaxation. The main idea consists of the following [10, 13].

Let V be the Banach space with norm $\|\cdot\|$ and $I : V \to \mathbb{R}$ be the coercive on V functional, i.e. $I(u) \to +\infty$ as $\|u\| \to \infty$. The standard minimization problem $\inf\{I(u) : u \in V\}$ is considered. The solution of this problem can be absent. However, a sequence $\{u_k\} \subset V$ may exist such that $u_k \to u_0$ almost everywhere and $I(u_k) \to I_0 \in \mathbb{R}$ as $k \to \infty$, where the limiting element $u_0 \notin V$. In this case we can construct a continuation $\overline{I}$ of the functional I into the class of functions $\overline{V} \supset V$ such that $\overline{I}(u_0) = I_0$ and $\overline{I}(u) = I(u)$ for every $u \in V$.

For variational problems with multiple integral functional of linear growth the appropriate space $\overline{V}$ equals the BV space of functions with bounded variations and generalized derivatives as the bounded Radon's measures [8, 13]. In numerical analysis only the finite dimension subspace of BV is used. Therefore, for the limiting analysis problem (4) we use *the partial relaxation* which is based on the special FEA with functions having breaks of the first type.

Let $\Omega \subset \mathbb{R}^n$ $(n = 1, 2, 3)$ and $\Omega_h = \bigcup_{i=1}^{M} T_h^i$, $\Gamma_h = \partial\Omega_h$ such that $\mu_n(\Omega \backslash \Omega_h) \to 0$ and $\mu_{n-1}(\partial\Omega \backslash \partial\Omega_h) \to 0$ as $h \to +0$, where h is the characteristic step of approximation [5]. Here and in what follows M is the number of simpleces T_h^i of a given triangulation (the simplex is a point, triangle and tetraeder for $n = 1, 2$ and 3, respectively), $\mu_k(D)$ is the k-dimension Lebeg's measure of set D.

Each FEA is characterized by the sets of active nodes $\{x^\alpha\}_{\alpha=1}^m$ and active facets $G_h = \left\{S_h^{ij} = T_h^i \cap T_h^j : \ i, j = 1, 2, \dots, M\right\}$ which contain all inside and boundary on Γ_h^1 nodes and facets. For $n = 1, 2$ and 3 the facet is a point, segment and triangle, respectively [5].

For the scalar field the following piecewise continuous approximation is used (Fig. 2)

$$u_h(x) = U_\alpha^i \Lambda_\alpha^i(x) \qquad (\alpha = 1, 2, \dots, m; \ i = 1, 2, \dots, M),$$

where U_α^i is the value of the field at node x^α of simplex T_h^i, $\Lambda_\alpha^i : \Omega_h \to \mathbb{R}$ is the piecewise linear discontinuous function such that $\operatorname{supp}(\Lambda_\alpha^i) = T_h^i$ and $\Lambda_\alpha^i(x^\beta) = \delta_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \dots, m;$ $i = 1, 2, \dots, M)$.

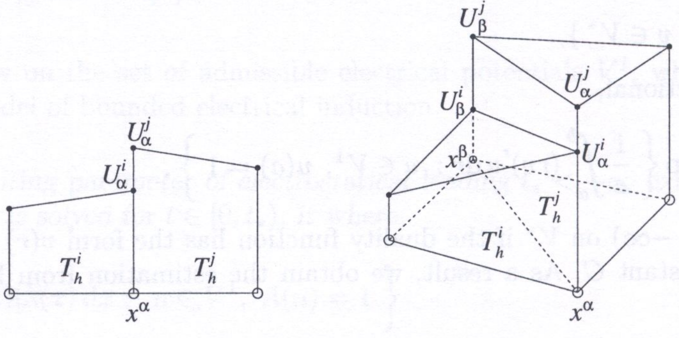

Fig. 2

In this case the set $V^1 \subset W^{1,1}(\Omega)$ is approximated by the set $V_h^1 \subset BV(\Omega)$ which is isomorphous to $\mathbb{R}^{(n+1)M}$. As a result, we have the special FEA with functions having breaks of the first type in every active node.

The partially relaxed limiting analysis problem (4) has the following form [3, 10],

$$t_h = \min\left\{ \int_{\Omega_h} |\nabla u(x)| \lambda(x)\, \mathrm{d}x + R_h(u) : \ u \in V_h, \ A_h(u) = 1 \right\},$$

$$R_h(u) = \sum_{S_h^{ij} \in G_h} \int_{S_h^{ij}} \left| u^i \lambda^i - u^j \lambda^j \right| \mathrm{d}\gamma, \qquad A_h(u) = \int_{\Omega_h} \rho_h u \, \mathrm{d}x + \int_{\Gamma_h^2} g_h u \, \mathrm{d}\gamma, \tag{7}$$

where indeces i and j correspond to functions on the neighbouring elements T_h^i and T_h^j having the common facet S_h^{ij} (Fig. 2). For facets on Γ_h^1 we have $u^j(x) = u_h^0(x)$ and $\lambda^i \equiv \lambda^j$. Here the standard piecewise linear continuous FEAs (u_h^0, ρ_h, g_h) of external influences are used [5]. According to the properties of FEA we have $t_h \searrow t_*$ as $h \to +0$.

From the computational point of view, the functional in problem (7) is singular because it has no classical derivative. Therefore, in practice the simplest approximation of the modulus $|z| \approx (z^2 + \mu^2)^{1/2}$ with *the regularization parameter* $\mu \ll 1$ is used [3].

By the necessary condition of stationarity, problem (7) transforms into a non-linear system of algebraic equations which can be *ill conditioned* [2, 3]. The main cause of this phenomenon consists of the following: the global stiffness matrix has lines with significantly different factors if the solution has a large gradient or breaks of the first type. For the regularization parameter $\mu \ll 1$ this situation is more difficult. As a result, the decomposition method of adaptive block relaxation is used for the numerical solution, because it practically disregards the condition number of the global stiffness matrix [9, 10]. The main idea of this method consists of the iterative improvement of zones with "proportional" fields by the special decomposition of variables and separate calculation on these variables.

5. Numerical results

The following electrostatical BVP is considered for numerical testing: an isotropic and homogeneous dielectric has a form of a finite round rod with radius a and length $2l$. The small round blocks of dielectric are covered by conductors having charges $\pm Q$.

The puncturing charge is $Q_* = t_* Q_0$, where $Q_0 = \pi a^2 \varepsilon E_0$ is the puncturing charge of the plane condenser [9, 11, 12] and t_* is the limiting parameter of electrostatical loading.

In view of the axial symmetry, the limiting analysis problem (4) has the following form,

$$t_* = \inf\{K(u) : u \in V^1, \, u(r,1) = 1\}, \tag{8}$$

where

$$K(u) = 2\int_0^1 \int_0^1 \left[\eta^2 \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right]^{1/2} r dr dz,$$

$$V^1 = \left\{ u \in W^{1,1}((0,1)\times(0,1)) : u(r,0) = 0, \ \frac{\partial u}{\partial z}(r,0) = 0, \ \frac{\partial u}{\partial r}(0,z) = 0 \right\}.$$

Here $\eta = l/a$ is a geometrical parameter.

For the plain condenser ($\eta \ll 1$) the field $u \approx u(z)$. As a result, the functional $K(u) = \int_0^1 |u'(z)| \, dz$ reaches its minimum $t_* = 1$ on the discontinuous field $u_*(z) = H(1-z) \notin V^1$.

According to the convexity of domain and axial symmetry of the problem (8), the minimizer can have a break of the first type only on line $z = 1$. Therefore, the following partial relaxation of the problem (8) is used,

$$t_* = \inf\{K(u) + R(u) : u \in V^1\},$$

where

$$R(u) = 2\int_0^1 |u(r,1) - 1| \, r dr.$$

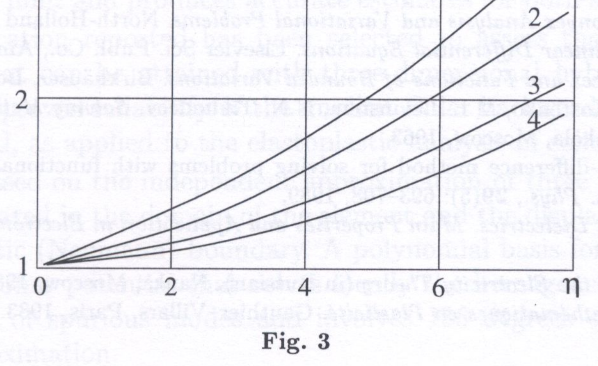

Fig. 3

In the computational experiments a uniform $N \times N$ triangulation of the domain $(0,1) \times (0,1)$ and a regularization parameter $\mu = 10^{-2}$ is used.

The experimental relations $\eta \mapsto t_*$ are shown in Fig. 3. Lines 1, 2 and 3 correspond to the continuous FEA with $N = 10$, $N = 20$ and $N = 40$, respectively. Line 4 corresponds to the discontinuous FEA with $N = 10$.

It is easily seen that continuous solutions converge to the discontinuous solution as the discretization of the domain is increased. The decrease of the regularization parameter μ down to 10^{-3} practically does not improve either the continuous or the discontinuous solutions.

CONCLUSIONS

In the paper the following main results are presented:

1. The limiting analysis problem in electrostatics is formulated. In the framework of this problem the limiting electrostatical load is calculated, which is very important for the estimation of practical electrical strength of dielectrics. The appropriate shape optimization problem is formulated for the dielectric of maximum electrical strength.

2. It is proved that the limiting analysis problem has discontinuous solutions with breaks of the first type and, therefore, needs a relaxation.

3. The partial relaxation of the limiting analysis problem is described. This relaxation is based on the special discontinuous finite-element approximation with functions having breaks of the first type in every active node.

4. The numerical results show that, for the calculation of the limiting electrostatical load, the proposed partial relaxation has qualitative advantages over standard continuous finite-element methods.

The analytical and numerical results presented are new. They have practical interest and need more theoretical and experimental research.

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