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General strategy of *h-p* adaptive solutions in structural Trefftz-type element analysis

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The paper deals with a strategy of reliable application of the Trefftz elements in the linear analysis of complex engineering structures with stress concentrators. The standard *p*-adaptivity is suggested in the low gradient areas. For selected large-gradient local zones certain specific T-element substructures are proposed. The *h-p* adaptive procedures for optimization of parameters of the substructures are numerically investigated.

1. Introduction

An accurate approximation of a solution with large local gradients is one of the major difficulties in the reliable finite element modelling of stress/displacement fields in any linear structural analysis. Such gradients occur in the vicinity of singularities and strong stress concentrators in the great majority of real engineering structures. In the standard finite element solutions the increase of the accuracy in the regions of large local gradients results mainly from the element mesh refinement. In the contemporary FE systems the improvement of the mesh (*h*-adaptivity) is introduced automatically, without any intervention of a user. This leads, however, to a large number of unknowns (degrees of freedom) and considerably increases the CPU time. If the structure consists of numerous concentrators and the basic solution must be repeated many times, e.g. in optimization loops, then the final numerical problem can be too large even for a high-performance computer.

The increase of the approximation degree of the applied elements (*p*-adaptivity) is another possibility of improvement of the solution accuracy. This is especially convenient in the case of the hierarchic elements [2] in which the high accuracy is achieved by increasing degree of their shape functions without the change of the nodal grid. The Trefftz-type finite elements (T-elements [9]), discussed in the present paper, also belong to this family. Additionally, according to the idea of Erich Trefftz [13], their shape functions fulfil exactly the differential equations of a given boundary value problem [15].

Unfortunately, the typical large hierarchic finite elements with high-degree polynomial shape functions do not model local singularities in a correct way. The application of the polynomial with the degree higher than $p = 3$ or 4 results in strong oscillations in the vicinity of the singular regions [11]. Also the investigations carried out for the T-elements [5] exhibited the similar behaviour (pollution effect) in this case. On the other hand, it was noted that the element boundaries attenuate the stress oscillations. This observation suggested introduction of a specific local mesh design in the singular region, advisable for the hierarchic or Trefftz-type elements.

Further investigations of the authors, continued for regular problems with strong concentrators [12, 14], proposed a global *h-p* algorithm ensuring highly accurate results in the whole region. The present paper, summarizing this research, introduces the way of proceeding for complex structures with numerous singularities and strong concentrators. Because of their variety this procedure is more general than the application of the special T-elements [9] with the local analytical solutions for each type of the concentrator.

2. Selective adaptivity in Trefftz-element solutions

The notion of the selective adaptivity is known in the commercial finite element codes. As we can see in Fig. 1 [1], it serves for elimination of the singular regions from the adaptive mesh refinement. Without such a selection it would be difficult to obtain reliable results in the vicinity of the hole.

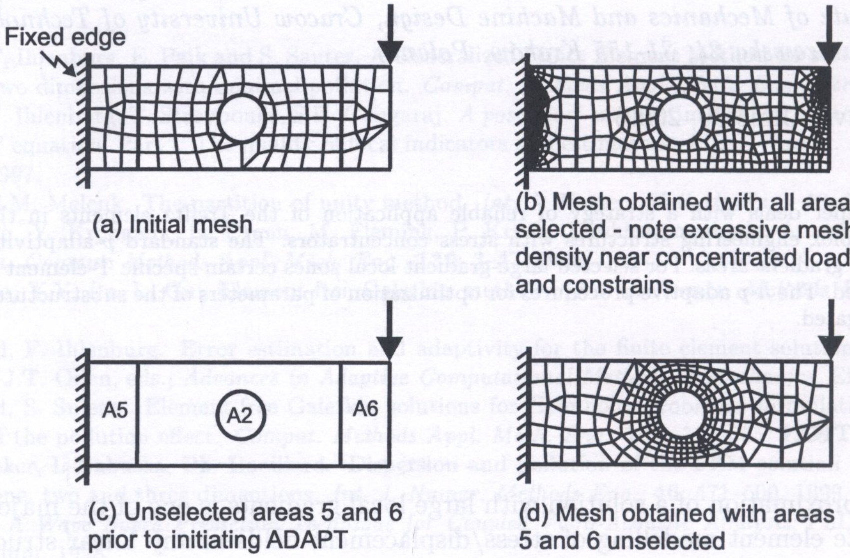

Fig. 1. Selective adaptivity used in ANSYS

This way of proceeding is, however, dangerous, decreasing large local gradients and excluding from the engineering analysis the regions of the highest effort.

A similar, wrong elimination of the strong local effect would exist in the Trefftz finite element approach if we divided the considered structure into large subregions, like in Fig. 2a. In spite of the fact that the T-elements should in principle be large, as substructures filled by analytical series-type solutions, in this case that 'natural' division would result in thoroughly false model of the structure even for a high approximation degree. The answer would be even worse than in the case of the standard h-elements because the concentrators would not only be reduced but would also cause the pollution oscillations inside the structure. To avoid this danger, we should in the very beginning select the local regions suspected of the large solution gradients (Fig. 2b).

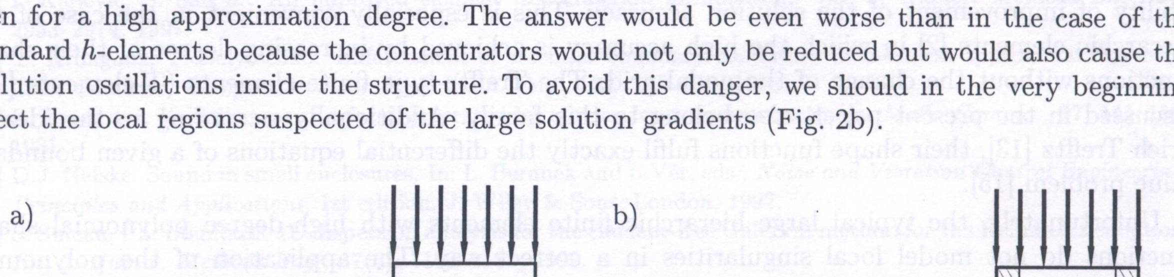

Fig. 2. Substructuring in T-element approach. Large elements degenerating strong local concentration effect (a); proposed mesh, prepared for the selective h-p adaptive refinement (b)

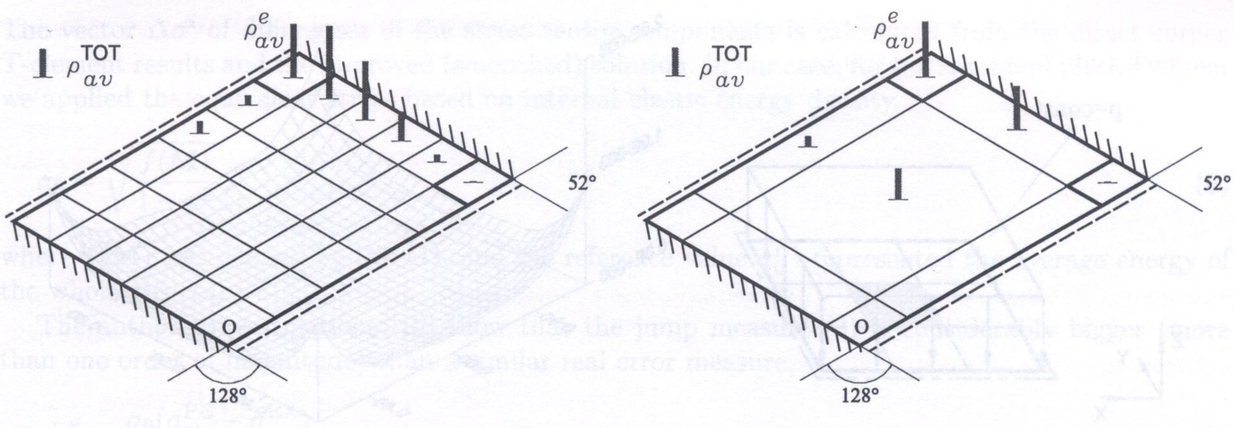

Fig. 3. Energy test for selection of local concentrators

These effects caused e.g. by concentrated forces, reentrant corners, fillets with small radii, changes in material properties or boundary conditions etc., can easily be defined by the system in an automatic way.

After the preliminary selection we should finally distinguish the local concentrators from the regions of small effort. In most cases the difference is obvious for an experienced engineer, however, to avoid a wrong choice and to automate the whole procedure the energy control is suggested.

Figure 3 explains the way of proceeding. In this example we considered a skew, Kirchhoff-type plate, uniformly loaded in the whole domain (element HTD [4]). The selected corner elements were suspected of the local concentrators. The comparison of the element average energy density ρ^e_{av} with the total average energy density ρ^{TOT}_{av} of the whole object evidently indicated the concentration regions. As we can also observe comparing Figs. 3a and 3b, the value of the energy measure in the corner elements did not practically depend of the substructuring.

The energy measure defined as

$$\rho^e_{av} = \frac{1}{2D\,(1-\nu^2)\,\Omega_e} \iint_{\Omega_e} f(\mathrm{M})\,\mathrm{d}\Omega \tag{1}$$

where

$f(\mathrm{M}) = M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)M_{xy}^2$,

Ω_e – element area,

D – plate bending stiffness,

ν – Poisson ratio,

M_x, M_y, M_{xy} – components of plate bending moment tensor,

was here calculated in a standard numerical way, because it served only as an indicator of the concentrated energy. In the second stage of the proposed strategy, this integration was eliminated and replaced by nodal error estimators, cheaper and more natural for the Trefftz-type solution.

3. Adaptive procedure for local concentrators

3.1. Nodal error estimators and indicators

The selection presented in the previous section divides the investigated object into regions of low gradients in which the standard *p*-extension is recommended, and the local zones of concentrated effects, where a special *h-p* procedure is necessary. In both cases the specific features of the T-elements should be taken into account — a large zone of superconvergence in the central part and the extreme errors in the element corners (Fig. 4)

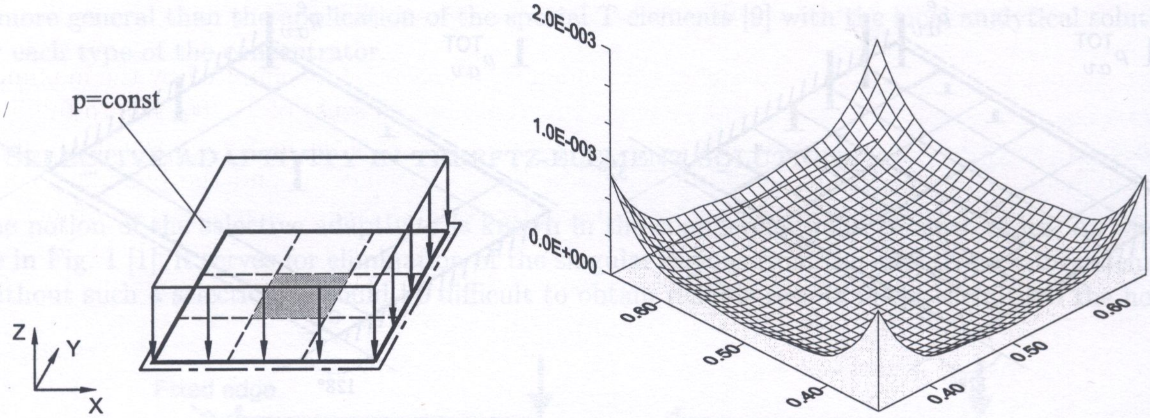

Fig. 4. Distribution of energy density error in plate T-element

In the classical hybrid-Trefftz finite element HTD [3] the analytical Trefftz-series solution $\mathbf{u}$ is residually fitted to the boundary 'frame' function $\widetilde{\mathbf{u}}$ in the integral way:

$$\int_{\Gamma_e} \mathbf{T}(\mathbf{u} - \widetilde{\mathbf{u}})\, d\Gamma = 0 \tag{2}$$

where $\mathbf{T}$ is the matrix of generalized traction weights. This type of the element was used in the present investigations, however, the other formulations like the least square element HTLS or the traction frame element HTT exhibit similar features [10]. The solution properties resulting from Eq. (2) presented in Fig. 4 suggested introduction of the corner error measures and estimators. In the T-element field the corner results are different for the particular elements meeting in this corner. The jumps of these results can be considered as an indicator of the error. Their measure is introduced below.

The corner results can be improved by taking their arithmetical average [6] or by different forms of smoothing with the help of data calculated from the Trefftz field inside the central, superconvergent element zone. Until now the smoothing was mainly done by quadratic polynomials and a procedure called 'krigeing' [6]. However, this complex procedure based on the probability calculus and used in cartography and treatment of experimental data occurred to be not very convenient in the practical application. Other smoothing procedures, also using Trefftz functions instead of standard polynomials, are now in the authors' research.

In the finite element solutions of linear elastic problems (also in the T-element approach) the displacement results are usually well convergent and sufficiently accurate. Therefore the quality of the approximation should be measured by stress-type quantities. In a complex stress state it is necessary to introduce a kind of an equivalent stress σ_0 representing the local effort of the structure. it can be based e.g. on the von Mises hypothesis, strain energy density measure or any other effort representative. Using σ_0 we can introduce the general jump measure in a point C in which n elements meet

$$\eta_C = \frac{1}{n}\sum_{i=1}^{n} (\eta_i)_C \,, \tag{3}$$

$$(\eta_i)_C = \frac{|\Delta\sigma_0^i|_C}{\sigma_0^{\mathrm{ref}}} \,, \tag{4}$$

where σ_0^{ref} is a certain normalizing reference value and $\Delta\sigma_0^i$ can be defined as

$$\Delta\sigma_0^i = \sigma_0(\Delta\sigma^i), \quad i = 1, 2, \ldots, n. \tag{5}$$

The vector $\Delta\sigma^i$ of differences of the stress tensor components is calculated from the direct corner T-element results and the improved (smoothed) solution. In our case, for the Kirchhoff plate problem we applied the equivalent stress based on internal elastic energy density,

$$\sigma_0 = \sqrt{\frac{f(\mathbf{M})}{1-\nu^2}}\,, \tag{6}$$

where $f(\mathbf{M})$ was defined by Eq. (1), and the reference value σ_0^{ref} represented the average energy of the whole structure.

The authors' investigations [12] show that the jump measure (3) is considerably bigger (more than one order of magnitude) than a similar real error measure,

$$\eta^{\text{EX}} = \frac{\sigma_0(\sigma_{av}^{\text{FE}} - \sigma^{\text{EX}})}{\sigma_0^{\text{ref}}}\,, \tag{7}$$

in which the equivalent stress σ_0 is calculated from the difference between the average corner moment values σ_{av}^{FE} and the exact solution σ^{EX} known for the particular test. However, the measure (3) served very well as an error indicator in the adaptive procedures, signalizing the necessary local improvement of the solution. Indeed, more accurate finite T-element solutions have evidently smaller corner jumps. The error estimators defined in [7, 8] in which the difference in (5) was taken between the average corner values and the results smoothed by 'krigeing' were not so reliable. By coincidence they could sometimes be smaller than the real solution errors.

3.2. Adaptive schemes

The large gradient zones selected in Section 2 should be filled by the special T-element substructures (Fig. 5). The parameters of the substructures — proportions of the element layers, mutual relations of the approximation degree of the layers, suggested number of the layers etc., should be carefully investigated before their application. The form of such investigation is presented below.

Of course, in certain cases, when a series-type analytical solution for a particular concentrator or singularity is known, one can introduce this solution into one special element and apply it to the selected zone [5]. However, taking into consideration a great variety of different concentrators, the procedure proposed in the present paper seems to be much more general.

A skew plate defined in Section 2 was chosen as a numerical example. For angle $\alpha > 129°$ the obtuse corner causes here singular effects. We investigated plates for $\alpha < 129°$, when the solution is

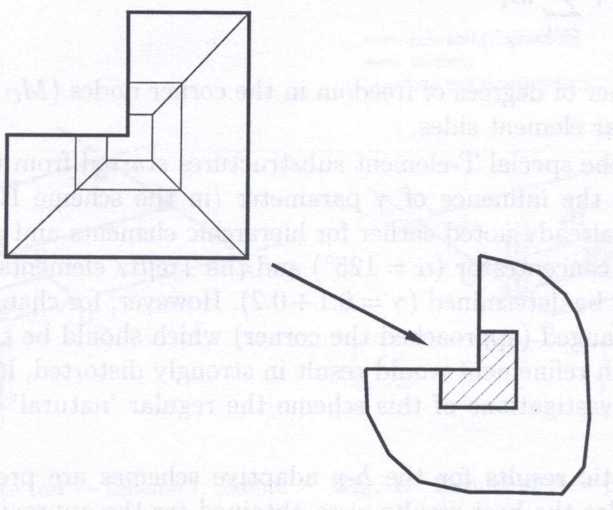

Fig. 5. Special T-element substructure

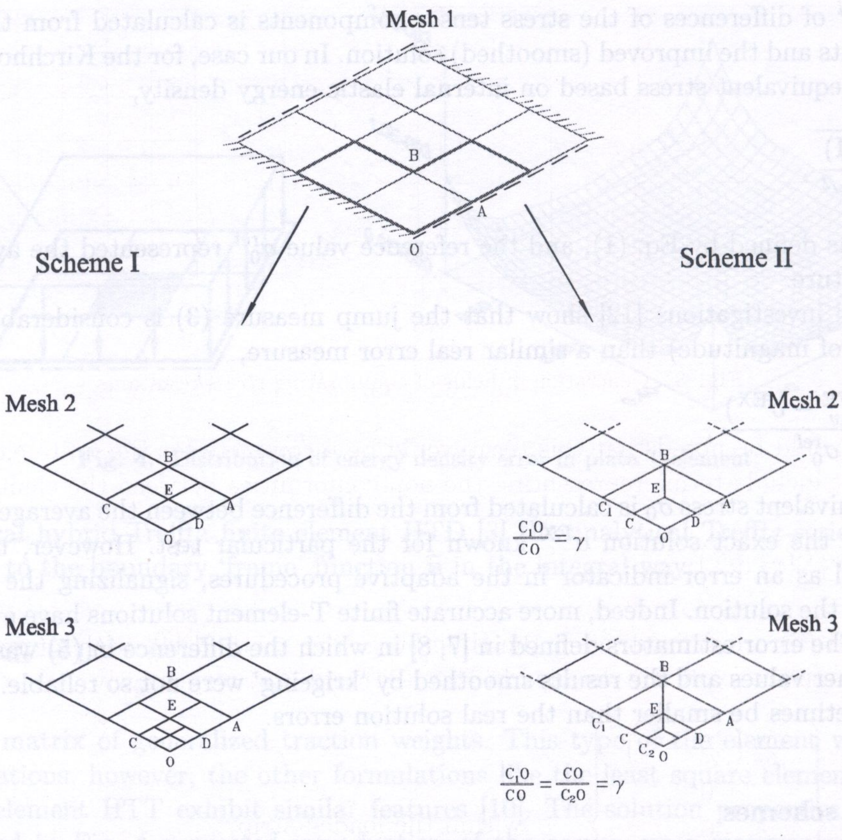

Fig. 6. Two proposed T-element substructures for selected concentrated zones

regular but strongly concentrated near this corner. Two different T-element substructures (Fig. 6), in which the h-adaptivity was possible, were taken into account.

Simultaneously, the p-adaptive procedure was introduced (Fig. 7). In this figure, M means the number of degrees of freedom in the mid-side node of the element, i.e. the number of nonlinear hierarchic frame functions defined along this side. Increasing M we also suitably increased the number N of the Trefftz shape functions in the internal element field. In our investigations, for quadrilateral elements, it was equal to

$$N = M_{\mathrm{TOT}} = 4 \cdot M_C + \sum_{i=1}^{4} M_i \tag{8}$$

where M_C was the number of degrees of freedom in the corner nodes ($M_C = 3$) and the number M_i changed for the particular element sides.

The investigation of the special T-element substructures started from the local element proportions. Figure 8 presents the influence of γ parameter (in the scheme II — Fig. 6) on the error estimator (4). As it was already noted earlier for hierarchic elements and a singular problem [5, 11] also here, for the strong concentrator ($\alpha = 125°$) and the Trefftz elements, a certain optimal range of the parameter γ could be determined ($\gamma = 0.1 \div 0.2$). However, for changing γ the position of the point D' (Fig. 8) also changed (approached the corner) which should be taken into account. In the scheme I the similar mesh refinement would result in strongly distorted, long rectangular elements. Therefore, for further investigations of this scheme the regular 'natural' element division (Fig. 6) was applied.

The most characteristic results for the h-p adaptive schemes are presented in Figs. 9–10. In all the numerical examples the best results were obtained for the approximation degree degressive towards the concentration point, near which $M = 3$ appeared to be optimal. This agreed with the

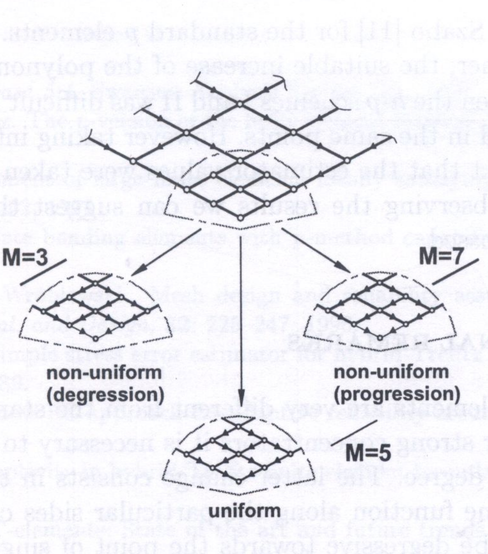

Fig. 7. *p*-adaptive schemes in concentration zones, in thick-line sides $M = 5$

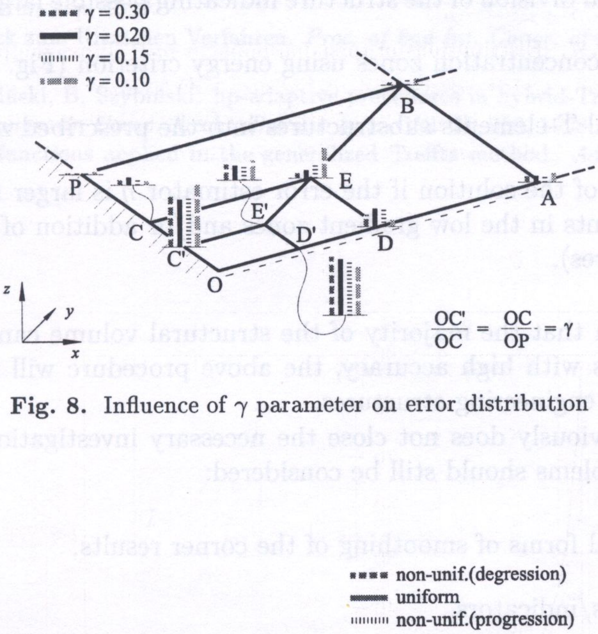

Fig. 8. Influence of γ parameter on error distribution

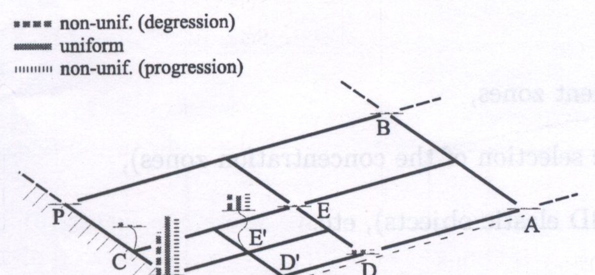

Fig. 9. Results of skew plate test — scheme I, mesh 3

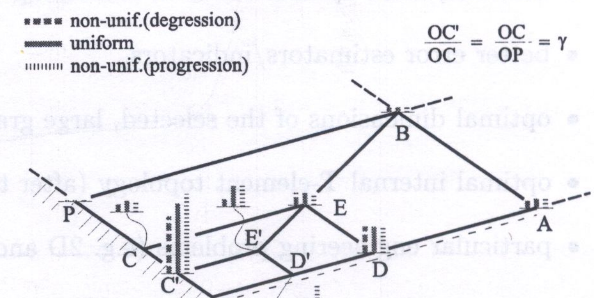

Fig. 10. Results of skew plate test — scheme II, mesh 3

preliminary suggestions of B. Szabo [11] for the standard p-elements. However, in the element sides not touching directly the corner, the suitable increase of the polynomial degree M was profitable.

The objective choice between the h-p schemes I and II was difficult because the error estimators in both cases were not calculated in the same points. However taking into consideration the additional 'free' parameter γ and the fact that the estimator values were taken in the second scheme (Fig. 6) much closer to the corner, observing the results we can suggest the irregular scheme II for the T-element substructures discussed.

4. Conclusions and final remarks

The adaptive rules of the T-elements are very different from the standard finite element approach. To obtain reliable results near strong concentrators it is necessary to improve simultaneously both, mesh and the approximation degree. The latter change consists in the increase or decrease of the polynomial degree in the frame function along the particular sides of the T-element. This change, against the intuition, should be degressive towards the point of singularity or strong concentrator not to cause the solution oscillations.

The authors propose the following general strategy for the Trefftz element linear analysis of a structure with large local stress gradients:

1. Preliminary topological division of the structure indicating possible large gradient areas (Fig. 2b),
2. Final selection of the concentration zones using energy criterion (Fig. 3),
3. Insertion of the special T-elements substructures into the prescribed zones,
4. Further improvement of the solution if the error estimator η is larger than the admissible value (e.g. division of elements in the low gradient zones and/or addition of the element layers in the T-element substructures).

Taking into consideration that the majority of the structural volume can be approximated by the low-cost large T-elements with high accuracy, the above procedure will result in a not expensive and reliable modelling of engineering structures.

The present paper obviously does not close the necessary investigations of the discussed algorithm. The following problems should still be considered:

- different, more optimal forms of smoothing of the corner results,
- better error estimators/indicators,
- optimal dimensions of the selected, large gradient zones,
- optimal internal T-element topology (after the selection of the concentration zones),
- particular engineering problems (e.g. 2D and 3D elastic objects), etc.

However, the general strategy of proceeding proposed in the paper seems to be inevitable in the fully automated, commercial code while applying to real complex engineering structures.

Acknowledgements

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