Computer Assisted Mechanics and Engineering Sciences, 8: 1–16, 2001.

Thin plate bending analysis using an indirect Trefftz collocation method

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(Received April 20, 1998)

In this work, the application of an indirect Trefftz collocation method to the analysis of bending of thin plates (Kirchhoff's theory) is described. The deflection field approximation is obtained with the use of a set of functions satisfying *a priori* the homogeneous part of the differential equation of the problem. Each of the approximating functions is derived from a known thick plate solution. The boundary conditions are imposed by means of continuous (integral) and discrete (collocation) least squares methods. Numerical examples are presented and the accuracy of the proposed technique is assessed.

Keywords: Trefftz method, thin plates, point collocation, complex trial functions, least squares

1. Introduction

In the engineering practice, the finite element method (FEM) is used widely to obtain numerical solutions of problems that are formulated in terms of differential equations and the corresponding boundary conditions (BC). The search for accurate and better performing elements lead some researchers to use, as trial functions for the internal fields, functions that are themselves solutions of the differential equations of the problem *a priori*. The interelement continuity and the BC may, then, be enforced pointwise or in an integral weighted residual sense.

While the conventional conforming FEM formulations may be viewed as particular applications of the Rayleigh-Ritz method, the alternative method of using actual solutions of the governing equations as trial functions is closely related to the method presented by Trefftz in 1926 [10]. Since 1977, when Jiroušek and Leon [5] published a thin plate bending element based on the variational Trefftz method, the range of applications has been extended to a broad variety of problems. Reviews on the subject may be found in Zieliński [15] and Kita and Kamiya [7].

It is well known that the thin plate theory, due to the assumptions that are made, requires C^1 continuity. That leads to difficulties when generating conventional FE (finite elements). On the other hand, for the usual practice of civil engineering it is not necessary to take into account the effect on bending of the transverse (shear) deformation. Reference [6] shows that it is possible to obtain quality results for Kirchhoff plates with the variational Trefftz method.

The formulations of Trefftz-type elements (T-elements) can include an auxiliar displacements frame defined in terms of the same degrees of freedom (DOF) used in the conventional FE. The main advantage is that these elements can be used by a standard FE program. On the other hand, the frameless T-elements (in which the DOF are only the parameters of the set of functions chosen to approximate the internal field) are easier to generate. The formulation described in this paper can be seen as a form of frameless T-elements. The choice of how to impose the BC is also an important one. Two options are available: point collocation or some form of weighted residuals (integration) on the boundary. A review of both was made by Eason [1], where references to applications in atomic physics, elasticity (composite and homogeneous materials, stress concentrations, thermal loadings), electromagnetic fields, fluid flow, heat transfer and mathematical are given.

A review of the analysis of problems of shells and plates using the boundary collocation method (BCM) was made by Hutchinson [4]. Discussion of several cases was made: thin plate bending, buckling and vibration, thick plate vibration, shallow spherical shells and thick shells of revolution. A extensive review of the application of the BCM to mechanics of continuous media is given by Kolodziej [8]. The possibility of using influence functions and collocation outside the domain of the problem under analysis was exploited by Wu and Altiero [13, 14] to solve thin plate bending problems.

The structure of this work includes a brief description of the Kirchhoff plate bending theory, followed by the definition of the complex polynomial set of functions used to approximate the deflection field. The technique proposed here, together with the definition of the BC, is then described. Reference to implementation aspects, numerical applications and corresponding comments and conclusions complete the work.

2. Classic formulation of thin plate theory

Consider the plate represented in Fig. 1 where "$^{-}$" stands for imposed quantities. The following notation will be used:

w – deflection of the middle surface of the plate;

$\frac{\partial w}{\partial n}$ – slope of the middle surface of the plate;

M_n – bending moments of a section of the plate perpendicular to the n direction;

M_{nt} – twisting moments of a section of the plate perpendicular to the n direction;

Q_n – shearing force parallel to z axis of a section of the plate perpendicular to the n direction;

$V_n = Q_n + \frac{\partial M_{nt}}{\partial s}$ – effective shear force parallel to z axis of a section of the plate perpendicular to the n direction;

x, y, z – cartesian co-ordinate system;

t – thickness of the plate;

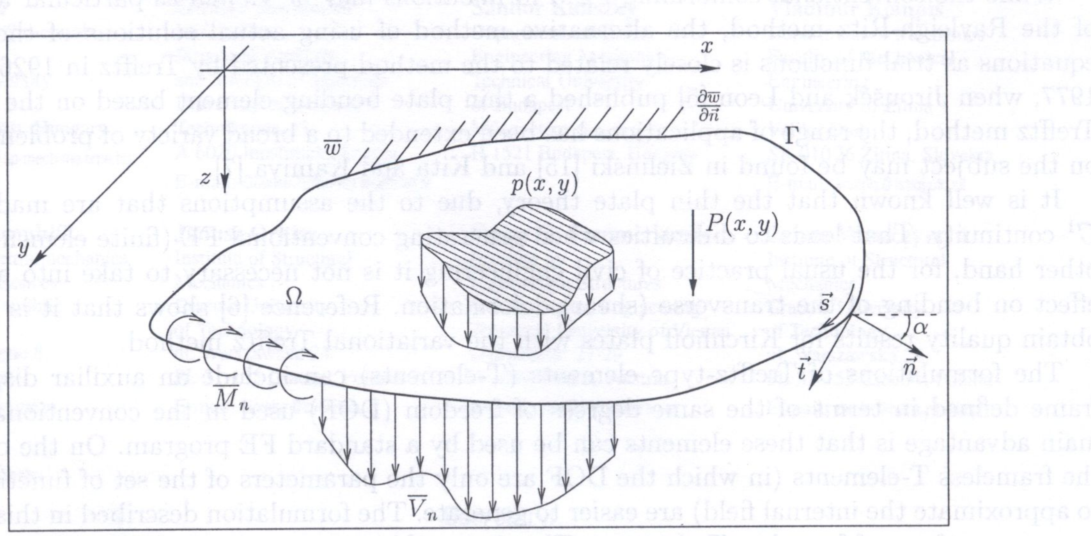

Fig. 1. Arbitrary thin plate

E – Young's modulus;

ν – Poisson's ratio;

p – uniform load;

P – concentrated load.

The differential equation of the deflection surface is the well known Lagrange equation

$$\nabla^4 w = \frac{p}{D}\,, \tag{1}$$

where ∇^4 is the biharmonic operator and $D = \frac{Et^3}{12(1-\nu^2)}$ is the flexural rigidity of the plate.

By neglecting the effect of deformation due to the shear stresses it is possible to identify the dependency between an additional shear force, Q'_n and $\frac{\partial M_{nt}}{\partial t}$.

The domain of the problem, Ω, may be split into several regions featuring some specific behaviour. For the i-th region the BC may be classified as being of the *support* or *continuity* type. Each point on the boundary of this region should fulfil *two* support BC: deflection or effective normal shear force and normal slope or normal bending moment,

$$w^i = \overline{w}^i \qquad t \in \Gamma^i_w\,, \tag{2a}$$

$$\frac{\partial w^i}{\partial n} = \frac{\partial \overline{w}^i}{\partial n} \qquad t \in \Gamma^i_{\frac{\partial w}{\partial n}}\,, \tag{2b}$$

$$M^i_n + GJ\frac{\partial^3 w^i}{\partial t^2 \partial n} = \overline{M}^i_n \qquad t \in \Gamma^i_{M_n}\,, \tag{2c}$$

$$Q^i_n + \frac{\partial M^i_{nt}}{\partial t} + EI\frac{\partial^4 w^i}{\partial t^4} = \overline{V}^i_n \qquad t \in \Gamma^i_{V_n}\,, \tag{2d}$$

The existence of beams on the boundary is allowed by means of the *flexural rigidity*, EI, and *torsional rigidity*, GJ.

The enforcement of continuity and equilibrium between the regions requires *four* boundary conditions to be satisfied at the interface between Ω_i and Ω_{i+1}, that is, $\Gamma_I = \Gamma_i \cap \Gamma_{i+1}$.

$$w^i - w^{i+1} = 0, \tag{3a}$$

$$\frac{\partial w^i}{\partial n} + \frac{\partial w^{i+1}}{\partial n} = 0, \tag{3b}$$

$$M^i_n + GJ\frac{\partial^3 w^i}{\partial t^2 \partial n} - M^{i+1}_n = 0, \tag{3c}$$

$$V^i_n + EI\frac{\partial^4 w^i}{\partial t^4} + V^{i+1}_n = 0. \tag{3d}$$

Once the displacement field is known it is straightforward to evaluate the bending and twisting moments

$$M_x = -D\left(\frac{\partial^2 w}{\partial^2 x} + \nu\frac{\partial^2 w}{\partial^2 y}\right), \tag{4a}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial^2 y} + \nu\frac{\partial^2 w}{\partial^2 x}\right), \tag{4b}$$

$$M_{xy} = M_{yx} = -D(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right) \tag{4c}$$

and the shear forces

$$Q_x = -D\frac{\partial}{\partial x}\nabla^2 w, \tag{5a}$$

$$Q_y = -D\frac{\partial}{\partial y}\nabla^2 w, \tag{5b}$$

where ∇^2 is the harmonic operator.

Noticing that

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha,$$

Eq. (2b) can be written in the form

$$\frac{\partial w^i}{\partial x}\cos\alpha + \frac{\partial w^i}{\partial y}\sin\alpha = \frac{\partial \overline{w}^i}{\partial n}. \tag{6}$$

The integration of the normal component of the stresses parallel to n direction, σ_n, and the shearing stresses perpendicular to n direction, τ_{nt}, gives

$$M_n = M_x\cos^2\alpha + M_y\sin^2\alpha + 2M_{xy}\sin\alpha\cos\alpha, \tag{7a}$$

$$M_{nt} = (M_y - M_x)\sin\alpha\cos\alpha + M_{xy}\left(\cos^2\alpha - \sin^2\alpha\right). \tag{7b}$$

The shear forces acting perpendicular to the middle plane are

$$Q_n = Q_x\cos\alpha + Q_y\sin\alpha. \tag{8}$$

Substituting (4) in (7a), and then in (2c), the following equation is obtained,

$$-D\left\{\nu\nabla^2 w^i + (1-\nu)\left(\cos^2\alpha\frac{\partial^2 w^i}{\partial x^2} + \sin^2\alpha\frac{\partial^2 w^i}{\partial y^2} + \sin 2\alpha\frac{\partial^2 w^i}{\partial x\partial y}\right)\right\} + GJ\frac{\partial^3 w^i}{\partial t^2\partial n} = \overline{M}_n^i. \tag{9}$$

Proceeding in the same way, substituting (5) in (8) and (4) in (7b), and then in (2d), leads to the following equation

$$-D\left\{\cos\alpha\frac{\partial}{\partial x}\nabla^2 w^i + \sin\alpha\frac{\partial}{\partial y}\nabla^2 w^i + (1-\nu)\frac{\partial}{\partial t}\left[\cos 2\alpha\frac{\partial^2 w^i}{\partial x\partial y} + \frac{1}{2}\sin 2\alpha\left(\frac{\partial^2 w^i}{\partial y^2} - \frac{\partial^2 w^i}{\partial x^2}\right)\right]\right\}$$
$$+ EI\frac{\partial^4 w^i}{\partial t^4} = \overline{V}_n^i, \tag{10}$$

where

$$\frac{\partial}{\partial t} = -\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}.$$

By using Eqs. (2a), (6), (9) and (10) all the support boundary conditions can be written in terms of w^i and its derivatives alone.

The same procedure is used for continuity type BC.

3. Approximating functions of the deflection field

The Trefftz method uses, as trial functions, actual solutions of the differential equation. The complex form representation is a very convenient way of describing the infinite set of linearly independent trial functions which form the complete basis of solutions of the governing differential equations of the plate bending problem. Due to the linearity of the differential equation it is possible to express the deflection function (the solution) as the sum of the solution of the homogeneous part (w_h) and a particular solution of the nonhomogeneous part ($\mathring{w}$):

$$w = \mathring{w} + w_h \, . \tag{11}$$

A thick plate solution is given in [9]. From this it is possible to obtain an appropriate thin plate solution after neglecting warping and change of thickness terms (all terms which involve h^2, z^2 and z^3), as follows,

$$w_h = \frac{1}{2\mu}\,\Re\left[\bar{\zeta}\Phi + \chi\right], \tag{12}$$

where $2\mu = \frac{E}{1+\nu}$, $\zeta = x + iy$, $\bar{\zeta} = x - iy$, $\Phi = \Phi(\zeta)$ and $\chi = \chi(\zeta)$. One possible choice for functions Φ and χ can be the following complex power series

$$\Phi(\zeta) = \sum_{j=1}^{m} a_j\,(\zeta)^j \, , \tag{13a}$$

$$\chi(\zeta) = \sum_{j=2}^{m} b_j\,(\zeta)^j \, , \tag{13b}$$

where $a_j = \alpha_j + i\beta_j$ and $b_j = \gamma_j + i\delta_j$. The term δ_0 is dropped out because its coefficient in the expansion (12) is zero. The terms γ_1 and δ_1 are, also, excluded since they represent rigid body motions that are already taken into consideration in the terms α_0 and β_0 . Expression (12) assumes the following matrix form,

$$w_h = \sum_{j=1}^{m} N_j\,c_j = \mathbf{N}_w\,\mathbf{c}, \tag{14}$$

where

$$\mathbf{N}_w = \left[\begin{array}{cccc} \mathbf{N}_0 & \mathbf{N}_1 & \cdots & \mathbf{N}_m \end{array}\right],$$

$$\mathbf{c}^{\mathrm{T}} = \left[\begin{array}{cccc} \mathbf{c}_0 & \mathbf{c}_1 & \cdots & \mathbf{c}_m \end{array}\right],$$

$$\mathbf{N}_0 = \frac{1}{2\mu}\left[\begin{array}{ccc} x & y & 1 \end{array}\right], \quad \mathbf{N}_1 = \frac{1}{2\mu}\left[\, r^2 \,\right],$$

$$\mathbf{N}_k = \frac{1}{2\mu}\left[\begin{array}{cccc} r^2\,\Re(\zeta)^{k-1} & -r^2\,\Im(\zeta)^{k-1} & \Re(\zeta)^k & -\Im(\zeta)^k \end{array}\right],$$

$$\mathbf{c}_0 = \left[\begin{array}{ccc} \alpha_0 & \beta_0 & \gamma_0 \end{array}\right], \quad \mathbf{c}_1 = \left[\, \alpha_1 \,\right],$$

$$\mathbf{c}_k = \left[\begin{array}{cccc} \alpha_k & \beta_k & \gamma_k & \delta_k \end{array}\right], \quad \text{where } r^2 = x^2 + y^2 \text{ and } k = 2, 3 \ldots m.$$

The number of terms in $\mathbf{c}$ is $4\,m$.

The particular solution, $\mathring{w}$, for a uniform load, p, and for a concentrated load, P, located at $(x_P\,,\,y_P)$, may be taken as

$$\mathring{w} = \frac{p\,r^4}{64D} + \frac{P\,r_P^2\ln r_P^2}{16\pi D} \tag{15}$$

where $r_P^2 = (x - x_P)^2 + (y - y_P)^2$.

If it is of interest to analyse the local effects of point loads may be advantageous to replace the actual load by a uniform one acting in a small circle of radius b. Taking d as an optional constant and using the definitions, $\rho = \frac{r}{d}$ and $\beta = \frac{b}{d}$, the solution for an infinite plate loaded uniformly over a circle with static resultant $\overline{P}$ is

$$\text{if } \rho \le \beta \qquad \overset{\circ}{w} = \left[\frac{\beta^2}{64}\left(4\ln\beta - 3\right) + \frac{1}{16} + \frac{\rho^2}{32}\left(4\ln\beta - \beta^2\right) + \frac{\rho^4}{64\beta^2}\right]\frac{\overline{P}d^2}{\pi D}, \tag{16a}$$

$$\text{if } \rho \ge \beta \qquad \overset{\circ}{w} = \left[\frac{1}{32}\left(2+\beta^2\right)\left(1-\rho^2\right) + \frac{1}{16}\left(\beta^2 + 2\rho^2\right)\ln\rho\right]\frac{\overline{P}d^2}{\pi D}. \tag{16b}$$

Solutions for linear loads acting on a patch domain or straight line are given by Fernandes [2].

For regions with singularities due to the geometry (regions containing holes, skew plates, etc) alternative trial functions for homogeneous and particular solutions may be used.

The region (or regions) may have an arbitrary shape as long as it is simply connected. For regions that are not simply connected, such as the ones with holes, it should be wise to use different sets of solutions for that specific case or to decompose the region into a number of simply connected ones.

4. Imposition of the Boundary Conditions

At each point located on the boundary {interface} two {four} boundary conditions should be satisfied.

The deflection field, w, is expressed by substituting (14), (15) and (16) in (11).

Substituting w in (2a), (6), (9) and (10), the equations relating the values of the boundary conditions to the undetermined parameters, $\mathbf{c}$, are obtained. In the process derivatives of $\overset{\circ}{w}$ and $\mathbf{N}$ arise. After symbolically evaluating the derivatives, the equations for the i-th region can be put in the form:

$$\overset{\circ}{w}{}^i + \mathbf{N}_w^i\,\mathbf{c}^i = \overline{w}^i, \tag{17a}$$

$$\frac{\partial \overset{\circ}{w}{}^i}{\partial n} + \mathbf{N}^i_{\frac{\partial w}{\partial n}}\,\mathbf{c}^i = \frac{\partial \overline{w}^i}{\partial n}, \tag{17b}$$

$$\overset{\circ}{M}{}^i_n + \mathbf{N}^i_{M_n}\,\mathbf{c}^i = \overline{M}^i_n, \tag{17c}$$

$$\overset{\circ}{V}{}^i_n + \mathbf{N}^i_{V_n}\,\mathbf{c}^i = \overline{V}^i_n. \tag{17d}$$

The corresponding equations for interfaces are derived in the same way.

Writing these equations at selected points along the boundary (collocation points or CP) the governing system can be represented in the form

$$\mathbf{D}\mathbf{c} = \overline{\mathbf{d}} - \overset{\circ}{\mathbf{d}} \tag{18}$$

where matrix $\mathbf{D}$ contains the expansion terms of the series $\mathbf{N}_w$, $\mathbf{N}_{\frac{\partial w}{\partial n}}$, $\mathbf{N}_{M_n}$, $\mathbf{N}_{V_n}$, and the right hand side $\overline{\mathbf{d}} - \overset{\circ}{\mathbf{d}}$ contain the imposed values and the particular solutions, $\overline{w} - \overset{\circ}{w}$, $\frac{\partial \overline{w}}{\partial n} - \frac{\partial \overset{\circ}{w}}{\partial n}$, $\overline{M}_n - \overset{\circ}{M}_n$, and $\overline{V}_n - \overset{\circ}{V}_n$.

A determined system of equations may be obtained when the number of conditions imposed equals the number of terms of the displacement field expansion. A non symmetric square system matrix, $\mathbf{D}$, is, thus, obtained.

As the boundary conditions are imposed at points, the residuals (the difference to actual BC values) may be unacceptable or exhibit unreasonable variation. To decrease the errors, it is possible to collocate at more points than the strictly necessary ones leading, in this manner, to an overdetermined system of equations. The least squares criteria is used here to define the optimal solution whereby the undetermined coefficients, $\mathbf{c}^i$, of region i are found from the minimisation of the square of the boundary residuals.

If a continuous approach is used instead of a discrete one, follows that:

Minimize $I(\mathbf{c}^i)$

$$I(\mathbf{c}) = W_w^2 \int_{\Gamma_w^i} \left(w^i - \overline{w}^i\right)^2 \mathrm{d}\Gamma_w^i + W_{\frac{\partial w}{\partial n}}^2 \int_{\Gamma_{\frac{\partial w}{\partial n}}^i} \left(\frac{\partial w^i}{\partial n} - \overline{\frac{\partial w^i}{\partial n}}\right)^2 \mathrm{d}\Gamma_{\frac{\partial w}{\partial n}}^i$$

$$+ W_{M_n}^2 \int_{\Gamma_{M_n}^i} \left(M_n^i + GJ\frac{\partial^3 w^i}{\partial t^2 \partial n} - \overline{M}_n^i\right)^2 \mathrm{d}\Gamma_{M_n}^i + W_{V_n}^2 \int_{\Gamma_{V_n}^i} \left(V_n^i + EI\frac{\partial^4 w^i}{\partial t^4} - \overline{V}_n^i\right)^2 \mathrm{d}\Gamma_{V_n}^i$$

$$+ W_w^2 \int_{\Gamma_I} \left(w^i - w^j\right)^2 \mathrm{d}\Gamma_I + W_{\frac{\partial w}{\partial n}}^2 \int_{\Gamma_I} \left(\frac{\partial w^i}{\partial n} + \frac{\partial w^j}{\partial n}\right)^2 \mathrm{d}\Gamma_I$$

$$+ W_{M_n}^2 \int_{\Gamma_I} \left(M_n^i + GJ\frac{\partial^3 w^i}{\partial t^2 \partial n} - M_n^j\right)^2 \mathrm{d}\Gamma_I + W_{V_n}^2 \int_{\Gamma_I} \left(V_n^i + EI\frac{\partial^4 w^i}{\partial t^4} + V_n^j\right)^2 \mathrm{d}\Gamma_I, \quad (19)$$

where $d\Gamma_I = \Omega^i \bigcap \Omega^j$ and W_w, $W_{\frac{\partial w}{\partial n}}$, W_{M_n}, and W_{V_n} are weights. The j superscript refers to a neighbouring region.

The first four terms represent the support BC's and the other four express the continuity and equilibrium between the i and j regions.

The weights are used to restore the homogeneity of the physical dimensions and the relative strength of the different boundary conditions. In this work equal weights are assumed.

By setting the first derivative of (19) with respect to $\mathbf{c}$ to zero, the following equation arises

$$\mathbf{K}\,\mathbf{c} = \mathbf{f}$$

where $\mathbf{K}$ is a square symmetric definite positive matrix.

The contribution of the i-th region to this system of equations is

$$\begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} \end{bmatrix} \begin{Bmatrix} \mathbf{c}_i \\ \mathbf{c}_j \end{Bmatrix} = \{f_i\}.$$

If summation at collocation points is used instead of integrations then the discrete approach is reached.

A method for finding the *optimal* set of weights is presented and illustrated in [3].

5. Remarks on implementation

An important issue in the implementation of the formulation is the choice of the number and the position of the collocation points. If some part of the boundary is more important to the problem (in the sense that some local effect occurs such as a load or a change in the geometry or the boundary conditions), then the density of CP's located on that part, compared to the density on the remaining boundary, should be increased. Another way of achieving this is to increase the relative weights of the CP's. In this case the weights should be moved inside the integration operators in (19). Also Eqs. (17) should be written in a local co-ordinate system located at the region centroid and in dimensionless (or normalized) form,

$$\xi = \frac{x}{d}, \qquad \eta = \frac{y}{d}$$

where d is a characteristic distance of the problem, e.g., the length of the largest side of all regions. Once the system is solved the effect of that normalizing factor has to be removed in order to determine all the relevant results.

6. Numerical examples

The cases of a simply supported rectangular plate and a clamped circular plate subjected to various types of loading are considered.

In all cases the following numerical data used is: $E = 1.0$, $p = 1.0$, $a = 1.0$, $\nu = 0.3$ and $t = \frac{a}{10}$. The values of the weights are as follows: $W_w = 1.0$, $W_{\frac{\partial w}{\partial n}} = 1.0$, $W_{M_n} = 1.0$ and $W_{V_n} = 1.0$.

6.1. Simply supported rectangular plate

Consider a simply supported plate with dimensions $b \times a \times t$, material properties E, ν. The Navier solution for this case is

$$w(x,y) = \frac{1}{\pi^4 D} \sum_{m=1,2,3,\ldots}^{\infty} \sum_{n=1,2,3,\ldots}^{\infty} \frac{a_{mn}}{\left(\frac{m^2}{b^2} + \frac{n^2}{a^2}\right)^2} \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{a}. \tag{20}$$

where the coefficients a_{mn} depend on the applied load.

The discretization adopted for the three loads treated is show in Fig. 2b.

6.1.1. Uniform load

The numerical example concerns a rectangular simply supported plate under uniform load. The results obtained by collocation and with the direct approach are compared to the exact solution.

Only a quarter of the plate was analysed (Fig. 2a).

The boundary conditions (in the local centroid co-ordinate system, Fig. 2b) are

i) at symmetry axes

$$\left.\overline{\frac{\partial w}{\partial n}}\right|_{x=0,\,0\le y\le\frac{a}{2}} = 0, \qquad \left.\overline{\frac{\partial w}{\partial n}}\right|_{0\le x\le\frac{b}{2},\,y=0} = 0,$$
$$\left.\overline{V}_n\right|_{x=0,\,0\le y\le\frac{a}{2}} = 0, \qquad \left.\overline{V}_n\right|_{0\le x\le\frac{b}{2},\,y=0} = 0,$$

ii) at simply supported edges

$$\left.\overline{w}\right|_{x=\frac{b}{2},\,0\le y\le\frac{a}{2}} = 0, \qquad \left.\overline{w}\right|_{0\le x\le\frac{b}{2},\,y=\frac{a}{2}} = 0,$$
$$\left.\overline{M}_n\right|_{x=\frac{b}{2},\,0\le y\le\frac{a}{2}} = 0, \qquad \left.\overline{M}_n\right|_{0\le x\le\frac{b}{2},\,y=\frac{a}{2}} = 0.$$

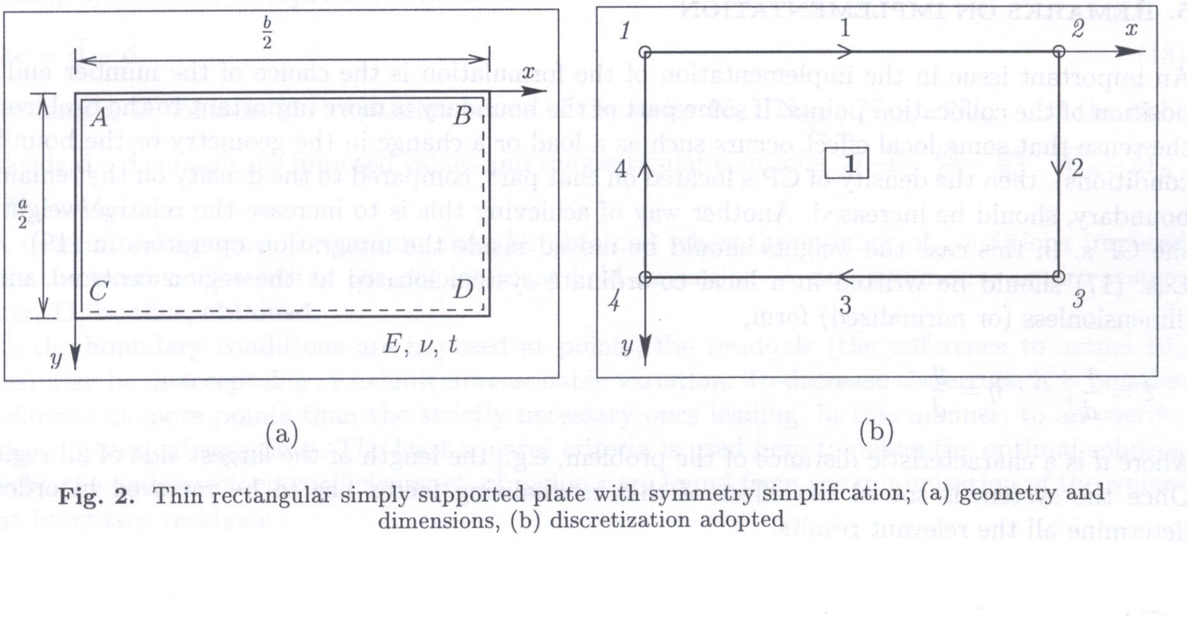

Fig. 2. Thin rectangular simply supported plate with symmetry simplification; (a) geometry and dimensions, (b) discretization adopted

The coefficients for the Navier solution for this case [11] are

$$a_{mn} = \frac{16\,p}{\pi^2\,m\,n}\,, \tag{21}$$

where $m = 1,3,5,\dots$ and $n = 1,3,5,\dots$.

The results obtained considering $b = 2a$ are presented in Tables 1 and 2. The Navier solution was found using 5000 terms in each direction.

The CPU time in creating the system matrices for both, discrete and continuous, approaches are listed in Table 3. The discrete approach is, as expected, faster than the continuous one as no integrations are required.

Table 1. Values of $w^{adim\,A}$, $Q_x^{adim\,B}$ and $Q_y^{adim\,C}$ for a simply supported plate subjected to uniform load.

	$10\,w^{adim\,A}$		$-Q_x^{adim\,B}$		$-Q_y^{adim\,C}$	
m	Disc.	Cont.	Disc.	Cont.	Disc.	Cont.
6	0.103029	0.104508	0.3425	0.3164	0.4694	0.4767
8	0.101157	0.101246	0.3787	0.3793	0.4629	0.4609
10	0.101237	0.101291	0.3727	0.3740	0.4649	0.4664
12	0.101280	0.101286	0.3672	0.3670	0.4645	0.4638
Navier	0.101287		0.3696		0.4650	

Table 2. Values of $M_{xx}^{adim\,A}$, $M_{yy}^{adim\,A}$ and $M_{xy}^{adim\,D}$ for a simply supported plate subjected to uniform load.

	$10\,M_{xx}^{adim\,A}$		$M_{yy}^{adim\,A}$		$-10\,M_{xy}^{adim\,D}$	
m	Disc.	Cont.	Disc.	Cont.	Disc.	Cont.
6	0.47312	0.47417	0.10355	0.10508	0.48884	0.52893
8	0.46149	0.46435	0.10165	0.10180	0.48206	0.47881
10	0.46342	0.46385	0.10164	0.10171	0.46945	0.46869
12	0.46349	0.46354	0.10168	0.10169	0.46579	0.46574
Navier	0.46350		0.10168		0.46267	

Table 3. CPU time (*sec*) in creating the system matrices.

m	Disc.	Cont.
6	0.05	0.6
8	0.06	1.15
10	0.11	1.81
12	0.17	1.98

6.1.2. Uniform load in a rectangular area

Consider the plate represented in Fig. 3, where $b = 2a$. The coefficients for the Navier solution for this case are

$$a_{mn} = \frac{16}{ab}\frac{P}{uv}\sin\frac{m\pi\xi}{b}\sin\frac{n\pi\eta}{a}\sin\frac{m\pi u}{2b}\sin\frac{n\pi v}{2a} \tag{22}$$

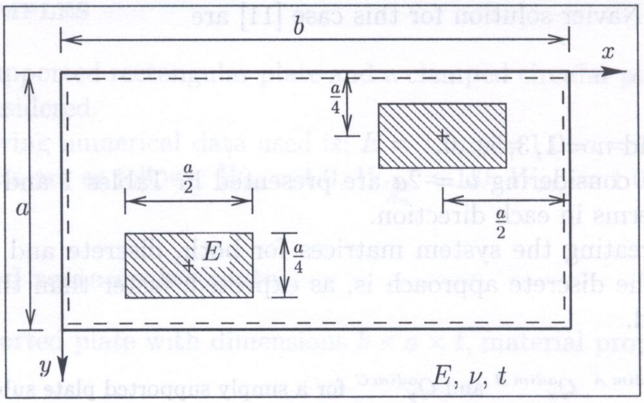

Fig. 3. Simply supported rectangular plate subjected to uniform load applied on two rectangular areas

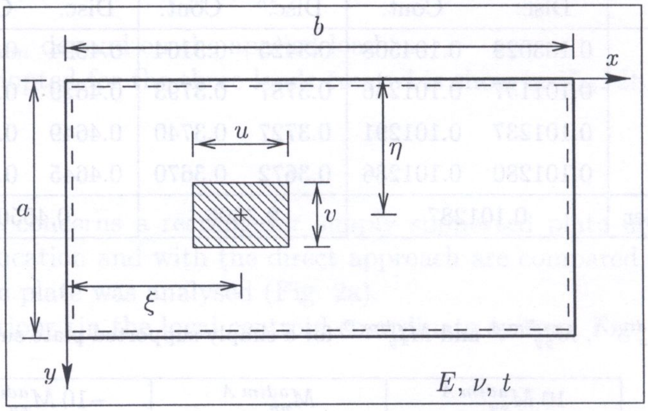

Fig. 4. Simply supported rectangular plate subjected to uniform load applied on rectangular area, see (22)

Table 4. Values of $w^{adim\,A}$ e $M_{xy}^{adim\,E}$ for a simply supported rectangular plate subjected to uniform load applied on rectangular areas

	$100\,w^{adim\,A}$		$100\,M_{xy}^{adim\,E}$	
m	Disc.	Cont.	Disc.	Cont.
10	0.139705	0.134997	0.73549	0.72029
14	0.134859	0.134675	0.75186	0.75121
18	0.134665	0.134659	0.75783	0.75845
22	0.134657	0.134659	0.75715	0.75726
Navier	0.134659		0.75727	

Table 5. Values of $M_{xx}^{adim\,E}$ e $M_{yy}^{adim\,E}$ for a simply supported rectangular plate subjected to uniform load applied on rectangular areas

	$10\,M_{xx}^{adim\,E}$		$10M_{yy}^{adim\,E}$	
m	Disc.	Cont.	Disc.	Cont.
10	0.49685	0.50000	0.78522	0.77242
14	0.49590	0.49531	0.80496	0.80427
18	0.49570	0.49567	0.80591	0.80603
22	0.49585	0.49586	0.80604	0.80603
Navier	0.49593		0.80601	

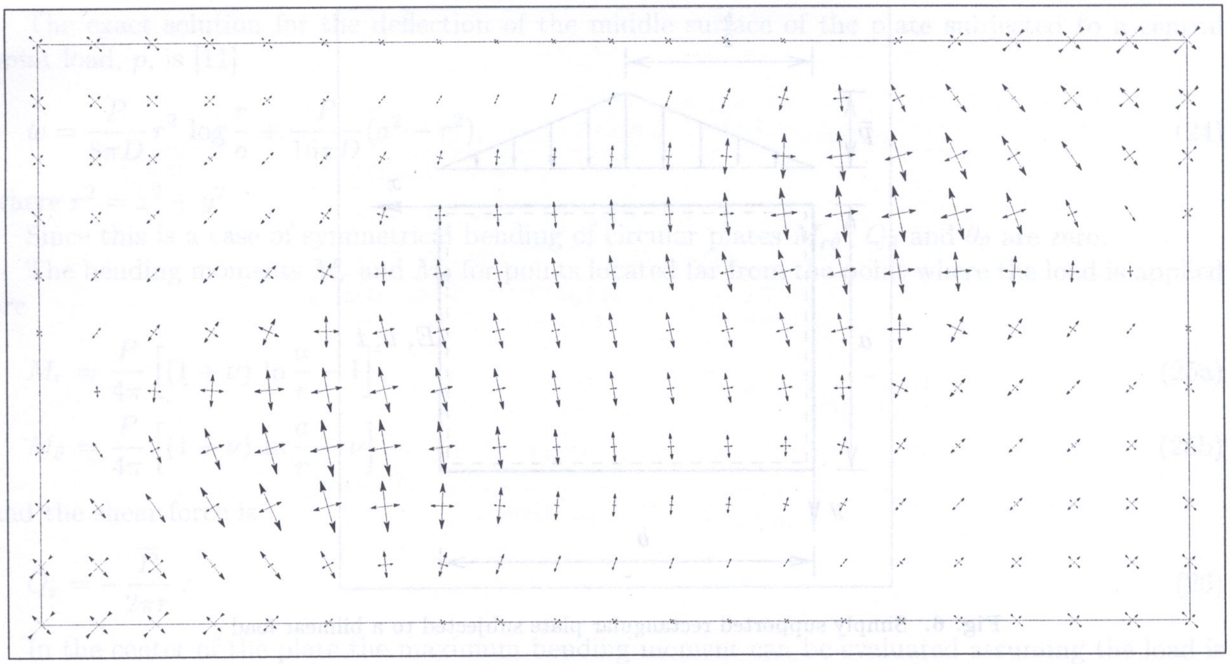

Fig. 5. Principal bending directions for a simply supported rectangular plate subjected to uniform load applied on rectangular areas

where $m = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$ and constants u, v, ξ and η are represented in Fig. 4. P denotes the total load, $P = u\,v\,p$.

The results are listed in Tables 4 and 5. Point E is denoted in Fig. 3. This case was studied by Venkatesh and Jiroušek [12] and similar results were obtained.

The principal bending directions are represented in Fig. 5.

6.1.3. Bilinear load

The model analysed is represented in Fig. 6.

If the load, $\overline{p}$, is expressed in the global reference system by $\overline{p}(x, y) = p_0 + p_1 x + p_2 y$, then the series coefficients a_{mn} that describe the load in the Navier method are $a_{mn} = a_{mn}^{p_0} + a_{mn}^{p_1} + a_{mn}^{p_2}$, where $a_{mn}^{p_0}$ is given by (22) and

$$a_{mn}^{p_1} = \frac{8\,p_1}{\pi^3 m^2 n} \sin\frac{n\pi\eta}{a} \sin\frac{n\pi v}{2a}\left[2b\cos\frac{m\pi\xi}{b}\sin\frac{m\pi u}{2b} + 2\xi m\pi \sin\frac{m\pi\xi}{b}\sin\frac{m\pi u}{2b} - u\,m\pi\cos\frac{m\pi\xi}{b}\cos\frac{m\pi u}{2b}\right]$$

$$a_{mn}^{p_2} = \frac{8\,p_2}{\pi^3 m n^2} \sin\frac{m\pi\xi}{b} \sin\frac{m\pi u}{2b}\left[2a\cos\frac{n\pi\eta}{a}\sin\frac{n\pi v}{2a} + 2\eta n\pi \sin\frac{n\pi\eta}{a}\sin\frac{n\pi v}{2a} - v\,n\pi\cos\frac{n\pi\eta}{a}\cos\frac{n\pi v}{2a}\right]$$

where u, v, ξ and η are the same as in the previous example and $\overline{P} = u\,v\,\overline{p_0}$.

The derivation of these expressions, as well as the particular solutions used in this case and in the previous example can be found in Fernandes [2].

The results obtained in the center of the plate for several a, b ratios are shown in Table 6.

The Navier solution was found using 500 terms in each direction. The discrete procedure was applied assuming $m = 10$ and $m = 30$ (m is the order of the approximation considered).

The results achieved prove the excellent agreement of the two different solutions.

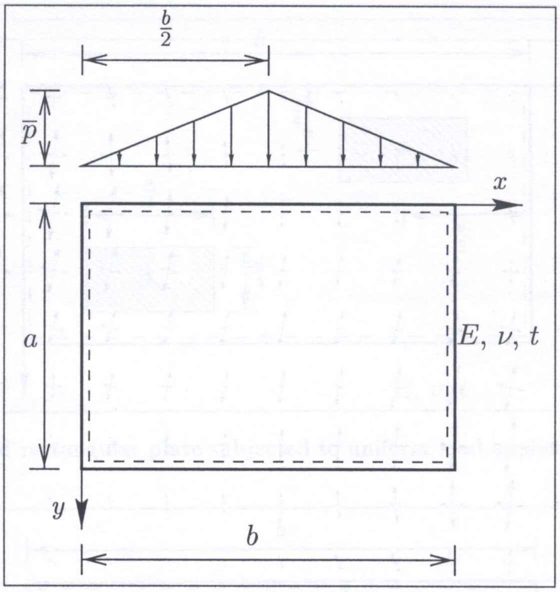

Fig. 6. Simply supported rectangular plate subjected to a bilinear load

Table 6. Values of $w^{adim\,A}$, $M_{xx}^{adim\,A}$ and $M_{yy}^{adim\,A}$ for a simply supported plate subjected to a bilinear load.

	$100\,w^{adim\,A}$			$10M_{xx}^{adim\,A}$			$10M_{yy}^{adim\,A}$		
$\frac{b}{a}$	$m=10$	$m=30$	Navier	$m=10$	$m=30$	Navier	$m=10$	$m=30$	Navier
1.0	0.262486	0.262744	0.262744	0.340630	0.340498	0.340498	0.316428	0.317095	0.317095
1.1	0.315525	0.315835	0.315835	0.356264	0.356065	0.356065	0.367775	0.368460	0.368460
1.2	0.367426	0.367751	0.367751	0.367773	0.367518	0.367518	0.417060	0.417699	0.417699
1.3	0.417229	0.417571	0.417571	0.376055	0.375779	0.375779	0.463633	0.464222	0.464222
1.4	0.464339	0.464729	0.464729	0.381857	0.381618	0.381618	0.507161	0.507729	0.507729
1.5	0.508440	0.508934	0.508934	0.385778	0.385649	0.385649	0.547528	0.548120	0.548120
1.6	0.549413	0.550089	0.550089	0.388284	0.388350	0.388350	0.584762	0.585438	0.585438
1.7	0.587273	0.588231	0.588230	0.389727	0.390085	0.390085	0.618976	0.619813	0.619813
1.8	0.622109	0.623479	0.623479	0.390367	0.391124	0.391124	0.650327	0.651426	0.651426
1.9	0.654051	0.656004	0.656003	0.390392	0.391670	0.391669	0.678990	0.680482	0.680482
2.0	0.683238	0.685998	0.685997	0.389926	0.391869	0.391868	0.705134	0.707193	0.707193
3.0	0.848359	0.886844	0.886844	0.369017	0.388367	0.388389	0.855320	0.884728	0.884716

6.2. Circular clamped plate

Consider a circular clamped plate of radius a, thickness t and material properties E, ν.

In this example, the description of the problem geometry, as well as the generalized forces, will be made in a cylindrical coordinate system.

The boundary conditions in the centroid local system of coordinates are

$$w|_{r=a} = 0, \tag{23a}$$

$$\left.\frac{\partial w}{\partial n}\right|_{r=a} = 0. \tag{23b}$$

The exact solution for the deflection of the middle surface of the plate subjected to a central point load, p, is [11]

$$w = \frac{P}{8\pi D}\, r^2 \log\frac{r}{a} + \frac{P}{16\pi D}(a^2 - r^2), \tag{24}$$

where $r^2 = x^2 + y^2$.

Since this is a case of symmetrical bending of circular plates $M_{r\theta}$, Q_θ and θ_θ are zero.

The bending moments M_r and M_θ for points located far from the point where the load is applied are

$$M_r = \frac{P}{4\pi}\left[(1+\nu)\,\ln\frac{a}{r} - 1\right] \tag{25a}$$

$$M_\theta = \frac{P}{4\pi}\left[(1+\nu)\,\ln\frac{a}{r} - \nu\right], \tag{25b}$$

and the shear force is

$$Q_r = -\frac{\overline{P}}{2\pi r}\,. \tag{26}$$

In the center of the plate the maximum bending moment can be evaluated assuming the load is uniformly distributed in a circle of small radius, c, by [11]

$$M_{\max} = \frac{P}{4\pi}\left[(1+\nu)\,\ln\frac{a}{c} - \frac{(1-\nu)c^2}{4a^2}\right]. \tag{27}$$

The discretization used, a single region with four sides (cubic splines), is represented in Fig. 7.

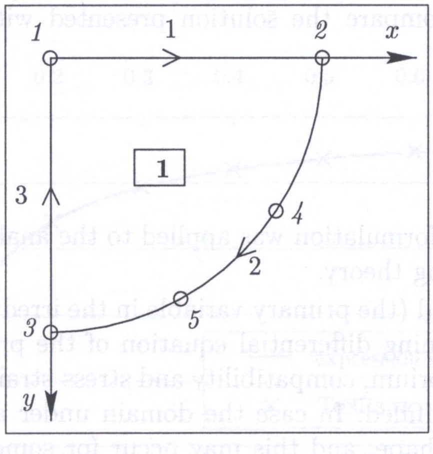

Fig. 7. Discretization used for a circular clamped plate

Table 7. Values of $w^{adim\,A}$ and $Q_r^{adim\,B}$ for a circular clamped plate subjected to a central point load

m	$n_{equations}$	DOF	$10\,w^{adim\,A}$ Cont.	$10\,w^{adim\,A}$ Disc.	$-Q_r^{adim\,B}$ Cont.	$-Q_r^{adim\,B}$ Disc.
1	8	4	0.19826	0.19826	0.15915	0.15915
3	24	12	0.19923	0.19942	0.15915	0.15915
5	40	20	0.19925	0.19921	0.15268	0.15374
		Exact	0.19894		0.15915	

Table 8. Values of $M_r^{adim\,B}$ and $M_\theta^{adim\,B}$ for a circular clamped plate subjected to a central point load

m	$n_{equations}$	DOF	$-10\,M_r^{adim\,B}$ Cont.	$-10\,M_r^{adim\,B}$ Disc.	$-10M_\theta^{adim\,B}$ Cont.	$-10M_\theta^{adim\,B}$ Disc.
1	8	4	0.79756	0.79756	0.24052	0.24052
3	24	12	0.79502	0.79452	0.23798	0.23748
5	40	20	0.78015	0.78269	0.23171	0.23279
		Exact	0.79577		0.23873	

Table 9. Values of $M_r^{adim\,A}$ and $M_\theta^{adim\,A}$ for a circular clamped plate subjected to a central load distributed in a circle of radius $c = a \cdot 10^{-5}$

m	$n_{equations}$	DOF	$\frac{M_r^{adim\,A}}{10} = \frac{M_\theta^{adim\,A}}{10}$ Cont.	Disc.
1	8	4	0.119084	0.119084
3	24	12	0.119110	0.119115
5	40	20	0.119110	0.119109
		Exact	0.119102	

The results, using the actual concentrated load, are in Tables 7 and 8, where points A and B are located in the center and in the boundary of the plate, respectively.

The bending moments in the center of the plate were found using the particular solution (16), and the results are in Table 9.

In Fig. 8 it is possible to compare the solution presented with the solution given by Timoshenko [11].

7. Conclusions

In this work an Indirect Trefftz formulation was applied to the analysis of bending of thin plates by using the Kirchhoff plate bending theory.

An assumed displacement field (the primary variable in the irreducible thin plate approximation) that satisfies *a priori* the governing differential equation of the problem (Trefftz method) ensures that all the requirements (equilibrium, compatibility and stress-strain relationship in solid mechanics problems) in the domain are fulfilled. In case the domain under analysis is discretized into more than one regions, of arbitrary shape, and this may occur for some specific problems, an assumed displacement field must be defined for each region and extra continuity/compatibility conditions must be enforced at the interfaces.

Other type of problems (domain configuration and/or boundary conditions) exist for which it is more convenient to use specifically devised displacement field approximation. These displacement fields are obtained in such a way that some prescribed boundary conditions are directly enforced. This is the case for skewed or perforated plates for example.

The approximating displacement field is expressed by a truncated series. The number of terms of the series which is used is variable. It depends mainly on the geometry of the domain and on the boundary conditions/loads.

In the present study the governing differential equation is of the fourth order. This implies that in the finite element analysis the primary variables are, at least, the displacement and the rotations of the middle surface. The present formulation only requires the use of the natural unknown, thus reducing the dimension of the sought solution.

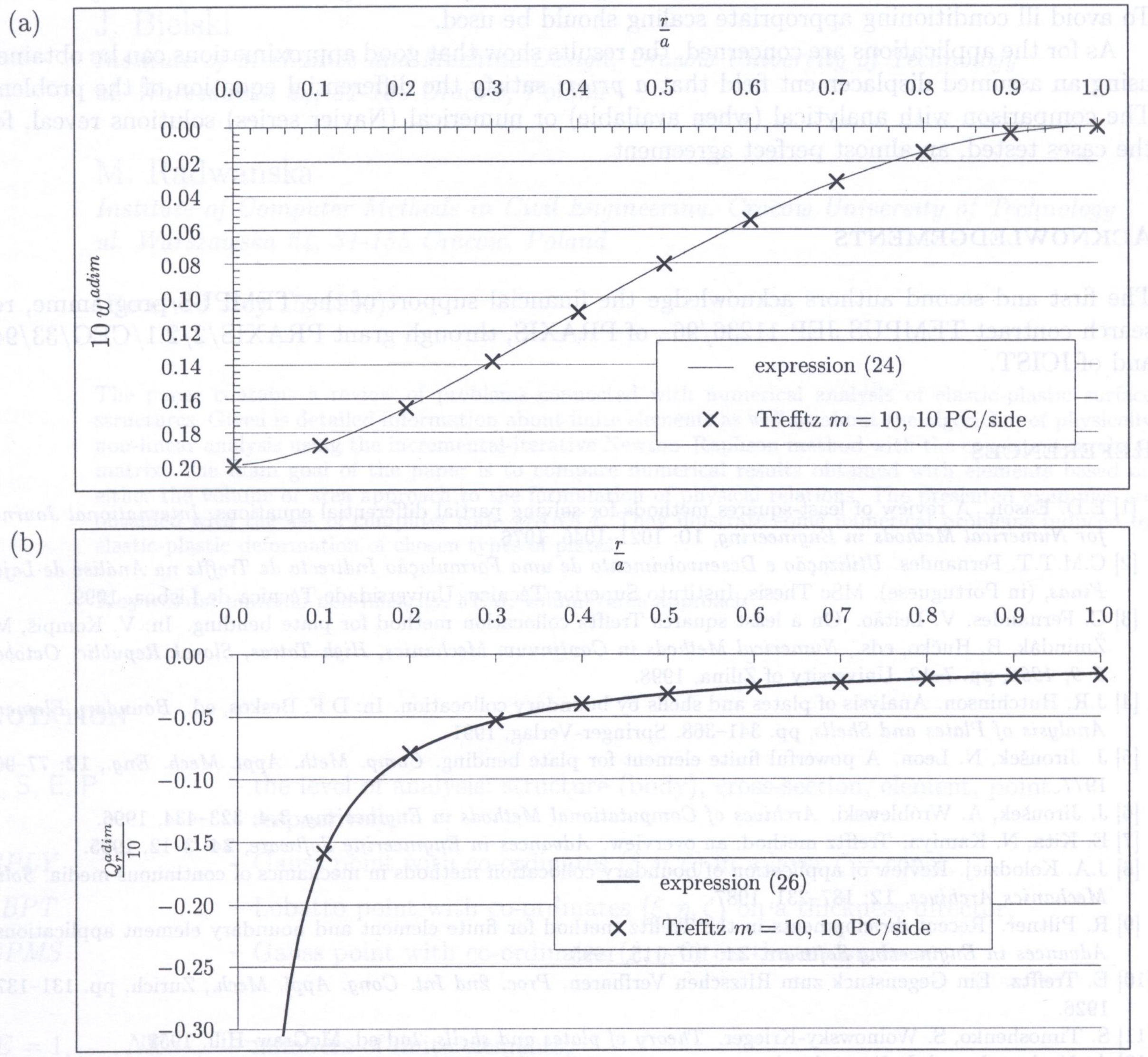

Fig. 8. Results for a circular clamped plate subjected to a central concentrated load; (a) Deflection, w^{adim}, (b) Shear force, Q_r^{adim}

The boundary conditions of each region are expressed in terms of the primary variable approximation and imposed by minimizing, in a conveniently weighed way, the errors of the generalized displacements and forces. To achieve this, a discrete (or collocation) and a continuous (or integration) approach for the minimization of the residues were studied.

The minimization process based on least squares proved to be an appropriate way of imposing the boundary conditions. On the other hand, the collocation procedure is more efficient as it requires less numerical calculations for similar accuracy.

The resulting system of equations (either obtained by collocation or by integration) is full (in case only one region is used) or almost full (as, in general, only a few regions are used) and symmetric. To avoid ill conditioning appropriate scaling should be used.

As for the applications are concerned, the results show that good approximations can be obtained using an assumed displacement field that *a priori* satisfy the differential equation of the problem. The comparison with analytical (when available) or numerical (Navier series) solutions reveal, for the cases tested, an almost perfect agreement.

Acknowledgements

The first and second authors acknowledge the financial support of the TEMPUS programme, research contract TEMPUS JEP 11236/96., of PRAXIS, through grant PRAXIS/2/2.1/CEG/33/94, and of ICIST.

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