

## Effectiveness of the Trefftz method in different engineering optimization procedures

Andrzej P. Zieliński, Henryk Sanecki and Marek Karaś  
*Institute of Mechanics and Machine Design, Cracow University of Technology  
ul. Warszawska 24, 31-155 Kraków, Poland*

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In a large class of linear, mathematically modelled engineering problems the Trefftz algorithms give accurate solutions in a relatively short computational time. Moreover, the Trefftz functions, fulfilling the governing differential equations, can be used as shape functions of finite elements (T-elements), also with openings and notches. This suggested the authors to investigate the advantages and limitations of the method in optimization of structures with the stress concentrators, e.g. perforated plates. Certain auxiliary object functions, which included simultaneously the objective of the optimization and the constraints, were introduced and investigated. Different optimization strategies were also taken into consideration. To improve the optimization task in case of a large number of variables the authors suggested an algorithm, which used the engineering sensitivity analysis to eliminate less important variables in the particular stages of the procedure.

### 1. INTRODUCTION

The paper deals with the strategy of optimization of complex engineering structures. The general theory of structural optimization considerably developed in recent years [2, 16]. However, the great majority of works concern design of relatively simple objects — plates, shells, arcs, trusses, frames in which the shape, thickness, configuration, reinforcements etc. are optimized [2]. The present work takes into consideration more complex, real engineering structures the geometry of which is in principle defined (Fig. 1). In such structures, however, numerous design parameters — certain dimensions, angles, local thickness, position of holes, corner radii etc., can be optimally modified. The objective functions and constraints include many different structural features like volume, strength, stiffness, dynamics (eigenfrequencies) or stability.

The above variety enlightens complexity of the engineering optimization problem. However, it should be underlined that the optimization algorithms based on numerical modelling of any structure are incomparably easier and less expensive than its experimental modifications. Therefore, it is necessary to improve the optimization procedures in different aspects, the most important of which, in opinion of the authors, are presented below.

#### 1. Decreasing of the computer time in a single solution inside the optimization loops.

In the standard numerical engineering analysis, the CPU time is less important than the effort of the engineer preparing the data. Therefore, the contemporary commercial finite element systems do not attach primary importance to the decrease of the computational time. In optimization algorithms the situation is very different. The basic, gradually modified solution must here be repeated even thousands times inside the optimization loops. In this case the Trefftz approach [6, 7, 15] seems to be a very convenient tool at least for linear models of the considered structures. The application of the Trefftz finite elements (T-elements [8, 10]) to optimization procedures is one of the most important aspects of the present paper (see also [14]).



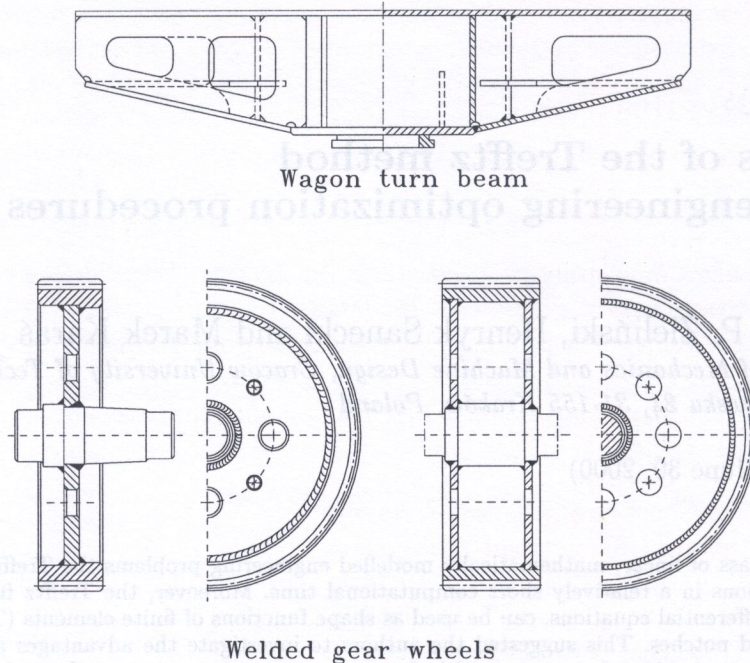


Fig. 1. Examples of complex engineering structures

## 2. Modifications of the objective functions and constraints.

The contemporary trends and also the numerical experiments carried out by the authors suggest introduction of the constraints in the weak, weighted form into the objective function. The discussion of different modifications of this function for a strength/stiffness problem is presented in Section 3.

## 3. Decreasing of the number of variables in a particular stage of the optimization algorithms.

In more complex structures the number of potential optimization variables can be very large. Also, as it was mentioned before, they are of different types and have different constraints. In this situation to successfully search for a global extremum of the objective function, it is necessary to eliminate (to fix) a certain number of variables in a particular stage of the procedure. This can be done with the help of the engineering sensitivity analysis, which is discussed in Section 4.

## 4. Improved search for extrema of the objective function.

Even after elimination of the majority of the potential variables from the optimization process, the stable search of a global extremum of the objective function can be difficult. The algorithms usually include partly stochastic, partly gradient-type procedures. In recent years, most successfully developed is the family of evolution methods including the genetic algorithms [3, 5]. This aspect of improvement of the optimization procedures is not considered in the present work.

## 2. EFFECTIVE APPLICATION OF THE TREFFTZ ELEMENTS IN THE OPTIMIZATION ALGORITHM

The T-elements are large finite elements or substructures, the shape functions of which (called T-functions) identically fulfil the governing differential equations of a given problem [17]. The neighbouring fields of the T-functions are non-conforming.

The minimization of the functional jumps disturbing the solution on the T-element interfaces can be achieved in different ways [4, 9, 11]. However, in most cases a certain boundary "frame" function, common for the neighbouring elements, is introduced [8, 10]. In our numerical investigations we



considered mainly 2D elastic structures and applied the well-established hybrid-Trefftz displacement (HTD) element [7]. The stiffness matrix of this element has the characteristic form (symmetric and positive definite),

$$\mathbf{k} = \mathbf{G}^T \mathbf{H}^{-1} \mathbf{G},$$

$$\mathbf{G} = \int_{\Gamma_e} \mathbf{G}^T \tilde{\mathbf{N}} d\Gamma, \quad \mathbf{H} = \int_{\Gamma_e} \mathbf{N}^T \mathbf{T} d\Gamma, \quad (1)$$

where  $\mathbf{N}$  and  $\mathbf{T}$  are Trefftz function matrices of displacements and tractions, respectively, and  $\tilde{\mathbf{N}}$  contains polynomial frame functions of hierarchic character. The relation between the coefficients (degrees of freedom)  $\mathbf{d}$  of the boundary frame function  $\tilde{\mathbf{u}}$  and the Trefftz coefficients  $\mathbf{c}$  of the internal displacement field  $\mathbf{u}$  is defined by the residual equation,

$$\int_{\Gamma_e} \mathbf{T}^T (\mathbf{u} - \tilde{\mathbf{u}}) d\Gamma = 0, \quad (2)$$

where  $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{Nc}$ ,  $\hat{\mathbf{u}}$  — particular solution,  $\tilde{\mathbf{u}} = \mathbf{Nd}$ , which leads to

$$\mathbf{c} = \mathbf{H}^{-1} (\mathbf{Gd} - \mathbf{g}), \quad \mathbf{g} = \int_{\Gamma_e} \mathbf{T}^T \hat{\mathbf{u}} d\Gamma. \quad (3)$$

As we can see, all the matrices in (1) and (3) are calculated only along the element boundaries  $\Gamma_e$ .

The finite element system SAFE used in the numerical investigations includes also certain "special" T-elements in which the Trefftz shape functions fulfil not only the governing differential equations of the problem but also the boundary conditions on certain internal boundaries  $\Gamma_s$  (Fig. 2).

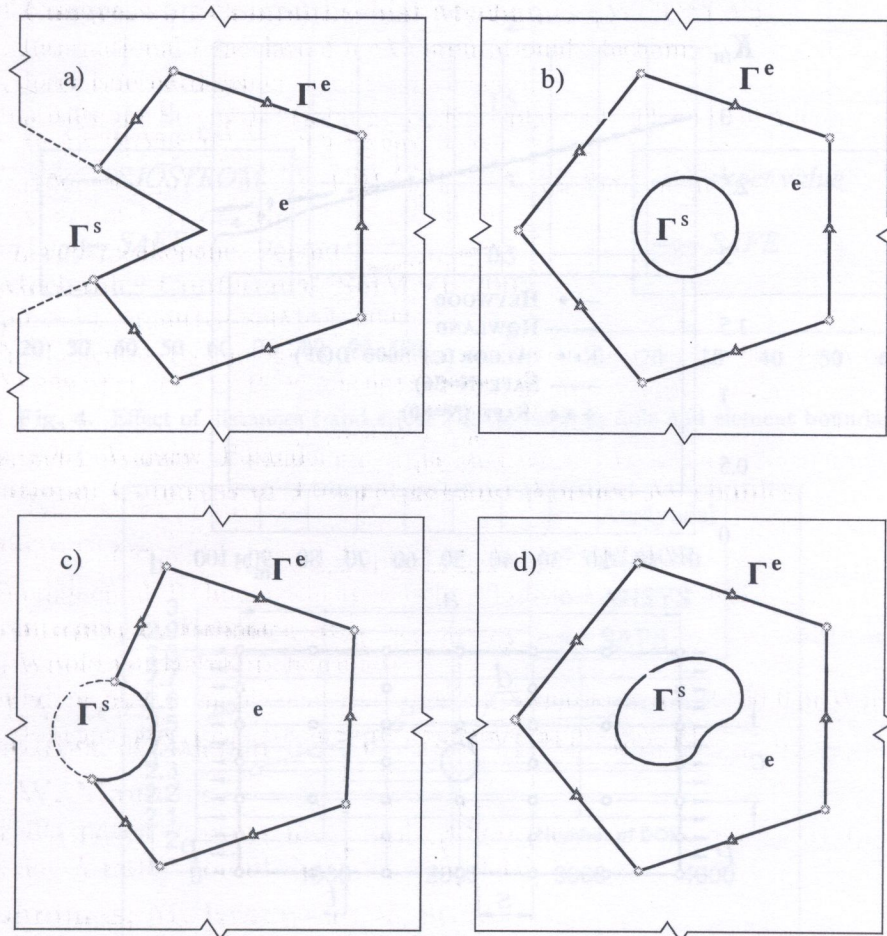


Fig. 2. Different types of special T-elements



In the case of an arbitrary contour  $\Gamma_s$  (Fig. 2d) the least square element HTS must be defined [18]. Unlike in the elements with the circular holes, the boundary conditions on  $\Gamma_s$  —  $\mathbf{u} = \bar{\mathbf{u}}$  or  $\mathbf{t} = \bar{\mathbf{t}}$  — cannot in general be satisfied exactly. To ensure symmetry of the stiffness matrices, the following functionals should be introduced [18],

$$J_1 = \int_{\Gamma_e} (\mathbf{u} - \tilde{\mathbf{u}})^2 d\Gamma + \alpha \int_{\Gamma_s} (\mathbf{u} - \bar{\mathbf{u}})^2 d\Gamma = \min \tag{4}$$

or

$$J_2 = \int_{\Gamma_e} (\mathbf{u} - \tilde{\mathbf{u}})^2 d\Gamma + \alpha\beta \int_{\Gamma_s} (\mathbf{t} - \bar{\mathbf{t}})^2 d\Gamma = \min, \tag{5}$$

where  $\alpha$  influences the relative accuracy of the fit on  $\Gamma_e$  and  $\Gamma_s$ , and  $\beta$  equalizes dimensions of both terms in  $J_2$ .

The standard T-elements were carefully examined showing evident superiority over the conventional FE formulation [8]. Therefore the authors paid more attention to the special Trefftz elements, very important in structural optimization. Their numerical investigations started from determination of admissible relations between the circular opening and the element dimensions [12]. For the diameters  $d$  larger than  $0.7c$  the solution error increased which could partly be diminished by the application of the higher order of the Trefftz approximation (Fig. 3). The above phenomenon is a certain limitation of the special T-elements while applying to structures with systems of holes very near to each other.

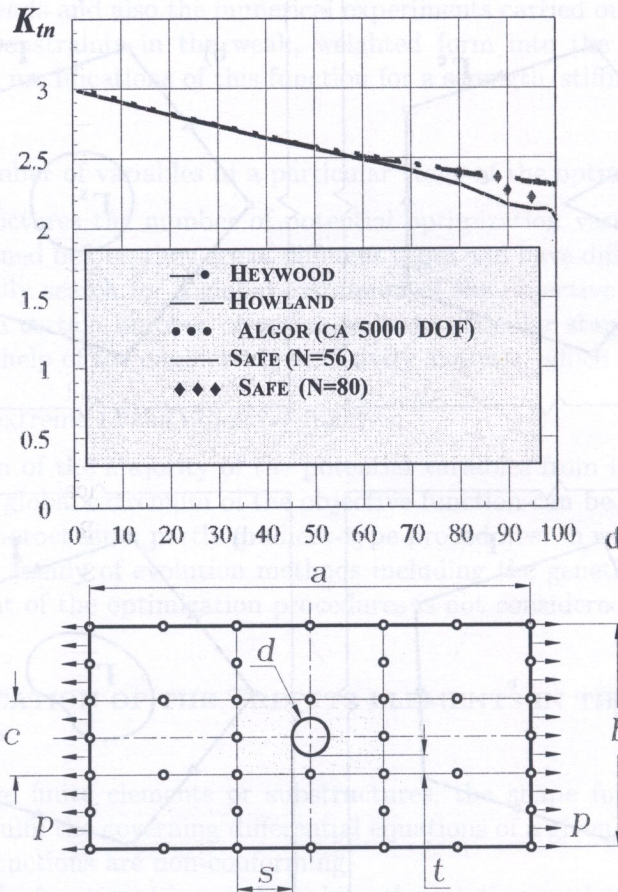


Fig. 3. Special T-element — influence of hole/element proportions on approximation accuracy;  $K_{tn}$  — concentration factor,  $a = 600$ ,  $b = 300$ ,  $c = b/3$



The approaching of the element boundary by a relatively smaller hole is less dangerous. Figure 4 presents the comparison of the stress concentration factor  $K_{tn}$  calculated by SAFE and analytically. Good accuracy (error less than 0.75%) was obtained for  $t > 5$  and arbitrary  $s$ .

The authors also carried out a series of examples comparing the efficiency of the special T-elements and the popular commercial FE codes. Figure 5 shows relations of numbers of degrees of freedom in the case of SAFE, ANSYS and ALGOR while solving the example defined in Fig. 3. The relations of the computer time were obviously not so spectacular because of more complex form of the stiffness matrix (1), however, the gain up to nine times was observed (it depended on the mesh design in the commercial codes).

Figures 6 and 7 present an example of a cantilever, which was also used in further investigations of the optimization procedures [13, 19]. The advantages of the application of the T-elements are here clearly visible. It should be underlined that the ALGOR mesh in the concentration regions was individually designed. And even in this case the CPU time was seven times smaller while applying the T-elements. Obviously, with the automatic h-adaptive mesh improvement the ALGOR computer time would be much larger.

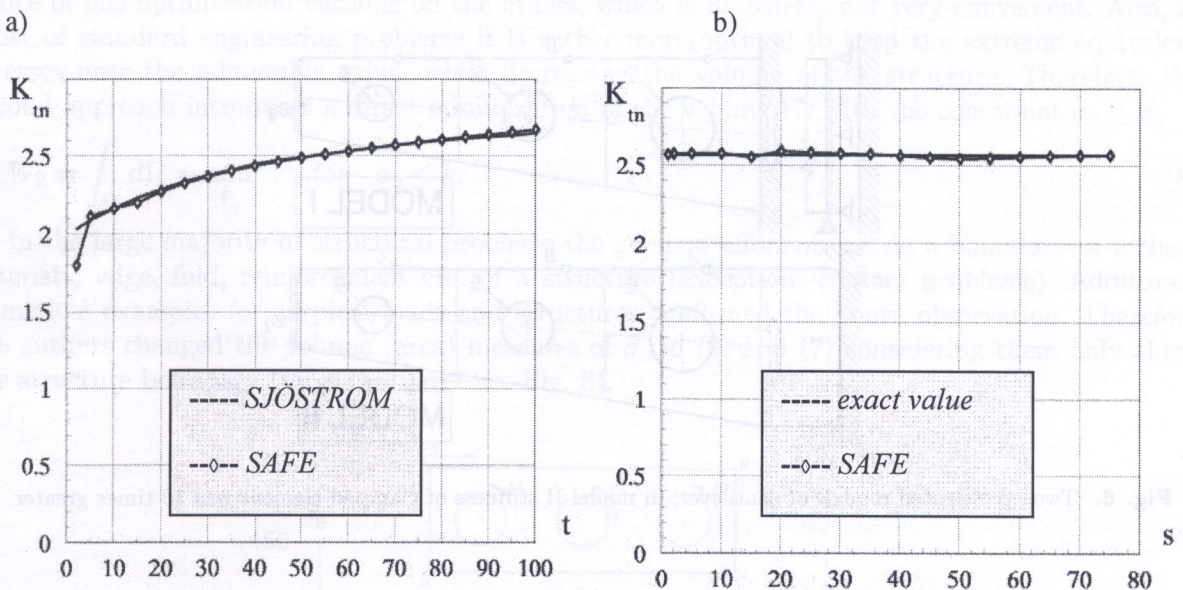


Fig. 4. Effect of distances  $t$  and  $s$  (see Fig. 3) between hole and element boundary

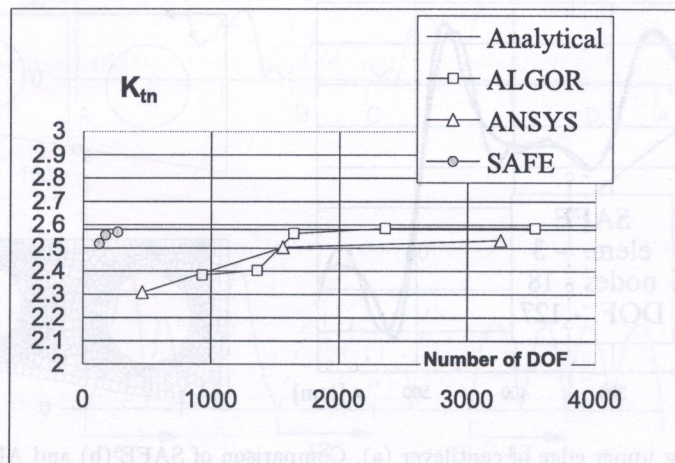


Fig. 5. Convergence of results for different FE codes (example defined in Fig. 3)



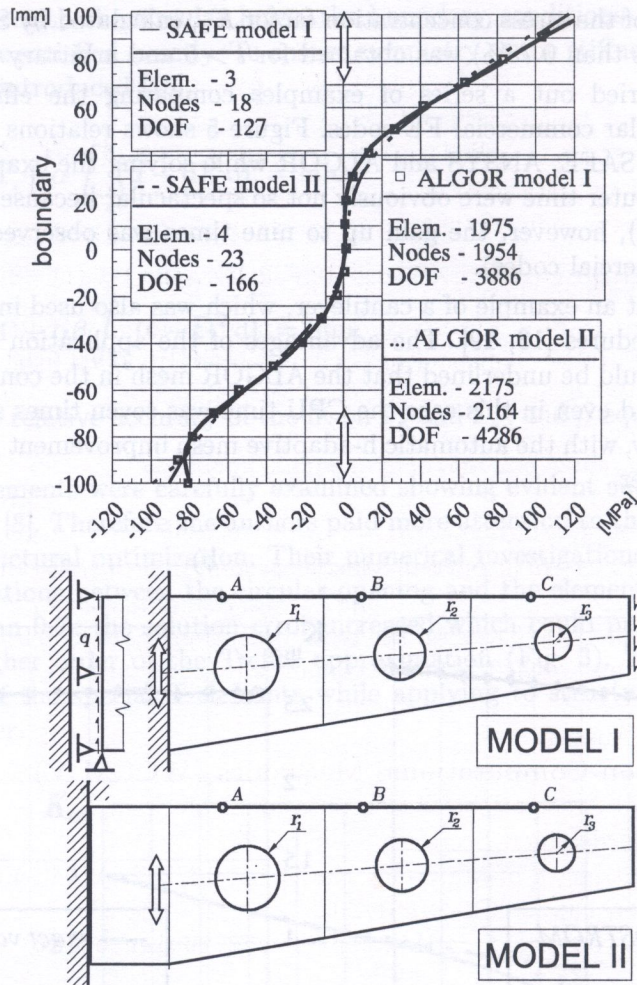


Fig. 6. Two investigated models of cantilever; in model II stiffness of clamped element was 10 times greater

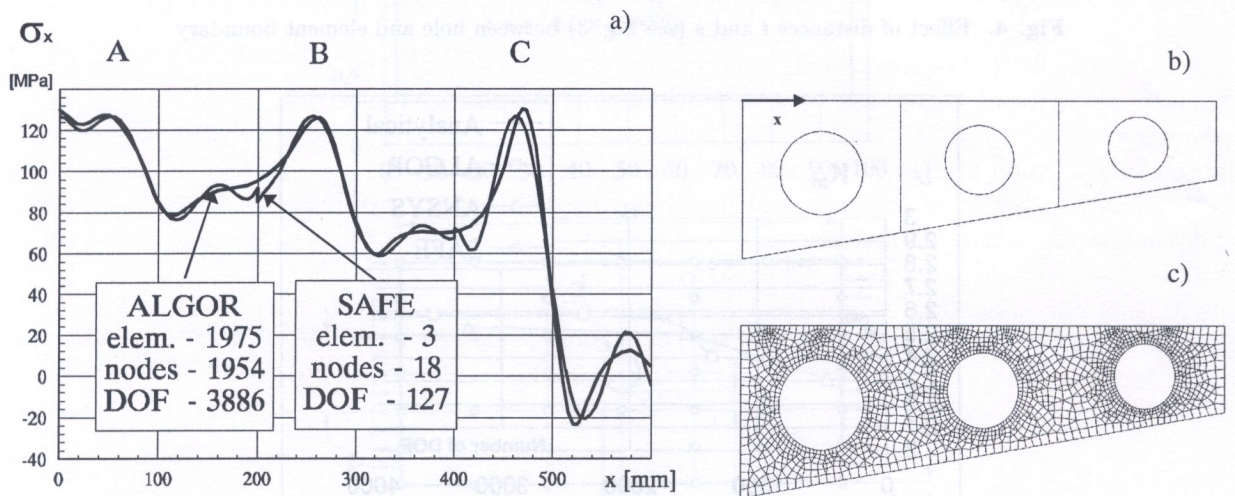


Fig. 7. Stress  $\sigma_x$  along upper edge of cantilever (a). Comparison of SAFE (b) and ALGOR (c) mesh and results for optimized radii of holes



### 3. DIFFERENT MODIFICATIONS OF OBJECTIVE FUNCTIONS

The second aspect of improvement of the optimization procedures is connected with different modified forms of the objective functions and constraints. In our investigations mainly the strength- and stiffness-type linear structural problems are considered. The classical strength formulation minimizes a certain global stress functional, for example

$$W_1 = \int_{\Omega} \left( \frac{\sigma_o}{k} \right)^K d\Omega = \min_{\mathbf{p}} \quad \text{for } V_{\Omega} = \text{const} \quad (6)$$

where  $\mathbf{p}$  is the vector of optimization variables,  $\sigma_o$  is an equivalent stress (according e.g. to the von Mises hypothesis),  $k$  is an admissible stress and  $V_{\Omega}$  is a volume of a thin-walled structure with the area  $\Omega$  (for 2D problems). If the exponent  $K \rightarrow \infty$  the integral (5) transforms into the Chebyshev-type norm and the formulation changes to

$$\overline{W}_1 = \max_{\Omega}(\sigma_o) = \min_{\mathbf{p}} \quad \text{for } V_{\Omega} = \text{const}. \quad (7)$$

The above optimization approach, the first one investigated by the authors, introduces dependence of one optimization variable on the others, which is in general not very convenient. Also, in most of standard engineering problems it is rather more optimal to keep the extreme equivalent stresses near the admissible value, while decreasing the volume of the structure. Therefore, the second approach introduces a direct minimization of the volume  $V_{\Omega}$  with the constraint  $\sigma_o \leq k$ ,

$$W_2 = \int_{\Omega} d\Omega = \min_{\mathbf{p}} \quad \text{for } \sigma_o \leq k. \quad (8)$$

In the large majority of structural problems the greatest effort occurs on a boundary or a characteristic edge, fold, reinforcement etc. of a structure (exception: contact problems). Additional numerical examples for atypical loads and structures confirmed the above observation. Therefore the authors changed the volume (area) measures of  $\sigma_o$  in (6) and (7) considering them only along the structure boundary (for a cantilever see Fig. 8).

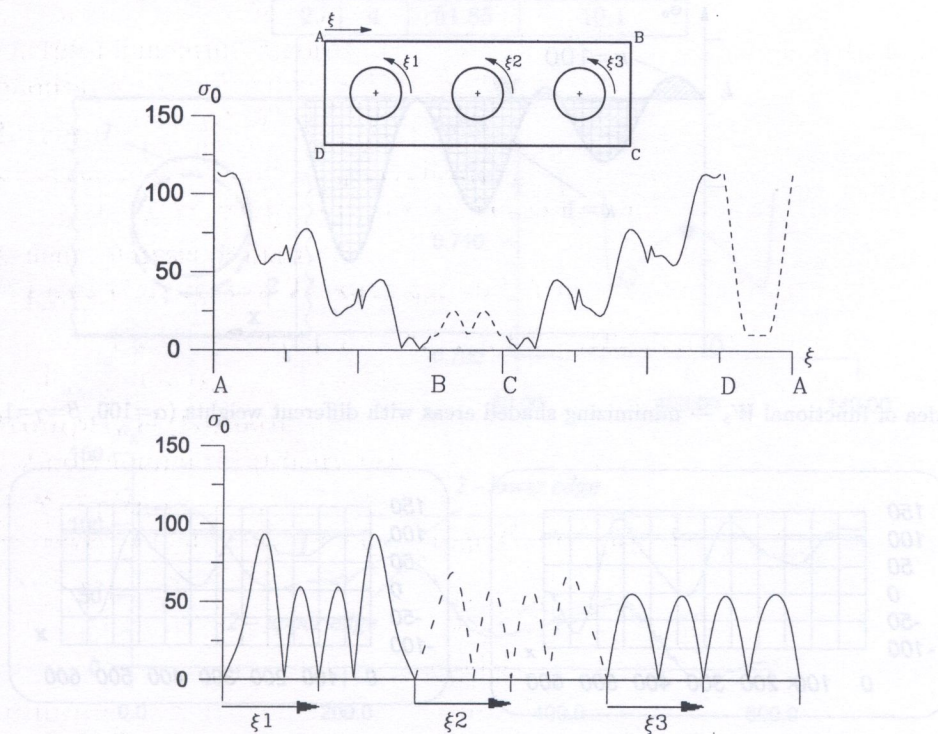


Fig. 8. Stresses  $\sigma_o$  along boundaries of rectangular cantilever loaded like in Fig. 6 (model I)



It should here be underlined that the *calculation of all the stress tensor components (not only tractions) is in the Trefftz approach very quick and simple*, which seems to be a considerable advantage in comparison to the boundary element method.

The formulation (7) of the objective function also occurred not to be convenient in the numerical algorithm because of the very different measure of the constraint. Therefore, the authors introduced the auxiliary functional  $W_3$  including both the optimization objective and the main constraint

$$W_3 = \frac{1}{\Omega} \int_{\Omega} \kappa \cdot \left| 1 - \left( \frac{\sigma_o}{k} \right)^{\beta} \right|^{\gamma} d\Omega = \min_{\mathbf{p}} \quad (9)$$

where

$$\kappa = \frac{1 + \alpha}{2} + \frac{1 - \alpha}{2} \cdot \text{sign} \left( 1 - \frac{\sigma_o}{k} \right). \quad (10)$$

The weighting constant  $\alpha$  minimized here the stress exceeding the admissible value  $k$ .

The interpretation of the functional (8) is illustrated in Fig. 9. The concept of minimization of the integral  $W_3$  assumes that the structure is more optimal from the strength point of view when the effort measure  $\sigma_o$  of the material is more uniformly distributed in the whole object and possibly close to the admissible value  $k$ . In this sense the idea is analogical to the concept of the fully stressed design [1]. Of course, in most structures the uniform effort is impossible. Hence, the difference existing in statically indeterminate frames between the optimal and fully stressed construction does not play here any important role.

The influence of the constant  $\alpha$  in formulation (1) is presented in Fig. 10. It can be observed that the introduction  $\alpha = 100$  is practically equivalent to the constraint  $\sigma_o \leq k$ . For  $\alpha = 10$ , the stress slightly crossed the admissible value. Obviously, this constant influences the final solution and can be problem dependent. However, the engineer who observes the final result of the optimization process can easily decide whether the small crossing of the admissible limit is in his opinion acceptable. Therefore, the exact choice of the penalty constant  $\alpha$  is not so important.

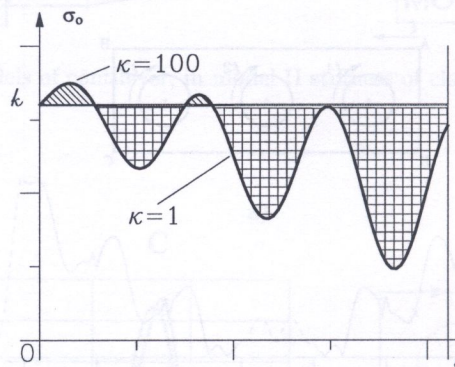


Fig. 9. Idea of functional  $W_3$  — minimizing shaded areas with different weights ( $\alpha=100$ ,  $\beta=\gamma=1$ ,  $\Omega \rightarrow l$ )

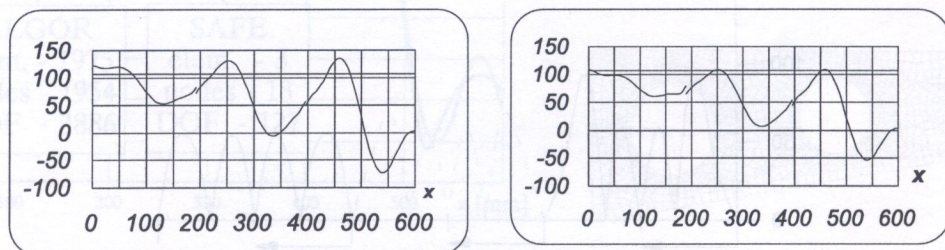


Fig. 10. Stresses (in MPa) along upper boundary of cantilever. Minimizing cantilever volume with objective function  $W_3$ : a)  $\alpha=1$ , b)  $\alpha=100$ ;  $\beta=2$ ,  $\gamma=2$ ; 3 Trefftz elements with 127 DOF



The constants  $\beta$  and  $\gamma$  obviously also influence the behaviour of the functional  $W_3$ . The higher values of these constants turned out to be not acceptable because they decreased the effect of the constant  $\alpha$  as a penalty-type constraint (Table 1). Practically,  $\beta = 2$  and  $\gamma = 1$  or 2 can be recommended.

The introduction of the functional  $W_3$  divided the optimization variables into two groups:

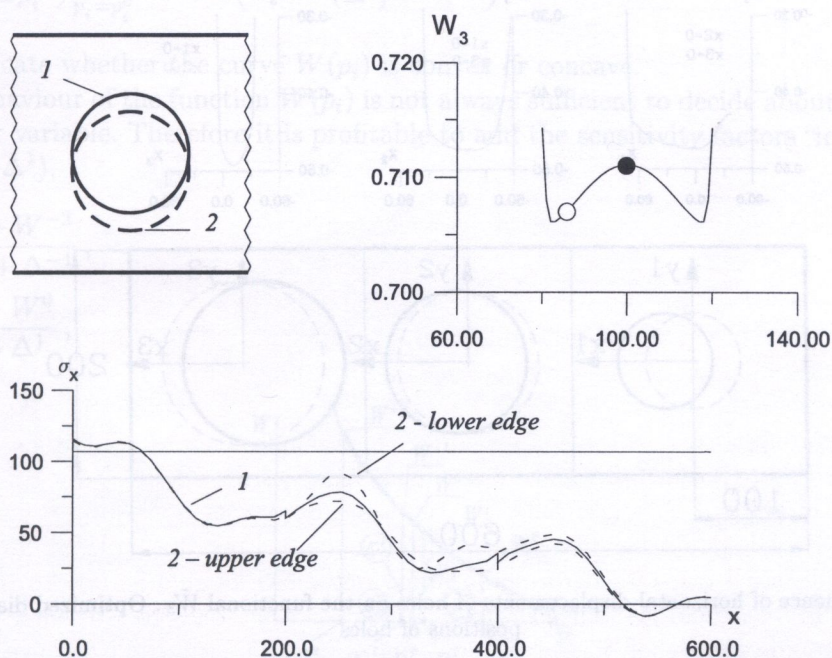
- I. variables directly changing the structure volume (e.g. diameters of holes),
- II. variables influencing the optimal shape of the structure without the direct change of the volume (e.g. positions of holes).

The definition (8)–(9) concerns the variables of the first kind ( $\kappa = \kappa_I$ ). In the optimization process they should be taken into consideration in the first stage. The variables II, introduced in the second turn, should rather maximize the functional  $W_3$  to make the variables I more effective in the consecutive stage. However, the constant  $\alpha$  penalizing the crossing of the admissible stress should be maintained. Hence, the definition (9) in the case of the variables II should have a form

$$\kappa_{II} = \frac{\alpha - 1}{2} - \frac{\alpha + 1}{2} \cdot \text{sign} \left( 1 - \frac{\sigma_o}{k} \right). \quad (11)$$

**Table 1.** Influence of parameters  $\beta$  and  $\gamma$  on optimization results in the first stage;  $r_{1opt}$  is optimal value of the radius,  $\sigma_{0max}$  is maximum equivalent stress calculated along the structure boundary

$\beta$	$\gamma$	$r_{1opt}$	$\sigma_{0max} - k$
1	1	44.94	0.0
2	1	44.94	0.0
4	1	44.94	0.0
4	2	46.59	1.9
2	2	47.31	2.8
1	2	48.29	4.0
4	4	49.33	5.4
2	4	51.85	10.1



**Fig. 11.** Influence of vertical shifting of a hole on stress distribution and the functional  $W_3$



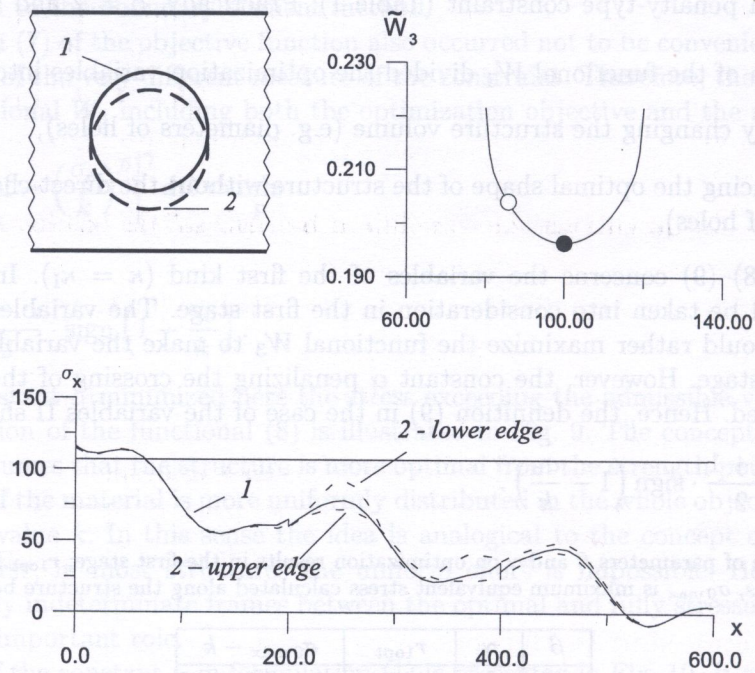


Fig. 12. Influence of vertical shifting of a hole on stress distribution and the corrected functional  $\tilde{W}_3$

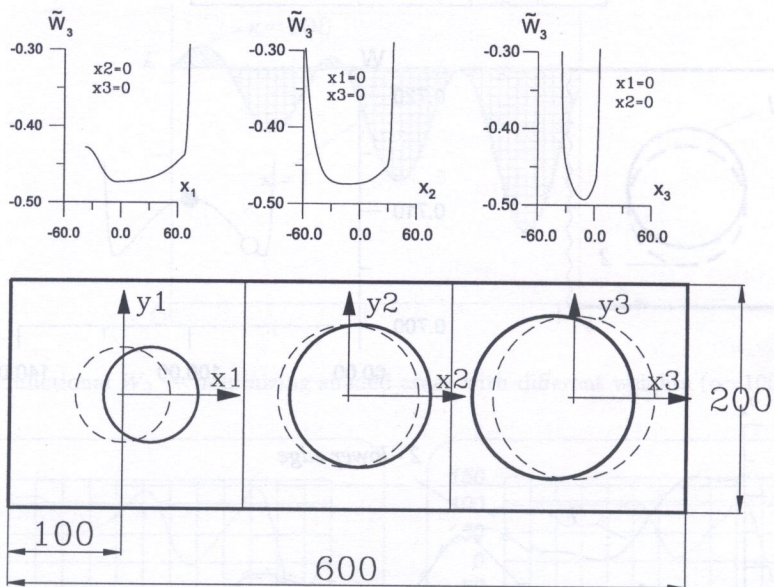


Fig. 13. Influence of horizontal displacements of holes on the functional  $\tilde{W}_3$ . Optimized diameters and positions of holes



The introduction of the functional  $\widetilde{W}_3$  (with  $\kappa = \kappa_{II}$ ) is justified in Figs 11, 12. For  $W_3$ , the vertical shifting of a hole caused a local maximum in the central, optimal position. The exchange of  $\kappa_I$  — for  $\kappa_{II}$  corrected the functional behaviour (Fig. 12). Figure 13 presents the results of the optimization of the rectangular cantilever (analogical to one defined in Fig. 6 model I) after the second stage. The functional  $\widetilde{W}_3$  includes the penalty constant (10). As we can see, this stage of optimization process requires shifting of the boundaries of the T-elements.

#### 4. ENGINEERING SENSITIVITY ANALYSIS DECREASING NUMBER OF OPTIMIZATION VARIABLES

Any objective function  $W$  (e.g.  $W_3$  or  $\widetilde{W}_3$ ) is not equally dependent on its optimization variables  $p_i$ . If the number of components of the vector  $\mathbf{p}$  is large, some of them usually could be removed from the searching process without any considerable effect on the optimum. Therefore, at least at some stages, it is profitable to eliminate such variables from the optimization algorithm. Figure 14 introduces certain sensitivity factors (numerical derivatives) estimating the local influence of the particular variable  $p_i$  on the objective function  $W$ . They are defined as

$$S_i^{-1} = \left. \frac{\Delta W}{\Delta p_i} \right|_{p_i=p_i^{-1}} = \frac{-3W^{-1} + 4W^0 - W^1}{\Delta^{-1} + \Delta^1}, \quad (12)$$

$$S_i^0 = \left. \frac{\Delta W}{\Delta p_i} \right|_{p_i=p_i^0} = \frac{W^1 - W^{-1}}{\Delta^{-1} + \Delta^1}, \quad (13)$$

$$S_i^1 = \left. \frac{\Delta W}{\Delta p_i} \right|_{p_i=p_i^1} = \frac{3W^1 - 4W^0 + W^{-1}}{\Delta^{-1} + \Delta^1}, \quad (14)$$

where  $\Delta^{-1}$  or  $\Delta^1$  are small steps of the variable in the closest vicinity of its current value. Using the three values of the objective function and observing the sign of the second derivative,

$$Z_i = \text{sign} \left( \left. \frac{\Delta^2 W}{\Delta p_i^2} \right)_{p_i=p_i^0} \right) = \text{sign} \left( \frac{W^{-1} - 2W^0 + W^1}{(\Delta^1)^2} \right), \quad (15)$$

we can also indicate whether the curve  $W(p_i)$  is convex or concave.

The local behaviour of the function  $W(p_i)$  is not always sufficient to decide about the elimination of the particular variable. Therefore it is profitable to add the sensitivity factors "in large" ( $\Delta^{-2} \gg \Delta^{-1}$  and  $\Delta^2 \gg \Delta^1$ ),

$$S_i^{-2} = \frac{W^0 - W^{-2}}{\Delta^{-2} + \Delta^{-1}}, \quad (16)$$

$$S_i^2 = \frac{W^2 - W^0}{\Delta^2 + \Delta^1}, \quad (17)$$

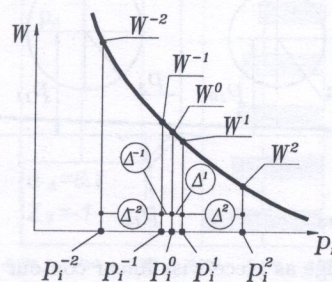


Fig. 14. Numerical estimation of sensitivity factors — defining scheme



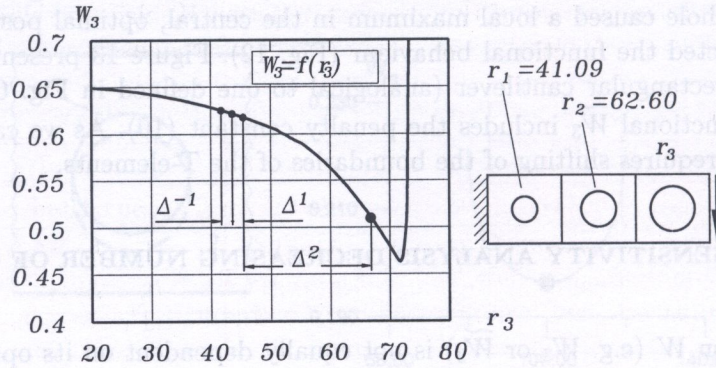


Fig. 15. Investigation of function  $W_3$  sensitivity in large

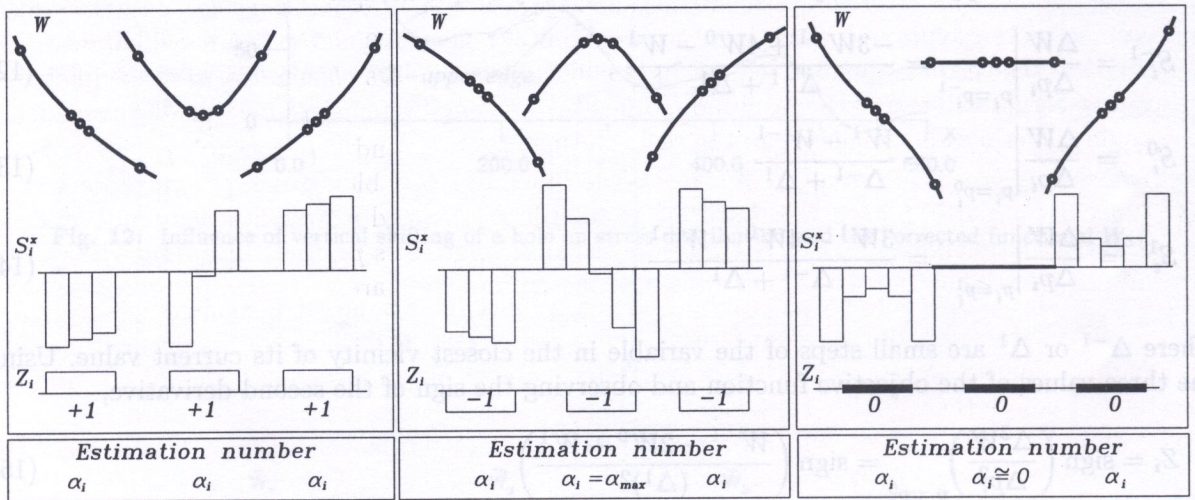


Fig. 16. Sensitivity estimation factors for different shape cases of objective function  $W$

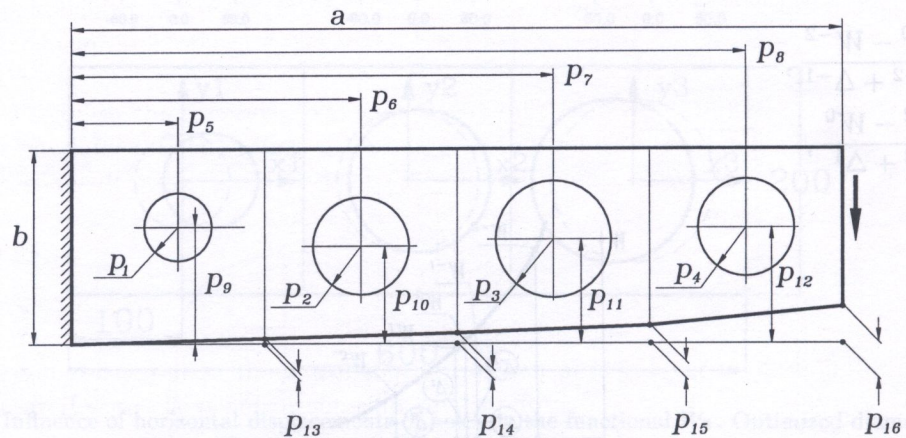


Fig. 17. Cantilever beam with lower edge as piece-wise linear contour and with 4 holes changing their radii and positions; 16 optimization parameters



which introduce more information about the global behaviour of  $W(p_i)$  (compare Fig. 15).

Figure 16 presents different cases of the objective function  $W$  classified due to the  $S_i^0$ ,  $S_i^{-1}$ , or  $S_i^1$  (denoted as  $S_i^x$ ) and also  $Z_i$ . For the sensitivity estimation we usually apply the average angle

$$\alpha_i = \frac{1}{3} |\arctan S_i^{-1} + \arctan S_i^0 + \arctan S_i^1|. \tag{18}$$

However, the specific value  $\alpha_i \cong 0$  should be investigated carefully — here the factor  $Z_i$  decides whether we are near maximum or minimum on the curve  $W(p_i)$ . If  $S_i^{-2}$  and  $S_i^2$  confirm e.g. a distinct maximum then the value  $\alpha_i = \alpha_{\max}$  (equal for instance 90) should instead be taken — we are far from the optimum.

As an example let us take a cantilever plate, Fig. 17, with 4 holes changing their radii and positions and with its lower edge defined as piece-wise linear contour. We have 16 optimization parameters: 8 of them are variables directly changing the structure volume and the remaining 8 are variables of the II-nd kind. The estimation (18) of the sensitivity of the objective function on the optimization variables was calculated and the “top variables” were first taken to minimization process. Fig. 18 presents bar diagrams with the  $S_i^x$  values for some of the I-st kind variables with the initial values:  $p_1 = p_2 = p_3 = p_4 = 50 \text{ mm}$  and  $p_{13} = p_{14} = p_{15} = p_{16} = 0$ . Maximum value of  $\alpha_i$  was obtained for the variable  $p_2$ : 10.6.

Figure 19 presents bar diagrams with the  $S_i^x$  values for the II-nd kind variables. The diagrams were plotted after reaching the minimum for the first kind variables. Here, the maximum value  $\alpha_8 = 29.8$  was obtained for the variable  $p_8$ . The co-ordinates  $p_5$  and  $p_6$  practically did not influence the objective function, so only the paths of the remaining variables  $p_8$  and  $p_7$  were searched at the current stage of the optimization. The results of the optimization are presented in Fig. 20.

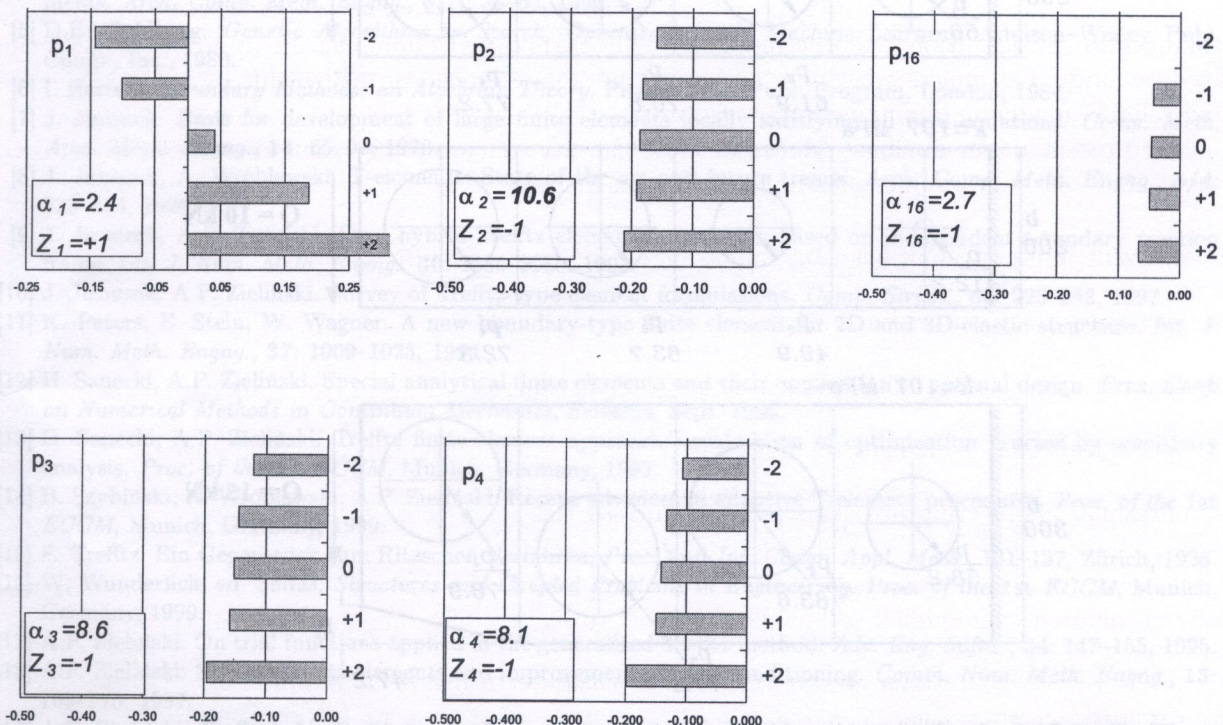


Fig. 18. Sensitivity factors for I-st kind variables at start of optimization process



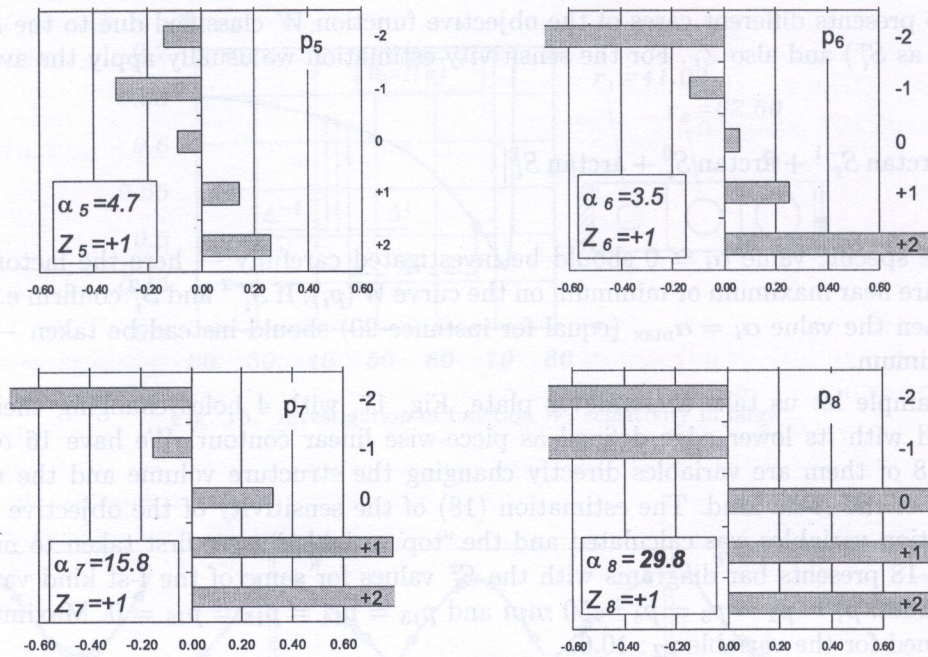


Fig. 19. Sensitivity factors for II-nd kind variables

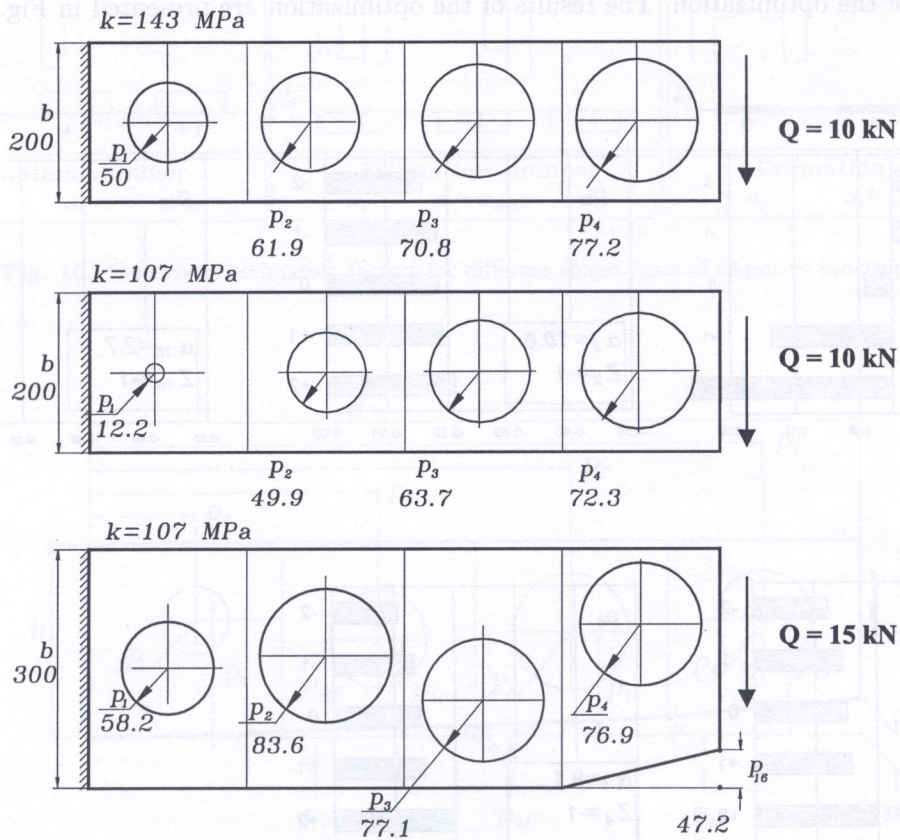


Fig. 20. Optimal radii, positions of holes and shape of lower edge for different admissible stresses  $k$  and cantilever dimensions



## 5. CONCLUSIONS AND FINAL REMARKS

The paper proposed certain improvements of the optimization procedure in case of complex engineering structures. The visible progress was observed while applying the Trefftz-type elements on each step of the optimization loops. The specific objective function  $W_3$ , including the main constraint in a weak form, appeared to be very convenient in the optimization process. Also a new form of the sensitivity analysis was proposed, which enabled elimination of less important optimization variables.

The diagrams and figures presented the characteristic features of the introduced procedures. However, they need further investigations to prove their applicability to a possibly large class of problems. In this stage of the study the optimization of complex engineering structures can be considered rather as a computer aided design in which the "on line" control of an engineer is necessary. However, the authors are convinced that, after several stages including the procedures proposed, the final form of the designed structure will be to a large extent optimized.

## ACKNOWLEDGEMENT

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