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# A-priori estimates of the hp-adaptive BEM in elastic scattering of acoustic waves

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In the paper some a-priori hp- adaptive error estimates, applied to the problem of acoustic wave scattering on an elastic body in the 2D space, solved by the Boundary Element Method, are presented. The estimate includes both the function- and boundary approximation errors.

**Keywords:** Boundary Element Method, acoustic scattering, hp- adaptive method, a-priori error estimate

## 1. Introduction

In the Finite Element Method (FEM) the hp- adaptive methods, as a consequence of h- and p- adaptive methods, were introduced first by Babuška *et al.* [2]. The Boundary Element Method (BEM), which accepted much of the FEM ideas, had included these methods, after some time, to its own tools. The pioneer works in the hp- adaptivity were done by E. Rank [18] and I. Babuška [3] with coworkers. In the first papers two-dimensional domains with piecewise straight boundaries were considered [7, 18, 19]. A plane with an extracted curve (screen problem) was a domain of the process in [17]. Infinite domains surrounding bounded regions were considered in [6]. An analogous approach to three-dimensional problems was a subject of some later papers, e.g. for open domains [8, 19], and for infinite domains with bounded 'holes' [10]. The exponential convergence was extended onto hp- adaptive BEM in [7, 8, 19]. A-posteriori local error indicators were proposed in [15]. Detailed discussion of mathematical aspects of the adaptive BEM may be found e.g. in [9, 10, 16].

The error estimates proved in the papers cited above include only errors caused by solution's approximation by shape functions, in general. It is assumed that all calculations are performed on the given boundary. It may be true e.g. for screen problems or polygonal domains, but for domains of higher regularity some approximation of the boundary has to be introduced. This approximation generates an additional error, which should be estimated too.

The plan of the paper is as follows: In Section 2 we remind some known hp- interpolation estimates. As an application we derive in Section 3 analogous estimates for boundary approximation. The problem of elastic scattering is posed in Section 4 and the method of the Galerkin BEM solution is presented there. Some properties of a fundamental solution are considered in Section 5. In Section 6 the final a-priori error estimates are proved.

## 2. HP- interpolation estimates

Let

$$\hat{T} = [0, 1] \tag{1}$$

be a pattern 1-D finite element,

$$P_d[a,b] = \left\{ p(t) = \sum_{j=0}^{d} a_j t^j \ , \ t \in [a,b] \right\} \tag{2}$$

the space of polynomials of order not greater than $d$ on $[a,b]$, $H^m(\Omega)$, $\| \ \|_{m,\Omega}$ – the Sobolev space with its norm. As a reference domain we consider an interval $G = [a,b]$ divided by points $t_i$ into subintervals

$$a = t_0 < t_1 < \cdots < t_n = b, \qquad T_i = [t_{i-1}, t_i], \qquad G = \bigcup_{i=1}^{n} T_i$$

with

$$h_i = \operatorname{diam}(T_i) = t_i - t_{i-1}, \qquad h = \sup h_i.$$

For each $T_i$ we define an affine, invertible mapping $B_i : \hat{T} \to T_i$. Let $\hat{v} \in H^m(\hat{T})$ be a real function. On each $T_i$ real functions $v_i \in H^m(T_i)$ are defined by this affine mapping

$$v_i = \hat{v} \circ B_i^{-1}.$$

The following theorems about interpolation are true (cf. [4, 5]):

**Theorem 1.** *By the previous notations for integers $m \geq 1$, $d \geq 0$ there is a constant $C = C(m,d)$ such that for any function $v_i \in H^m(T_i)$ there is an interpolating polynomial $v_i^d \in P_d(T_i)$ such that for any $\alpha < \mu$,*

$$\sup \left\{ \left| \frac{d^\alpha}{dt^\alpha}(v_i - v_i^d)(t) \right| , \ t \in T_i \right\} \leq C \, \frac{h_i^{\mu-\alpha-1}}{d^{m-\alpha-1}} \|v_i\|_{m,T_i} \, , \tag{3}$$

*where*

$$\mu = \min\{d+1, m\}. \tag{4}$$

**Theorem 2.** *By the previous notations for $m \geq 1$, $d \geq 0$ there is a constant $C = C(m,d,G)$ such that for any function $v \in H^m(G)$ there is an interpolating function $v \in C^0(G)$, being a polynomial $v_i^d \in P_d(T_i)$ on each $T_i$ such that for any $\alpha < \mu$,*

$$\sup \left\{ \left| \frac{d^\alpha}{dt^\alpha}(v - v^d)(t) \right| , \ t \in G \right\} \leq C \, \frac{h^{\mu-\alpha-1}}{d^{m-\alpha-1}} \|v\|_{m,G} \, . \tag{5}$$

## 3. APPROXIMATION OF THE BOUNDARY

Let the bounded domain $\Omega \in \mathbb{R}^2$ with a boundary $\Gamma$ be given. There is a function

$$\boldsymbol{X} = (X^1, X^2) : G \to \mathbb{R}^2 \tag{6}$$

which describes the boundary as

$$\Gamma = \boldsymbol{X}(G).$$

We assume that $\boldsymbol{X} \in H^m(G) \times H^m(G)$ with the usual norm $\|\boldsymbol{X}\|^2_{m,G} = \|X^1\|^2_{m,G} + \|X^2\|^2_{m,G}$. Let $\boldsymbol{X}_h$ be an interpolant of $\boldsymbol{X}$ i.e. $X^i_h$ are interpolants of $X^i$ in the sense of Theorem 2 and $\Gamma_h = \boldsymbol{X}_h(G)$. Let $I_d$ denotes a class of vector interpolants of the order $d$

$$I_d = \{\boldsymbol{Y}_h = (Y^1_h, Y^2_h) : G \to \mathbb{R}^2 : Y^j_h \in C^0(G), \quad Y^j_h|_{T_i} \in P_d(T_i), \quad j = 1,2, \quad i = 1, \ldots, n\}.$$

We assume in what follows that derivatives of functions $X^i$, $X_h^i$ exist and are bounded from below and from above by positive constants $c_1$, $C_1$:

$$0 < c_1 \leq \left|\frac{d\boldsymbol{X}}{dt}(t)\right|, \left|\frac{d\boldsymbol{X}_h}{dt}(t)\right| \leq C_1, \quad \forall t \in G, \tag{7}$$

and they satisfy the Hölder condition

$$\left|\frac{d\boldsymbol{X}}{dt}(t) - \frac{d\boldsymbol{X}}{dt}(t+\Delta t)\right| \leq C_1|\Delta t|^\gamma, \quad \forall t, t+\Delta t \in G, \tag{8}$$

for some $\gamma \in (0,1]$.

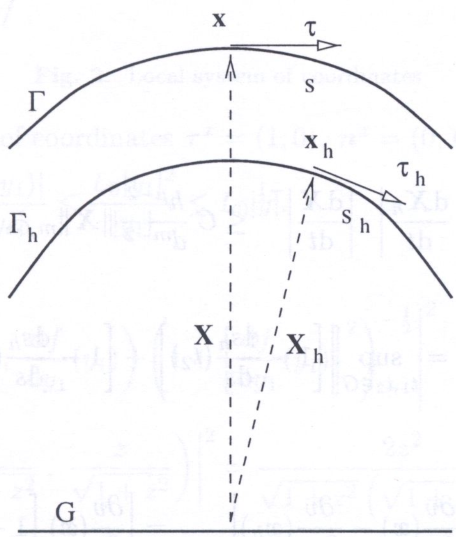


**Fig. 1.** Boundary and its approximation

A simple corollary of Theorem 2 is

**Lemma 1.** *By the previous assumptions for any $m \geq 1$, $d \geq 0$ there is a constant $C > 0$ such that the interpolation error for any vector function $\boldsymbol{X} \in H^m(G) \times H^m(G)$ may be estimated as*

$$\sup_{t\in G} |\boldsymbol{X}(t) - \boldsymbol{X}_h(t)| \leq C\,\frac{h^{\mu-1}}{d^{m-1}}\,\|\boldsymbol{X}\|_{m,G}, \tag{9}$$

$$\sup_{t\in G}\left|\frac{d\boldsymbol{X}}{dt}(t) - \frac{d\boldsymbol{X}_h}{dt}(t)\right| \leq C\,\frac{h^{\mu-2}}{d^{m-2}}\,\|\boldsymbol{X}\|_{m,G}, \tag{10}$$

*where $\boldsymbol{X}_h \in I_d$.*

We extend any function $v \in C^1(\Gamma)$ onto a neighbourhood $U_\Gamma$ of $\Gamma$ to define $\partial v/\partial\boldsymbol{\tau}(\boldsymbol{x})$, where $\boldsymbol{\tau}$ is a vector tangent to $\Gamma$ at $\boldsymbol{x}$. For the arc length parameter $s$ of $\Gamma$ we have

$$\frac{\partial v}{\partial \tau}(\boldsymbol{x}) = \frac{\partial v}{\partial s}(\boldsymbol{x}). \tag{11}$$

On the approximate boundary $\Gamma_h$ we define $v$ as

$$v(\boldsymbol{x}_h) = (v \circ \boldsymbol{X} \circ \boldsymbol{X}_h^{-1})(\boldsymbol{x}_h) \tag{12}$$

In the same way we obtain, in regular points of $\Gamma_h$, for its arc length parameter $s_h$

$$\frac{\partial v}{\partial \tau_h}(\boldsymbol{x}_h) = \frac{\partial v}{\partial s_h}(\boldsymbol{x}_h). \tag{13}$$

By the definition we have, if $\boldsymbol{x} = \boldsymbol{X}(t_0)$, $\boldsymbol{x}_h = \boldsymbol{X}_h(t_0)$,

$$v(\boldsymbol{x}) = v(\boldsymbol{x}_h). \tag{14}$$

**Lemma 2.** *By the previous assumptions there is a constant $C = C(m, G)$ such that*

$$\sup_{t\in G}\left|1-\frac{\mathrm{d}s_h}{\mathrm{d}s}(t)\right| \le C\,\frac{h^{\mu-2}}{d^{m-2}}\,\|\boldsymbol{X}\|_{m,G}, \tag{15}$$

$$\sup_{t_1,t_2\in G}\left|1-\frac{\mathrm{d}s_h}{\mathrm{d}s}(t_1)\,\frac{\mathrm{d}s_h}{\mathrm{d}s}(t_2)\right| \le C\,\frac{h^{\mu-2}}{d^{m-2}}\,\|\boldsymbol{X}\|_{m,G}, \tag{16}$$

*and for any function $v \in C^1(\Gamma)$ extended as above*

$$\left|\frac{\partial v}{\partial\tau}(\boldsymbol{x})-\frac{\partial v}{\partial\tau_h}(\boldsymbol{x}_h)\right|_{-\frac{1}{2},\Gamma} \le C\,\frac{h^{\mu-2}}{d^{m-2}}\,\|\boldsymbol{X}\|_{m,G}\cdot\|v\|_{\frac{1}{2},\Gamma}. \tag{17}$$

Proof.

$$\frac{\mathrm{d}s_h}{\mathrm{d}s}=\left|\frac{\mathrm{d}\boldsymbol{X}_h}{\mathrm{d}t}\right|\cdot\left|\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t}\right|^{-1},$$

$$\sup_{t\in G}\left|1-\frac{\mathrm{d}s_h}{\mathrm{d}s}\right| \le \sup_{t\in G}\left|\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t}-\frac{\mathrm{d}\boldsymbol{X}_h}{\mathrm{d}t}\right|\cdot\left|\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t}\right|^{-1} \le C\,\frac{h^{\mu-2}}{d^{m-2}}\,\|\boldsymbol{X}\|_{m,G}, \tag{18}$$

because of (7) and (10).

$$\sup_{t_1,t_2\in G}\left|1-\frac{\mathrm{d}s_h}{\mathrm{d}s}(t_1)\,\frac{\mathrm{d}s_h}{\mathrm{d}s}(t_2)\right| = \sup_{t_1,t_2\in G}\left|\left[1-\frac{\mathrm{d}s_h}{\mathrm{d}s}(t_2)\right]+\left[1-\frac{\mathrm{d}s_h}{\mathrm{d}s}(t_1)\right]\frac{\mathrm{d}s_h}{\mathrm{d}s}(t_2)\right| \le C\,\frac{h^{\mu-2}}{d^{m-2}}\,\|\boldsymbol{X}\|_{m,G},$$

$$\begin{aligned}
\left|\frac{\partial v}{\partial\tau}(\boldsymbol{x})-\frac{\partial v}{\partial\tau_h}(\boldsymbol{x}_h)\right|_{-\frac{1}{2},\Gamma} &= \left|\frac{\partial v}{\partial s}(\boldsymbol{x})-\frac{\partial v}{\partial s_h}(\boldsymbol{x}_h)\right|_{-\frac{1}{2},\Gamma} = \left|\frac{\partial v}{\partial s}(\boldsymbol{x})\left[1-\frac{\mathrm{d}s_h}{\mathrm{d}s}(\boldsymbol{x})\right]\right|_{-\frac{1}{2},\Gamma}\\
&= \sup_{q\in H^{(\Gamma)}}\int_\Gamma\frac{\partial v}{\partial s}\left[1-\frac{\mathrm{d}s_h}{\mathrm{d}s}\right](\boldsymbol{x})q(\boldsymbol{x})\mathrm{d}s\cdot\left(\|q\|_{\frac{1}{2},\Gamma}\right)^{-1}\\
&\le \sup_{\boldsymbol{x}\in\Gamma}\left|1-\frac{\mathrm{d}s_h}{\mathrm{d}s}(\boldsymbol{x})\right|\left\|\frac{\partial v}{\partial s}\right\|_{-\frac{1}{2},\Gamma} \le C\,\frac{h^{\mu-2}}{d^{m-2}}\,\|\boldsymbol{X}\|_{m,G}\cdot\|v\|_{\frac{1}{2},\Gamma}.
\end{aligned}$$

At any point $\boldsymbol{x}\in\Gamma$ a tangential–normal coordinate system may be established. We assume that in this system and in some neighbourhood $U_x$ of $\boldsymbol{x}$ the boundary $\Gamma$ can be described by a function $f$:

$$\begin{aligned}
&(y_1,y_2)\in U_{\boldsymbol{x}} &&\Rightarrow y_1\in(a_1,a_2), \quad a_1<0<a_2,\\
&(y_1,y_2)\in U_{\boldsymbol{x}}\cap\Gamma &&\Leftrightarrow y_2=f(y_1), \quad f(0)=0.
\end{aligned} \tag{19}$$

We suppose that $f\in C^1(a_1,a_2)$ and $\mathrm{d}f/\mathrm{d}y_1$ is Lipschitz-continuous at 0, i.e.

$$\left|\frac{\mathrm{d}f}{\mathrm{d}y_1}(y_1)\right| \le L_0\,y_1, \quad \forall y_1\in(a_1,a_2). \tag{20}$$

By $\boldsymbol{\tau}^x,\boldsymbol{\tau}^y,\boldsymbol{n}^x,\boldsymbol{n}^y$ we denote tangent and outward normal unit vectors at $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively (cf. Fig.2).

**Lemma 3.** *By the above assumptions and notations*

$$\left|n^x\cdot\frac{y}{|y|}\right| \le L_0|y|, \tag{21}$$

$$|\tau^x-\tau^y| \le L_0|y|, \tag{22}$$

$$|\tau^x(\tau^x-\tau^y)| \le \frac{1}{2}L_0|y|^2. \tag{23}$$

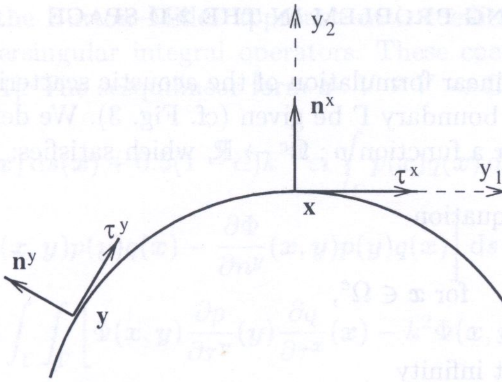


**Fig. 2.** Local system of coordinates

Proof. In the proposed system of coordinates $\boldsymbol{\tau}^x = (1,0)$, $\boldsymbol{n}^x = (0,1)$.

$$\left| n^x \cdot \frac{y}{|y|} \right| = \frac{|y_2|}{\sqrt{y_1^2 + y_2^2}} \le \frac{|f(y_1)|}{|y_1|} \le \frac{L_0 |y_1|^2}{|y_1|} \le L_0 |y|. \tag{24}$$

For $z = \mathrm{d}f/\mathrm{d}y_1(y_1)$

$$|\tau^x - \tau^y|^2 = \left| (1,0) - \left(1, \frac{\mathrm{d}f}{\mathrm{d}y_1}(y_1)\right)\left(1 + \left[\frac{\mathrm{d}f}{\mathrm{d}y_1}(y_1)\right]^2\right)^{-\frac{1}{2}} \right|^2$$

$$= \left| \left(1 - \frac{1}{\sqrt{1+z^2}}, \frac{z}{\sqrt{1+z^2}}\right) \right|^2 = \frac{2z^2}{\sqrt{1+z^2}\left(\sqrt{1+z^2}+1\right)} \le z^2 \le L_0^2 |y_1|^2.$$

$$|\tau^x(\tau^x - \tau^y)| = \left| (1,0)\left(1 - \frac{1}{\sqrt{1+z^2}}, \frac{z}{\sqrt{1+z^2}}\right) \right| = \frac{z^2}{\sqrt{1+z^2}\left(\sqrt{1+z^2}+1\right)} \le \frac{1}{2} L_0 |y_1|^2.$$

**Lemma 4.** *Let points $\boldsymbol{x}, \boldsymbol{y} \in \Gamma$ and their hp- interpolants $\boldsymbol{x}_h$, $\boldsymbol{y}_h \in \Gamma_h$ be given. Tangent and normal vectors at $\boldsymbol{x}_h$, $\boldsymbol{y}_h$ are correspondingly denoted by $\tau_h^x$, $n_h^x$, $\tau_h^y$, $n_h^y$. In addition $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, $r = |\boldsymbol{r}|$, $\boldsymbol{r}_h = \boldsymbol{x}_h - \boldsymbol{y}_h$, $r_h = |\boldsymbol{r}_h|$. For points $\boldsymbol{x}, \boldsymbol{y}$ near enough, by previous assumptions about interpolation, the following inequalities hold:*

$$|r - r_h| \le |\boldsymbol{r} - \boldsymbol{r}_h| \le Cr\,\frac{h^{\mu-1}}{d^{m-1}} \|\boldsymbol{X}\|_{m,G}, \tag{25}$$

$$\left| n^x \cdot \frac{r}{|r|} - n_h^x \cdot \frac{r_h}{|r_h|} \right| \le Cr\,\frac{h^{\mu-2}}{d^{m-2}} \|\boldsymbol{X}\|_{m,G}, \tag{26}$$

$$\left| n^y \cdot \frac{r}{|r|} - n_h^y \cdot \frac{r_h}{|r_h|} \right| \le Cr\,\frac{h^{\mu-2}}{d^{m-2}} \|\boldsymbol{X}\|_{m,G}, \tag{27}$$

$$|\tau^x - \tau_h^x| \le C\,\frac{h^{\mu-2}}{d^{m-2}} \|\boldsymbol{X}\|_{m,G}, \tag{28}$$

$$|\tau^x \tau^y - \tau_h^x \tau_h^y| \le Cr^2\,\frac{h^{\mu-2}}{d^{m-2}} \|\boldsymbol{X}\|_{m,G}. \tag{29}$$

Proof. It is easy to note that

$$|\boldsymbol{\tau}^x - \boldsymbol{\tau}_h^x| = |\boldsymbol{n}^x - \boldsymbol{n}_h^x|, \tag{30}$$

$$|\boldsymbol{\tau}^x \boldsymbol{\tau}^y - \boldsymbol{\tau}_h^x \boldsymbol{\tau}_h^y| = |\boldsymbol{n}^x \boldsymbol{n}^y - \boldsymbol{n}_h^x \boldsymbol{n}_h^y|. \tag{31}$$

Inequalities (25)–(27) were proved in [11]. The next ones were proved there for right-hand sides of (30), (31).

## 4. THE ELASTIC SCATTERING PROBLEM IN THE 2-D SPACE

We consider here the classical linear formulation of the acoustic scattering problem. Let a bounded domain $\Omega \in \mathbb{R}^2$ with a smooth boundary $\Gamma$ be given (cf. Fig. 3). We define the exterior domain $\Omega^e$ which completes $\Omega$ and look for a function $p : \bar{\Omega}^e \to \mathbb{R}$, which satisfies:

- the Helmholtz differential equation

$$-\Delta p(\boldsymbol{x}) - k^2 p(\boldsymbol{x}) = 0, \qquad \text{for } \boldsymbol{x} \in \Omega^e, \tag{32}$$

- the Sommerfeld condition at infinity

$$\left| \frac{\partial p^s}{\partial r} - ikp^s \right| = o(r^{-\frac{1}{2}}), \qquad \text{for } r = |\boldsymbol{x}| \to \infty, \tag{33}$$

- a simplified boundary condition on $\Gamma$

$$\frac{\partial p}{\partial n_x}(\boldsymbol{x}) = \varepsilon p(\boldsymbol{x}), \qquad \text{for } \boldsymbol{x} \in \Gamma, \tag{34}$$

where $k$ is the wavenumber. $p$ is interpreted as a total pressure of the acoustic wave

$$p(\boldsymbol{x}) = p^{inc}(\boldsymbol{x}) + p^s(\boldsymbol{x}). \tag{35}$$

Terms $p^{inc}$ and $p^s$ mean an incident and scattered wave pressure. The boundary condition (34) is a "spring-like" scatterer, which corresponds to a rubber layer on the surface of the body, which occupies the domain $\Omega$. For details of this model see e.g. [6] or [10].

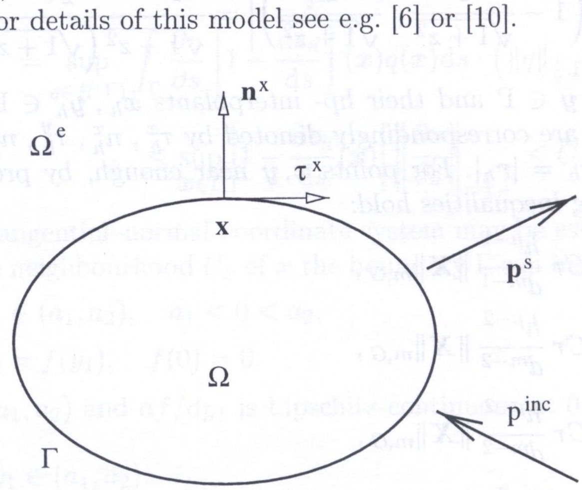


**Fig. 3.** The elastic scattering problem

The fundamental solution of Eq. (32) in the two-dimensional space is

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \frac{i}{4} H_0^1(kr), \tag{36}$$

where $i^2 = -1$, $r = |\boldsymbol{x} - \boldsymbol{y}|$ and $H_0^1(x)$ is the Hankel function of the first kind. Using this fundamental solution we may replace the boundary-value problem (32)–(34) by a variational equation

$$a(p, q) = l(q), \quad \forall q \in V = H^{\frac{1}{2}}(\Gamma), \tag{37}$$

which was obtained using the Burton–Miller approach with coefficients $\alpha \in [0,1]$ and $(1-\alpha)k^{-1}$ for the Helmholtz and hipersingular integral operators. These coefficients were proved to be the optimal ones if $\alpha = 0.5$ in [1]. The sesquilinear form $a : V \times V \to \mathbb{C}$ is given by

$$\begin{aligned} a(p,q) = {} & 0.5\alpha \int_\Gamma p(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}s(\boldsymbol{x}) + 0.5(1-\alpha)k^{-1}\varepsilon i \int_\Gamma p(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}s(\boldsymbol{x}) \\ & + \alpha \int_\Gamma\int_\Gamma \left[\varepsilon\Phi(\boldsymbol{x},\boldsymbol{y})p(\boldsymbol{y})q(\boldsymbol{x}) - \frac{\partial\Phi}{\partial n^y}(\boldsymbol{x},\boldsymbol{y})p(\boldsymbol{y})q(\boldsymbol{x})\right] \mathrm{d}s(\boldsymbol{y})\mathrm{d}s(\boldsymbol{x}) \\ & + (1-\alpha)k^{-1}i \int_\Gamma\int_\Gamma \left[\Phi(\boldsymbol{x},\boldsymbol{y})\frac{\partial p}{\partial \tau^y}(\boldsymbol{y})\frac{\partial q}{\partial \tau^x}(\boldsymbol{x}) - k^2\Phi(\boldsymbol{x},\boldsymbol{y})p(\boldsymbol{y})q(\boldsymbol{x})\tau^x\tau^y \right. \\ & \left. + \varepsilon\frac{\partial\Phi}{\partial n^x}(\boldsymbol{x},\boldsymbol{y})p(\boldsymbol{y})q(\boldsymbol{x})\right] \mathrm{d}s(\boldsymbol{y})\mathrm{d}s(\boldsymbol{x}) \end{aligned} \tag{38}$$

and the semilinear form $l : V \to \mathbb{C}$ by

$$l(q) = \alpha \int_\Gamma p^{inc}(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}s(\boldsymbol{x}) + (1-\alpha)k^{-1}i \int_\Gamma \frac{\partial p^{inc}}{\partial n^x}(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}s(\boldsymbol{x}). \tag{39}$$

Equation (37) is equivalent to (32)–(34) if $p$ is its regular solution. For details see e.g. [12]. It is known that the both forms (38) and (39) are continuous and $a$ fulfils the Gårding inequality

$$\mathrm{Re}[a(v,v) + c(v,v)] \geq \gamma\|v\|^2, \quad \forall v \in V. \tag{40}$$

for some sesquilinear compact form $c$ and constant $\gamma > 0$ (cf. [20]).

## 5. Properties of the fundamental solution

**Lemma 5.** *For Hankel functions $H_0^{(1)}$, $H_1^{(1)}$ the following estimate is true:*

$$|[H_\nu^{(1)}(x) - H_\nu^{(1)}(y)][H_\nu^{(1)}(x)]^{-1}| \leq C\frac{|x-y|}{y}, \quad \nu = 0,1, \quad x,y > 0. \tag{41}$$

Proof. We apply here an integral formula for these functions, valid for $\mathrm{Re}\nu > -0.5$, $-\pi/2 < arg\, z < \pi$, $i = \sqrt{-1}$ [14, p. 181]:

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \exp\{i[z - 0.25\pi(2\nu-1)]\} \left[\Gamma\left(\nu + \frac{1}{2}\right)\right]^{-1} \int_0^\infty e^{-s}s^{\nu-\frac{1}{2}}\left(1 - \frac{s}{2iz}\right)^{\nu-\frac{1}{2}} \mathrm{d}s. \tag{42}$$

In this proof $\Gamma$ means the Euler Gamma Function. This formula states, that $|H_\nu^{(1)}(x)| > 0$ for $\nu = 0,1$, $x > 0$. We have then, for $\nu = 0$,

$$H_0^{(1)}(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \exp(-0.25i\pi) \left[\Gamma\left(\frac{1}{2}\right)\right]^{-1} \int_0^\infty e^{iz-s}s^{-\frac{1}{2}}\left(z + \frac{is}{2}\right)^{-\frac{1}{2}} \mathrm{d}s. \tag{43}$$

$$\begin{aligned} H_0^{(1)}(x) - H_0^{(1)}(y) = {} & \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \exp(-0.25i\pi) \left[\Gamma\left(\frac{1}{2}\right)\right]^{-1} \\ & \times \int_0^\infty e^{-s}s^{-\frac{1}{2}} \frac{e^{ix}}{\sqrt{x+\frac{is}{2}}} \left[\left(\frac{e^{ix}}{\sqrt{x+\frac{is}{2}}}\right)^{-1} \left(\frac{e^{ix}}{\sqrt{x+\frac{is}{2}}} - \frac{e^{iy}}{\sqrt{y+\frac{is}{2}}}\right)\right] \mathrm{d}s. \end{aligned} \tag{44}$$

We pay attention to the term in the square brackets

$$\left|\left[\left(\frac{e^{ix}}{\sqrt{x+\frac{is}{2}}}\right)^{-1}\left(\frac{e^{ix}}{\sqrt{x+\frac{is}{2}}}-\frac{e^{iy}}{\sqrt{y+\frac{is}{2}}}\right)\right]\right| \le \left|1-e^{i(y-x)}\sqrt{\frac{x+\frac{is}{2}}{y+\frac{is}{2}}}\right| < \left|1-e^{i(y-x)}\sqrt{\frac{x}{y}}\right|$$
$$< \left|1-e^{i(y-x)}\right| + \left|1-\sqrt{\frac{x}{y}}\right| + \left|1-e^{i(y-x)}\right|\cdot\left|1-\sqrt{\frac{x}{y}}\right|$$
$$\le C|y-x| + \frac{C|y-x|}{\sqrt{y}(\sqrt{x}+\sqrt{y})} + \frac{C|y-x|^2}{\sqrt{y}(\sqrt{x}+\sqrt{y})} \le \frac{C|y-x|}{y}. \tag{45}$$

Inequalities (44) and (45) imply (41).

Let us consider the case $\nu = 1$ now.

$$H_1^{(1)}(x) - H_1^{(1)}(y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \exp(-0.75i\pi)\left[\Gamma\left(\frac{3}{2}\right)\right]^{-1}$$
$$\times \int_0^\infty e^{-s} s^{\frac{1}{2}} \frac{e^{ix}\sqrt{x+\frac{is}{2}}}{x}\left[\left(\frac{e^{ix}\sqrt{x+\frac{is}{2}}}{x}\right)^{-1}\left(\frac{e^{ix}\sqrt{x+\frac{is}{2}}}{x}-\frac{e^{iy}\sqrt{y+\frac{is}{2}}}{y}\right)\right] ds \tag{46}$$

and

$$\left|\left[\left(\frac{e^{ix}\sqrt{x+\frac{is}{2}}}{x}\right)^{-1}\left(\frac{e^{ix}\sqrt{x+\frac{is}{2}}}{x}-\frac{e^{iy}\sqrt{y+\frac{is}{2}}}{y}\right)\right]\right| \le \left|1-e^{i(y-x)}\frac{x}{y}\sqrt{\frac{y+\frac{is}{2}}{x+\frac{is}{2}}}\right|$$
$$< \left|1-e^{i(y-x)}\sqrt{\frac{x}{y}}\right| \le \frac{C|y-x|}{y} \tag{47}$$

analogously to (45).

We have applied here the algebraic formulas:

$$1-\delta\varepsilon = (1-\delta)+(1-\varepsilon)-(1-\delta)(1-\varepsilon), \qquad \forall \delta,\varepsilon\in\mathbb{R}, \tag{48}$$
$$\left|\frac{a^2(b+ic)}{b^2(a+ic)}-1\right| < \left|\frac{a}{b}-1\right|, \qquad \forall a,b,c>0, \tag{49}$$
$$\left|\frac{a+ic}{b+ic}-1\right| < \left|\frac{a}{b}-1\right|, \qquad \forall a,b,c>0. \tag{50}$$

**Lemma 6.** *There is a constant $C>0$ such that for any $\boldsymbol{x},\boldsymbol{y}\in\Gamma$, $\boldsymbol{x}\ne\boldsymbol{y}$*

$$\|\Phi(\boldsymbol{x},\boldsymbol{y})-\Phi(\boldsymbol{x}_h,\boldsymbol{y}_h)\| \le C|\Phi(\boldsymbol{x},\boldsymbol{y})|\cdot\frac{h^{\mu-1}}{d^{m-1}}\|\boldsymbol{X}\|_{m,G}, \tag{51}$$
$$\left\|\frac{\partial\Phi}{\partial n^y}(\boldsymbol{x},\boldsymbol{y})-\frac{\partial\Phi}{\partial n_h^y}(\boldsymbol{x}_h,\boldsymbol{y}_h)\right\| \le C\left[\left|\frac{\partial\Phi}{\partial n^y}(\boldsymbol{x},\boldsymbol{y})\right|+1\right]\cdot\frac{h^{\mu-2}}{d^{m-2}}\|\boldsymbol{X}\|_{m,G}. \tag{52}$$

Proof. Formulas (25), (36) and (41) imply (51). We know that

$$\frac{\partial\Phi}{\partial n^y}(\boldsymbol{x},\boldsymbol{y}) = \frac{i}{4}\frac{\mathrm{d}}{\mathrm{d}r}H_0^1(kr)\cdot\frac{\boldsymbol{r}\cdot\boldsymbol{n}^y}{r} = \frac{k}{4i}H_1^1(kr)\frac{\boldsymbol{r}\cdot\boldsymbol{n}^y}{r}. \tag{53}$$

Therefore

$$\left|\frac{\partial\Phi}{\partial n^y}(\boldsymbol{x},\boldsymbol{y})-\frac{\partial\Phi}{\partial n_h^y}(\boldsymbol{x}_h,\boldsymbol{y}_h)\right| \le \frac{k}{4}\left|\left[H_1^1(kr)-H_1^1(kr_h)\right]\frac{\boldsymbol{r}\cdot\boldsymbol{n}^y}{r}\right| + \frac{k}{4}H_1^1(kr)\left|\frac{\boldsymbol{r}\cdot\boldsymbol{n}^y}{r}-\frac{\boldsymbol{r}_h\cdot\boldsymbol{n}_h^y}{r_h}\right|$$
$$\le C\,H_1^1(kr)\left|\frac{\boldsymbol{r}\cdot\boldsymbol{n}^y}{r}\right|\frac{|r-r_h|}{r_h} + C\,H_1^1(kr)r\,\frac{h^{\mu-2}}{d^{m-2}}\|\boldsymbol{X}\|_{m,G}. \tag{54}$$

Expression $H_1^1(kr)r$ is bounded because of (41). (25) and (27) imply then

$$(54) \le C\left|1+\left|\frac{\partial\Phi}{\partial n^y}(\boldsymbol{x},\boldsymbol{y})\right|\right|\frac{h^{\mu-2}}{d^{m-2}}\|\boldsymbol{X}\|_{m,G} \tag{55}$$

and (52) is proved.

## 6. Convergence estimation

We solve Eq. (37) by the Galerkin method, although we compare in it approximate forms $a_h$, $l_h$, being forms $a$ and $l$ obtained on the approximate boundary using numerical integration

$$a_h(p_h,q_h)=l_h(q_h), \quad \forall q_h\in V_h. \tag{56}$$

The convergence estimates can be stated as follows:

**Theorem 3.** *We assume that*

- *solution $p$ of (37) belongs to $H^{m-1}(\Gamma)$, $m>2$;*
- $V_h\subset\{v\in C^0(G):\ \forall i\in\{1,\dots,I\}\ v|_{T_i}\in P_{d-1}(T_i)\},\ d>0;$
- $\boldsymbol{X}\in H^m(G)\times H^m(G);$
- $\boldsymbol{X}_h\in I_d;$
- *terms $p^{inc}$ and $\partial p^{inc}/\partial n^x$ are interpolated by $p_h^{inc}$ and $\partial p_h^{inc}/\partial n_h^x$ on $\Gamma_h$;*
- *the hp- adaptive BEM described above is used to solve the problem (37) and $p_h$ is the solution of (56).*

*Then there is a positive constant $C$, for which*

$$\|p-p_h\|_{\frac12,\Gamma}\le C\left[\|p^{inc}-p_h^{inc}\|_{-\frac12,\Gamma}+\left\|\frac{\partial p^{inc}}{\partial n^x}-\frac{\partial p_h^{inc}}{\partial n_h^x}\right\|_{-\frac12,\Gamma}+\frac{h^{\mu-2}}{d^{m-2}}\left(\|\boldsymbol{X}\|_{m,G}\|p\|_{\frac12,\Gamma}+\|p\|_{m,\Gamma}\right)\right], \tag{57}$$

*where $C$ does not depend on $h$ and $d$.*

**Corollary 1.** *If $p^{inc}$ and $\partial p^{inc}/\partial n^x$ are hp - interpolated like $p$, inequality (57) obtains the form*

$$\|p-p_h\|_{\frac12,\Gamma}\le C\frac{h^{\mu-2}}{d^{m-2}}\left[\|p^{inc}\|_{m,\Gamma}+\left\|\frac{\partial p^{inc}}{\partial n^x}\right\|_{m,\Gamma}+\|\boldsymbol{X}\|_{m,G}\|p\|_{\frac12,\Gamma}+\|p\|_{m,\Gamma}\right]. \tag{58}$$

Proof of Theorem 3. To the modified Galerkin equation (56) we can apply the Second Strang Lemma (cf. [11]):

**Lemma 7.** *By the above assumptions there is a constant $C$, such that*

$$\|p-p_h\|\le C\left\{\inf_{v_h\in V_h}\left[\|p-v_h\|+\sup_{q_h\in V_h}\frac{|a_h(v_h,q_h)-a(v_h,q_h)|}{\|q_h\|}\right]+\sup_{q_h\in V_h}\frac{|l_h(q_h)-l(q_h)|}{\|q_h\|}\right\}. \tag{59}$$

We should estimate two right-hand side terms of inequality (59)

$$\begin{aligned}|a_h(v_h,q_h)-a(v_h,q_h)| = \Bigg| \int_{\Gamma_h} &[\alpha v_h(\boldsymbol{x}_h)+i(1-\alpha)k^{-1}\varepsilon v_h(\boldsymbol{x}_h)]q_h(\boldsymbol{x}_h)\,\mathrm{d}s_x \\ &+\int_{\Gamma_h}\int_{\Gamma_h}\Big\{\alpha\varepsilon\,\Phi(\boldsymbol{x}_h,\boldsymbol{y}_h)\,v_h(\boldsymbol{y}_h)\,q_h(\boldsymbol{x}_h)-\alpha\,\frac{\partial\Phi}{\partial n_h^y}(\boldsymbol{x}_h,\boldsymbol{y}_h)\,v_h(\boldsymbol{y}_h)\,q_h(\boldsymbol{x}_h) \\ &\qquad+i(1-\alpha)k^{-1}\varepsilon\,\frac{\partial\Phi}{\partial n_h^x}(\boldsymbol{x}_h,\boldsymbol{y}_h)\,v_h(\boldsymbol{y}_h)\,q_h(\boldsymbol{x}_h) \\ &\qquad-i(1-\alpha)k\,\Phi(\boldsymbol{x}_h,\boldsymbol{y}_h)\,\tau_h^x\tau_h^y\,v_h(\boldsymbol{y}_h)\,q_h(\boldsymbol{x}_h) \\ &\qquad+i(1-\alpha)k^{-1}\,\Phi(\boldsymbol{x}_h,\boldsymbol{y}_h)\,\frac{\partial v_h}{\partial\tau_h^y}(\boldsymbol{y}_h)\,\frac{\partial q_h}{\partial\tau_h^x}(\boldsymbol{x}_h)\Big\}\,\mathrm{d}s_y\mathrm{d}s_x \\ &-\int_{\Gamma}[\alpha v_h(\boldsymbol{x})+i(1-\alpha)k^{-1}\varepsilon v_h(\boldsymbol{x})]\,q_h(\boldsymbol{x})\,\mathrm{d}s_x \\ &-\int_{\Gamma}\int_{\Gamma}\Big\{\alpha\varepsilon\,\Phi(\boldsymbol{x},\boldsymbol{y})\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x})-\alpha\,\frac{\partial\Phi}{\partial n^y}(\boldsymbol{x},\boldsymbol{y})\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x}) \\ &\qquad+i(1-\alpha)k^{-1}\varepsilon\,\frac{\partial\Phi}{\partial n^x}(\boldsymbol{x},\boldsymbol{y})\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x}) \\ &\qquad-i(1-\alpha)k\,\Phi(\boldsymbol{x},\boldsymbol{y})\,\tau^x\tau^y\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x}) \\ &\qquad+i(1-\alpha)k^{-1}\,\Phi(\boldsymbol{x},\boldsymbol{y})\,\frac{\partial v_h}{\partial\tau^y}(\boldsymbol{y})\,\frac{\partial q_h}{\partial\tau^x}(\boldsymbol{x})\Big\}\,\mathrm{d}s_y\mathrm{d}s_x\Bigg|. \qquad (60)\end{aligned}$$

We take into account that

$$v_h(\boldsymbol{y})=v_h(\boldsymbol{y}_h),\quad q_h(\boldsymbol{x})=q_h(\boldsymbol{x}_h) \qquad (61)$$

$$\begin{aligned}(60)\le\Bigg|&\int_{\Gamma}[\alpha v_h(\boldsymbol{x})+i(1-\alpha)k^{-1}\varepsilon v_h(\boldsymbol{x})]\,q_h(\boldsymbol{x})\left[1-\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})}\right]\mathrm{d}s_x \\ &+\int_{\Gamma}\int_{\Gamma}\Big\{\alpha\varepsilon\,\Phi(\boldsymbol{x},\boldsymbol{y})\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x})\left[1-\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})}\right] \\ &\qquad+\alpha\varepsilon[\Phi(\boldsymbol{x}_h,\boldsymbol{y}_h)-\Phi(\boldsymbol{x},\boldsymbol{y})]\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x})\,\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})} \\ &\qquad-\alpha\,\frac{\partial\Phi}{\partial n^y}(\boldsymbol{x},\boldsymbol{y})\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x})\left[1-\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})}\right] \\ &\qquad-\alpha\left[\frac{\partial\Phi}{\partial n^y}(\boldsymbol{x},\boldsymbol{y})-\frac{\partial\Phi}{\partial n_h^y}(\boldsymbol{x}_h,\boldsymbol{y}_h)\right]v_h(\boldsymbol{y})\,q_h(\boldsymbol{x})\,\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})} \\ &\qquad+i(1-\alpha)k^{-1}\varepsilon\,\frac{\partial\Phi}{\partial n_h^x}(\boldsymbol{x}_h,\boldsymbol{y}_h)\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x})\left[1-\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})}\right] \\ &\qquad+i(1-\alpha)k^{-1}\varepsilon\left[\frac{\partial\Phi}{\partial n^x}(\boldsymbol{x},\boldsymbol{y})-\frac{\partial\Phi}{\partial n_h^x}(\boldsymbol{x}_h,\boldsymbol{y}_h)\right]v_h(\boldsymbol{y})\,q_h(\boldsymbol{x})\,\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})} \\ &\qquad-i(1-\alpha)k\,\Phi(\boldsymbol{x},\boldsymbol{y})\,\tau^x\tau^y\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x})\left[1-\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})}\right] \\ &\qquad-i(1-\alpha)k\,\Phi(\boldsymbol{x},\boldsymbol{y})[\tau^x\tau^y-\tau_h^x\tau_h^y]\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x})\,\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})} \\ &\qquad-i(1-\alpha)k[\Phi(\boldsymbol{x}_h,\boldsymbol{y}_h)-\Phi(\boldsymbol{x},\boldsymbol{y})]\,\tau_h^x\tau_h^y\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x})\,\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})}\end{aligned}$$

$$+ i(1-\alpha)k^{-1}\Phi(\boldsymbol{x},\boldsymbol{y})\,\frac{\partial v_h}{\partial \tau^y}(\boldsymbol{y})\,\frac{\partial q_h}{\partial \tau^x}(\boldsymbol{x})\left[1-\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\,\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})}\right]$$
$$+ i(1-\alpha)k^{-1}\Phi(\boldsymbol{x},\boldsymbol{y})\left[\frac{\partial v_h}{\partial \tau^y}(\boldsymbol{y})\,\frac{\partial q_h}{\partial \tau^x}(\boldsymbol{x})-\frac{\partial v_h}{\partial \tau^y_h}(\boldsymbol{y}_h)\,\frac{\partial q_h}{\partial \tau^x_h}(\boldsymbol{x}_h)\right]\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\,\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})}$$
$$\left.\left. + i(1-\alpha)k^{-1}[\Phi(\boldsymbol{x}_h,\boldsymbol{y}_h)-\Phi(\boldsymbol{x},\boldsymbol{y})]\,\frac{\partial v_h}{\partial \tau^y}(\boldsymbol{y})\,\frac{\partial q_h}{\partial \tau^x}(\boldsymbol{x})\,\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\,\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})}\right\}\mathrm{d}s_y\mathrm{d}s_x\right|. \tag{62}$$

All terms in square brackets are bounded by $Ch^{\mu-2}d^{2-m}\|\boldsymbol{X}\|_{m,G}$, cf. (15), (16), (17), (29), (51), (52). It is well-known, that the single- and double layer potential operators and the adjoint one,

$$V(v)(\boldsymbol{x}) = \int_\Gamma \Phi(\boldsymbol{x},\boldsymbol{y})\,v(\boldsymbol{y})\,\mathrm{d}s_y\,, \qquad V: H^{-\frac{1}{2}}(\Gamma)\to H^{\frac{1}{2}}(\Gamma); \tag{63}$$

$$K(v)(\boldsymbol{x}) = \int_\Gamma \frac{\partial\Phi}{\partial n^y}(\boldsymbol{x},\boldsymbol{y})\,v(\boldsymbol{y})\,\mathrm{d}s_y\,, \qquad K: H^{\frac{1}{2}}(\Gamma)\to H^{\frac{1}{2}}(\Gamma); \tag{64}$$

$$K'(v)(\boldsymbol{x}) = \int_\Gamma \frac{\partial\Phi}{\partial n^x}(\boldsymbol{x},\boldsymbol{y})\,v(\boldsymbol{y})\,\mathrm{d}s_y\,, \qquad K': H^{\frac{1}{2}}(\Gamma)\to H^{\frac{1}{2}}(\Gamma); \tag{65}$$

are linear and continuous in the indicated spaces. First terms of (62) may be estimated using the Schwarz inequality

$$\left|\int_\Gamma [\alpha v_h(\boldsymbol{x}) + i(1-\alpha)k^{-1}\varepsilon v_h(\boldsymbol{x})]\,q_h(\boldsymbol{x})\left[1-\frac{\mathrm{d}s_h}{\mathrm{d}s}(\boldsymbol{x})\right]\mathrm{d}s_x\right| \le C\,\frac{h^{\mu-2}}{d^{m-2}}\|\boldsymbol{X}\|_{m,G}\|v_h\|_{\frac{1}{2},\Gamma}\|q_h\|_{\frac{1}{2},\Gamma} \tag{66}$$

and last ones using properties of potentials

$$\left|\int_\Gamma\int_\Gamma \alpha\varepsilon\Phi(\boldsymbol{x},\boldsymbol{y})\,v_h(\boldsymbol{y})\,q_h(\boldsymbol{x})\left[1-\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\,\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})}\right]\right|$$
$$\le C\,\frac{h^{\mu-2}}{d^{m-2}}\|\boldsymbol{X}\|_{m,G}\|V(v_h)\|_{0,\Gamma}\|q_h\|_{0,\Gamma} \le C\,\frac{h^{\mu-2}}{d^{m-2}}\|\boldsymbol{X}\|_{m,G}\|v_h\|_{\frac{1}{2},\Gamma}\|q_h\|_{\frac{1}{2},\Gamma}\,; \tag{67}$$

$$\left|i(1-\alpha)k^{-1}\,\Phi(\boldsymbol{x},\boldsymbol{y})\,\frac{\partial v_h}{\partial \tau^y}(\boldsymbol{y})\,\frac{\partial q_h}{\partial \tau^x}(\boldsymbol{x})\left[1-\frac{\mathrm{d}s_h(\boldsymbol{y}_h)}{\mathrm{d}s(\boldsymbol{y})}\,\frac{\mathrm{d}s_h(\boldsymbol{x}_h)}{\mathrm{d}s(\boldsymbol{x})}\right]\right|$$
$$\le C\,\frac{h^{\mu-2}}{d^{m-2}}\|\boldsymbol{X}\|_{m,G}\left\|V\left(\frac{\partial v_h}{\partial \tau^y}\right)\right\|_{\frac{1}{2},\Gamma}\left\|\frac{\partial q_h}{\partial \tau^x}\right\|_{-\frac{1}{2},\Gamma} \le C\,\frac{h^{\mu-2}}{d^{m-2}}\|\boldsymbol{X}\|_{m,G}\|v_h\|_{\frac{1}{2},\Gamma}\|q_h\|_{\frac{1}{2},\Gamma}\,. \tag{68}$$

We have therefore

$$|a_h(v_h,q_h) - a(v_h,q_h)| \le C\,\frac{h^{\mu-2}}{d^{m-2}}\|\boldsymbol{X}\|_{m,G}\|v_h\|_{\frac{1}{2},\Gamma}\|q_h\|_{\frac{1}{2},\Gamma}\,. \tag{69}$$

The term $|l_h(q_h) - l(q_h)|$ may be estimated analogously.

## 7. Conclusions

The presented paper delivers an estimate of the hp- adaptive Boundary Element Method. It confirms a known fact that the optimal convergence rate is attainable, if the boundary interpolation order is greater by 1 than the function interpolation order.

Convergence tests are reported in [6, 10]. Analogous estimates over three-dimensional surfaces are given in [11]. Integrals arising in forms $a_h$ and $l_h$ are evaluated by appropriate quadratures, which are sources of another errors. This aspect of the problem is a subject of the paper [13].

As a continuation of this paper some numerical tests verifying "real" convergence rates for problems of this type are needed. An optimal way of hp-adaptation is still not fixed out. Other external problems and nonlinear tasks are a great research field, where the practice leaves the theory far behind.

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