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The algorithm of adaptation by using graded meshes generator

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An algorithm of remeshing based on graded meshes generator is presented. The algorithm starts with an initial grid, which is iteratively improved taking into account error estimate. Mesh density functions are used to generate grid over domain on which boundary value problem is solved. It is observed, that succussive meshes are convergent and especially they become denser near singularities. For unstructured grid generation the advancing front technique combined with Delaunay triangulation is used. The boundary of 2-D domain may be represented by B-spline curves. It may be multiconnected.

1. Introduction

The paper presents an algorithm of remeshing by using plane domain generator of graded meshes [7, 4]. The main task of the adaptation procedure is based upon iterative generation of successive meshes leading to the the reduction of error estimate. That main feature of the generator is taking into account a positively defined function. Roughly speaking, the function defines the size of the mesh. For the sake of grid points generation the advancing front method is used, during a front modification a mesh density function is taken into account to insert a new point and then front is updated. Some stop criteria are introduced to keep a good mesh quality. After internal points generation a Delaunay [6] advancing front approach is applied to triangularize the obtained set of points. After triangulation the obtained mesh is smoothed (usually 3–4 steps of iterations).

The algorithm of adaptation is based on the previously described generator. It iteratively modifies the mesh density function, then the problem is solved, the procedure continues until a satisfactory error estimate is obtained.

The next step of the adaptation is based on a construction of a new mesh density function. The mesh density function is obtained as a function being a product of ρ_0 and a piece-wise linear function.

There are other methods of adaptation based upon for remeshing [14] or local refinement [3]. The method presented in [14] is based upon knot insertion and rearranging the whole mesh, thus method presented in [3] enriches existing mesh by new elements obtained by structured division of chosen elements.

2. Model problem

The problem on which the illustrated algorithm is presented can be formulated as follows: find $u \in V \subset H_0^1(\Omega)$, where $H_0^1(\Omega)$ is Sobolev space of order 1 of functions satisfying boundary conditions, such that u is a stationary point of the functional

$$I(u) = \int_\Omega F(u(x,y), u_x(x,y), u_y(x,y), \ldots)\, \mathrm{d}\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a domain of the variational problem. For finite element solution to the problem, finite element grid $\mathcal{T}_0$ is generated with the given mesh density function

$$\rho_0 : \Omega \mapsto \mathbb{R}, \qquad \text{positively defined.} \tag{2}$$

Then the approximation space is defined as

$$V^0 = \left\{ v : \bigcup_{i=1}^{n_0} \overline{T}_i^0 \mapsto \mathbb{R}, \quad v \text{ continuous}, \quad v|_T \in P_1, \quad \forall T \in \mathcal{T}_0 \right\}, \tag{3}$$

where $\mathcal{T}_0 = \{T_i : i = 1, \ldots n_0\}$ is the set of non-intersecting triangles covering the domain. In the paper, for the sake of illustration of the proposed method, the Poisson's equations is solved, which means, that the following functional is minimized,

$$I(u) = \int_\Omega \left(u_x^2(x,y) + u_y^2(x,y) - 2u(x,y) \right) \mathrm{d}\Omega. \tag{4}$$

It corresponds to the equation

$$\Delta u + 1 = 0, \tag{5}$$

and the boundary conditions

$$u|_\Gamma = 0. \tag{6}$$

3. Error estimators and error indicators

The finite element solution accuracy depends on the mesh size and order of polynomials of shape functions. Very often near singularities it is useful to make the mesh denser or to raise the order of polynomials. The method based upon mesh refinement is called h-method, thus method based upon raising the order of polynomials is called p-method.

For the sake of adaptation it is necessary to know the value of error or its estimates [3, 11]. There are two main kinds of error estimates:

- *a priori* error estimates,
- *a posteriori* error estimates.

A priori error estimate gives information about convergence of the FEM and its rate. It is rather difficult to apply it directly to mesh refinement. We define error as

$$e_h = u - u_h, \tag{7}$$

where u is the solution of the considered problem and u_h is an approximated solution.

For h-refinement *a priori* error estimate is given by the following inequality [12],

$$\|e_h\| \leq C h^{\min(p,\lambda)}. \tag{8}$$

In the above formula p is order of approximation polynom and λ is the measure of the strength [12] of the singularity.

In practice the a priori error esimate is not known and then more useful is *a posteriori* error estimate.

A posteriori error is obtained on just existing approximated solution. There are variety kinds of *a posteriori* error estimates, and they fall in two categories. The first category is based on norm of residual and taking into account the jump of derivatives of $p-1$-order derivatives at boundaries. The second category [14] is based upon smoothing of usually $p-1$-order derivatives obtained from the solution. Then the smoothed solution is used to replace the exact solution in Eq. (7), so all the component defining error are known.

In the paper, regarding the adapation strategy we need quantitative description of the error. In practive value of the error is never known so the error estimates lead to error indicators. In the paper the error indicators [1, 14] based on the first category of *a posteriori error* estimate were applied and tested.

The applied indicators are defined as follows:

Let e_i for $i = 1, \ldots, n_0$ be an error indicator at i-th apex of the mesh $\mathcal{T}_0$, and $\mathcal{P}_0 = \{P_i, i = 1, \ldots, n_P\}$ – set of nodes. We define a patch of elements for every node P_i as

$$L_i = \{k : P_i \in \overline{T}_k\} \tag{9}$$

for $i = 1, \ldots, n_P$.

1. Let N_i be a set of neighbors of i-th element,

$$N_i = \{k : T_k \text{ has a common edge with } T_i\}. \tag{10}$$

Then, following [1], the value of indicator at i-th node is

$$\tilde{e}_i = \sqrt{\sum_{k \in N_i} \left(\frac{\partial u_i}{\partial n_k} - \frac{\partial u_k}{\partial n_k} \right)^2} \tag{11}$$

where u_i is the restriction of the solution to i-th element and n_k is unit normal the edge common with k-th and i-th element.

2. In our case the functional (1) takes the form

$$I(u) = \int_\Omega \left(u_x^2(x,y) + u_y^2(x,y) - 2u(x,y) \right) \mathrm{d}\Omega. \tag{12}$$

It corresponds to the equation $\Delta u + 1 = 0$, then it is taken

$$\tilde{e}_i = \tilde{\Delta} u_h(P_i) + 1 \quad \text{at } i\text{-th node}, \tag{13}$$

where $\tilde{\Delta}$ is an approximation of Δ.

3. In this case it is suggested to introduce directlty values of error indicator at every node of the mesh. The error indicator is suggested by the author. From the numerical analyses it follows that the application of this error indicator causes generation similar meshes to the firstly defined,

$$\tilde{e}_i = \sqrt{\sum_{k \in L_i,\, l \in L_i,\, l \neq k} \left(\frac{\partial u_l}{\partial x} - \frac{\partial u_k}{\partial x} \right)^2 + \left(\frac{\partial u_l}{\partial y} - \frac{\partial u_k}{\partial y} \right)^2}, \tag{14}$$

where L_i is the set of numbers of elements meeting at i-th node.

For more complicated problems solved by FEM it may be useful to apply some specific error indicator taking into account some additional information about the problem.

4. Generation of unstructured meshes with mesh density function

For the sake of grid generation the 2-D generator is made [7, 13]. The algorithm based upon advancing front technique method and Delaunay triangulation is worked out. The algorithm and appriopriate computer code assumes, that a domain being triangulated can be multiconnected with arbitrary finite number of loops. Its boundary can be represented by a collection of curves, provided that every one of them is

- straight line segment,
- arch of circle,
- B-spline curve.

The characteristic feature of the algorithm of grid generation is taking into account a positively defined function as mesh density [13]. At present state the considered mesh density function is a scalar function defined in the closure of the considered domain. Thanks to that, it is possible to defined mesh density function for curves consisting the boundary and then to generate points on the boundary taking into account requested points distribution. For internal points generation advancing front technique [7, 9] is applied. For triangulation of the considered domain on the previously obtained set of points Delaunay triangulation with advancing front technique [5, 10] is applied. After triangulation the obtained grid is smoothed. The whole algorithm can be divided into the following steps:

1. generation of points on boundary curves with mesh density function,
2. internal points generation with mesh density function by advancing front technique
3. Delaunay advancing front technique triangulation of obtained set of points,
4. Laplacian smoothing of the mesh.

5. Algorithm of adaptation

The whole algorithm of adaptation is realized in successive generation of sequence of meshes $\{\mathcal{T}_\nu\}$, where $\nu = 0, 1, 2, \ldots$, with modified mesh density function. On every mesh of the sequence the problem is solved and appropriate error indicator at every element is obtained. The value of the error indicator are led to the nodes by averaging method. Having values of errors at nodes a continuous error function in the whole domain is constructed by using piecewise linear interpolation at all the elements. The error function is appropriately transformed to obtain a multiplier for mesh density function.

The proposed approach gives us the possibility to solve the considered problem on well-conditioned meshes and to obtain optimal graded meshes.

5.1. Remeshing scheme

The algorithm of remeshing can be divided into the following steps:

1. prepare the information about the geometry and boundary conditions,
2. fix an initial mesh density function,
3. generate the mesh with the mesh density function,
4. solve the problem on the generated mesh,

5. find error indicator at every element,

6. calculate nodal error indicators values by using averaging method,

7. define the new mesh density function by using the error calculated every point,

8. if error not satisfactory go to point 3.,

9. end of computations.

In the examples solved by the author of the paper it was enough to make 3 to 4 steps of adaptation.

6. Modification of the mesh density function

The modification of the mesh density is performed at every adaptation step for the realization of the next. The main idea of this part of the algorithm relies on reduction of the values of the mesh density function by an appriopriately chosen function. The chosen function is continuous, linear on every element and has the smallest value at node where the value of the error indicator is maximal and greatest where the value the error is minimal. It increases when error decreases. To describe the algorithm of the mesh density function modification, it is necessary to lead the values of the error to the node. For every node P_i a weighted value of the indicator is defined as follows,

$$\tilde{e}_i = \frac{\sum_{k\in L_i} \text{area}(T_i)\, e_i}{\sum_{k\in L_i} \text{area}(T_i)} . \tag{15}$$

In such a way a set of values of the error at every nodal point is given.

$$\alpha = \min_{k=1,2,\ldots,n_0} \tilde{e}_k , \tag{16}$$

$$\beta = \max_{k=1,2,\ldots,n_0} \tilde{e}_k . \tag{17}$$

Obviously, $\alpha \le \tilde{e}_k \le \beta$ for $k = 1, \ldots, n_0$. Introducing the following new values,

λ – a value indicating the greatest mesh density function reduction,

μ – a value indicating the smallest mesh density function reduction.

Usually λ and μ have positive values less than 1, and additionally $\mu \le \lambda$.

Let us define the transformation

$$l : [\alpha, \beta] \mapsto [\mu, \lambda] \tag{18}$$

satisfying conditions $l(\alpha) = \lambda$ and $l(\beta) = \mu$. By these assumptions

$$\mu \le l(x) \le \lambda. \tag{19}$$

Provided that $Q_i = l(\tilde{e}_i)$, we have

$$\min_{i=1,2,\ldots,n_0} Q_i = \mu, \tag{20}$$

$$\max_{i=1,2,\ldots,n_0} Q_i = \lambda. \tag{21}$$

Introducing the function $\kappa : \overline{D} \mapsto \mathbb{R}$ as follows,

$$\kappa(\overline{x}) = \Pi_{ABC}(\overline{x}), \quad \text{if } \overline{x} \in \overline{T}_s , \tag{22}$$

where Π_{ABC} is the linear function of two variables satisfying the following conditions,

$$\Pi_{ABC}(P_i) = Q_i \quad \text{for } i = 1, 2, 3, \tag{23}$$

where P_1, P_2, P_3 are the vertices of the triangle T_s of the triangulation of Ω. The function $\kappa(\overline{x})$ is defined in the whole domain because triangles $\overline{T}_s$ cover it. The new mesh density function is defined as follows,

$$\rho_{i+1}(\overline{x}) = \rho_i(\overline{x})\kappa(\overline{x}). \tag{24}$$

As $\mu \le \kappa(\overline{x}) \le \lambda$ then $\mu\rho_i(\overline{x}) \le \rho_{i+1}(\overline{x}) \le \lambda\rho_i(\overline{x})$.

It is easy to show that $\exists\, \overline{x}, \overline{y} \in \Omega$ such, that

$$\mu\rho_i(\overline{x}) = \rho_{i+1}(\overline{x}) \qquad \text{and} \qquad \rho_{i+1}(\overline{y}) = \lambda\rho_i(\overline{y}). \tag{25}$$

It can be shown that

$$\|\rho_{i+1} - \rho_i\|_{max} = \max\{1-\mu, \lambda\}, \tag{26}$$

where

$$\|\rho\|_{max} := \max_{\overline{x}\in\Omega}\{|\rho(\overline{x})|\}. \tag{27}$$

7. Numerical examples

In Fig. 1 the initial mesh for square is given. After 5 steps of remeshing for error indicator (11) the final mesh is shown in Fig. 2, thus for error indicator (13) the final mesh in presented in Fig. 4. In this case there are no singularities at corners, it was taken $\mu = 0.5$ and $\lambda = 1.0$. The sequence of meshes is convergent, but final mesh depends upon error indicator.

In Figs. 5–7 there are appropriately initial mesh for L-shaped domain, final mesh after 3 steps of adaptation and line contours for modulus of gradient of the solution, it was taken $\mu = 0.5$ and $\lambda = 0.8$.

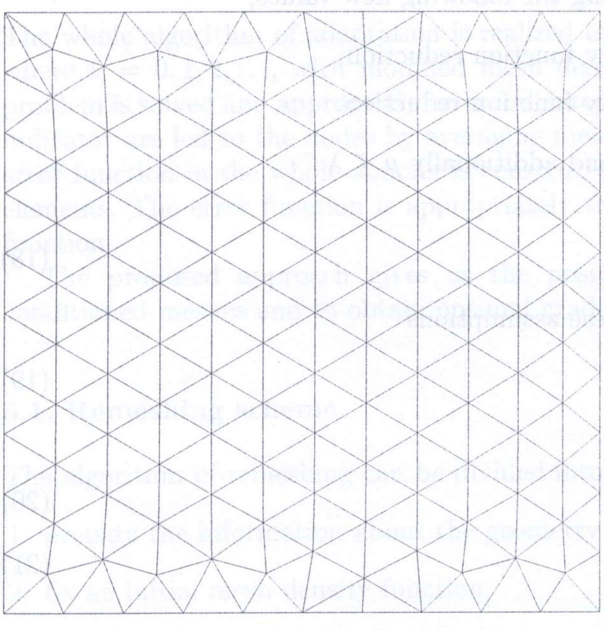

Fig. 1. Initial mesh over square generated with constant density

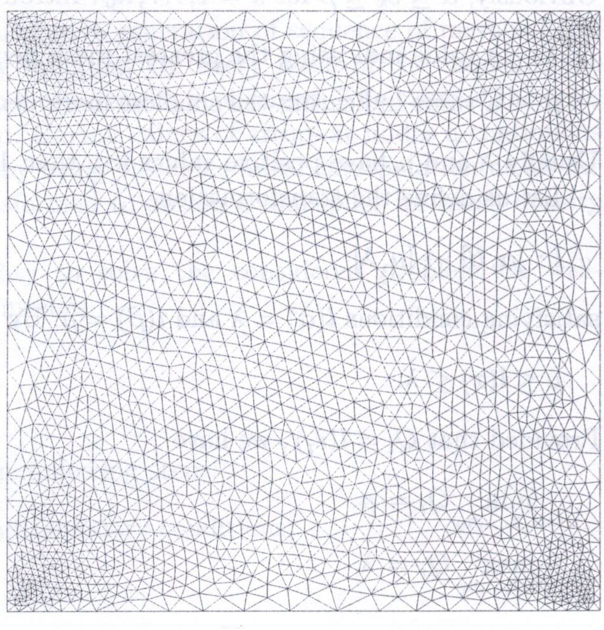

Fig. 2. Final mesh after 5 steps for error indicator given by Eq. (11)

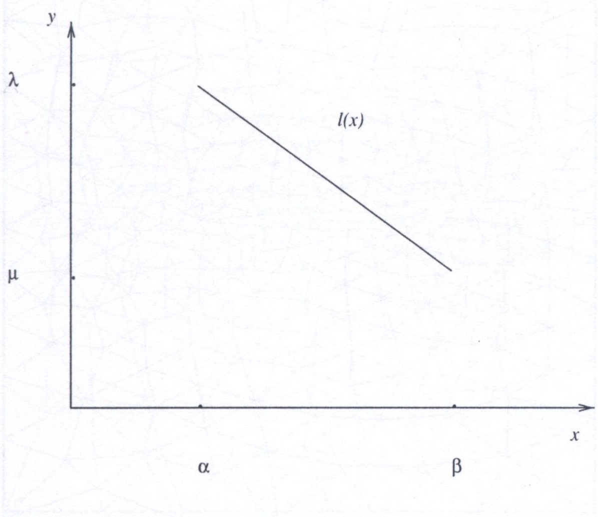

Fig. 3. The mapping $l(x)$

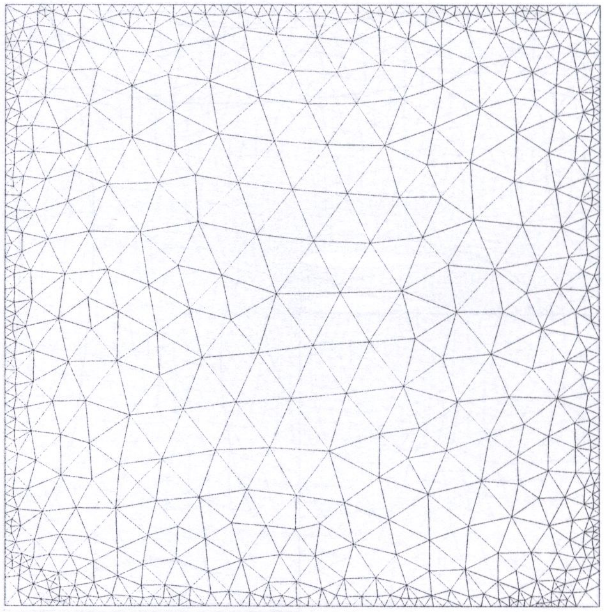

Fig. 4. Final mesh after 4 steps for error indicator (13)

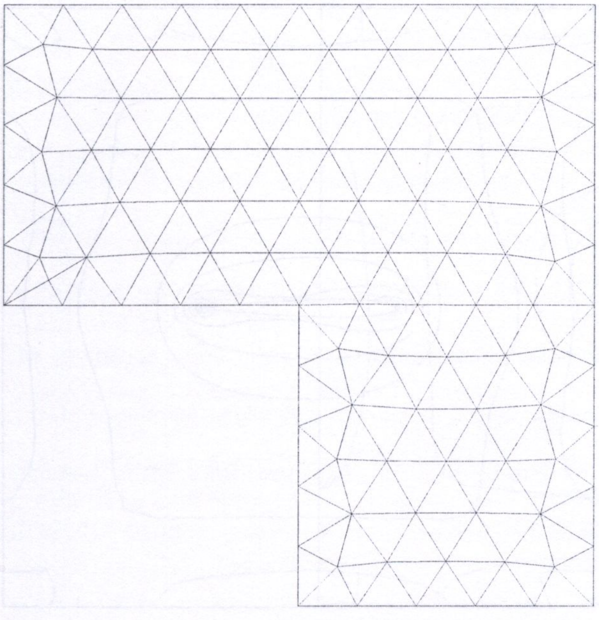

Fig. 5. Initial mesh for L-shaped domain

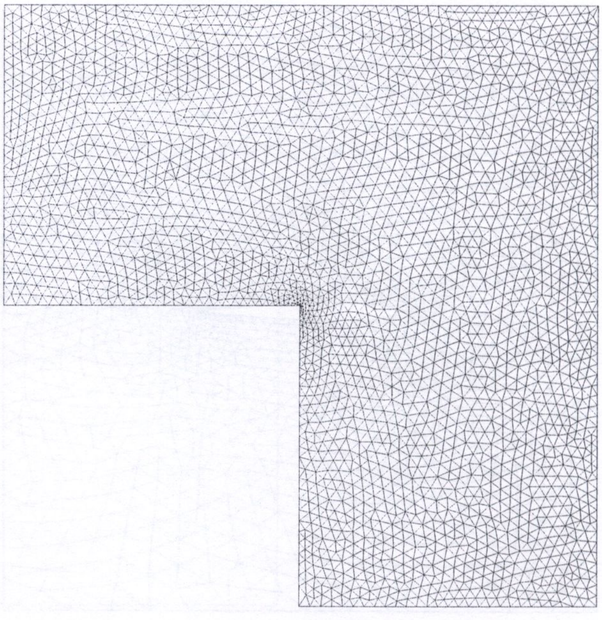

Fig. 6. Final mesh after 4 steps of adaptation

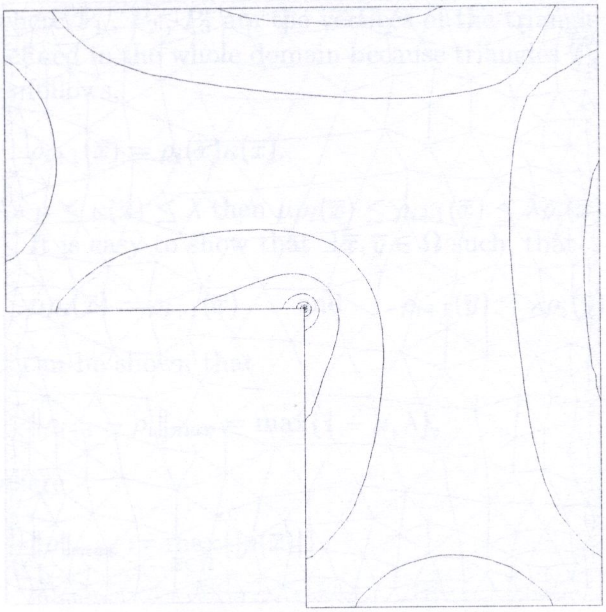

Fig. 7. Line contours for modulus of gradient

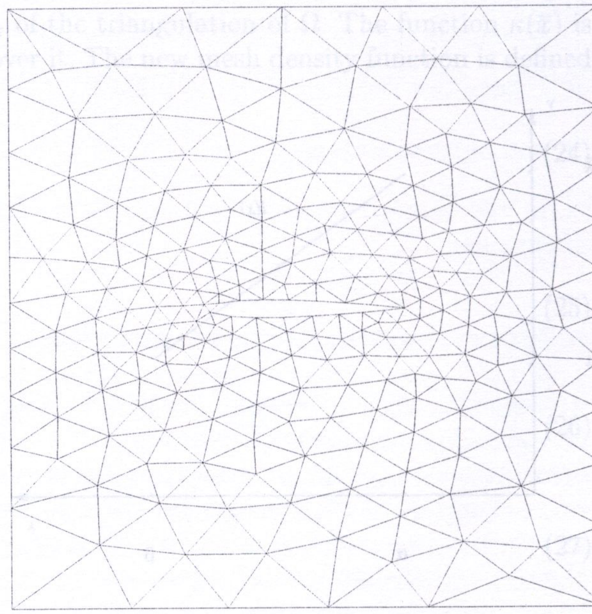

Fig. 8. Initial mesh

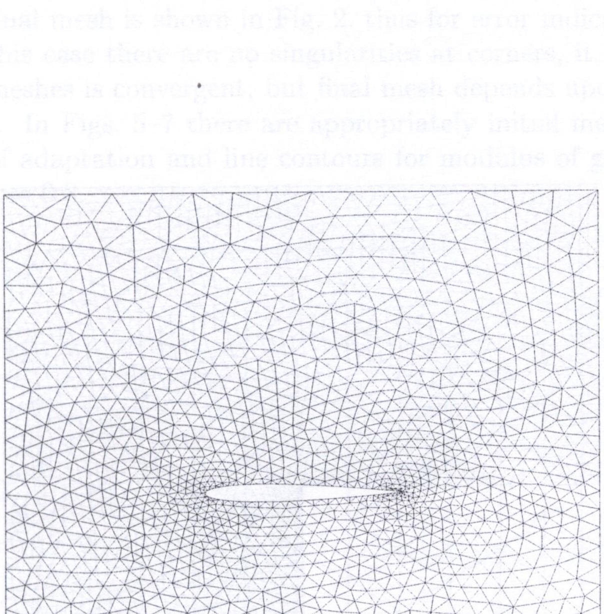

Fig. 9. Final mesh after 3 steps

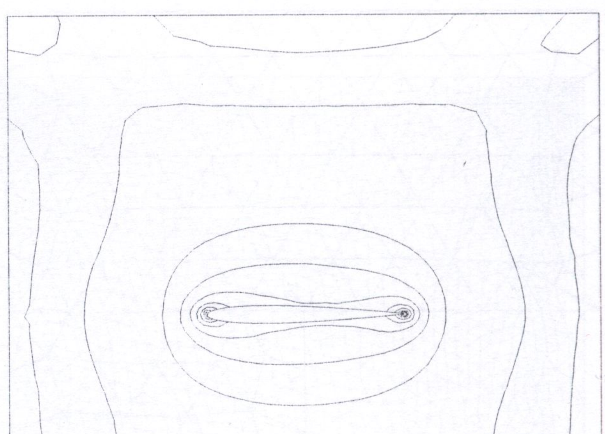

Fig. 10. Line contours for modulus of gradient

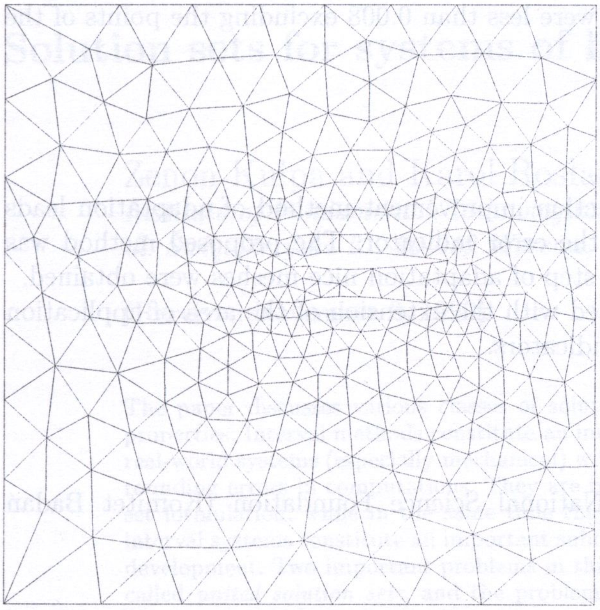

Fig. 11. Initial mesh

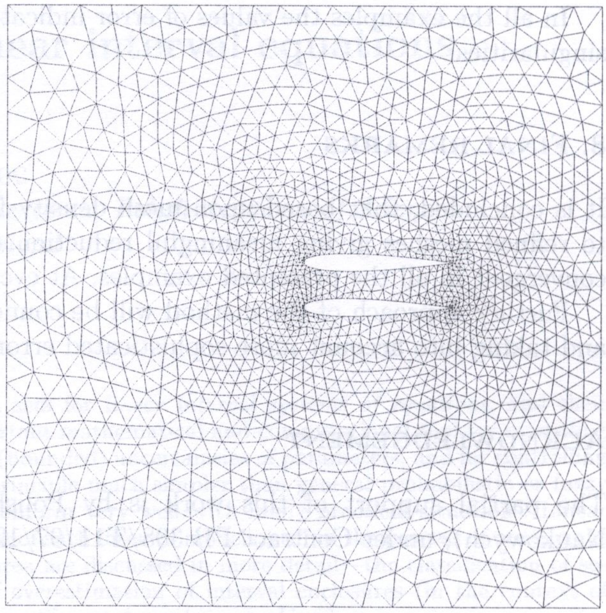

Fig. 12. Final mesh after 3 steps

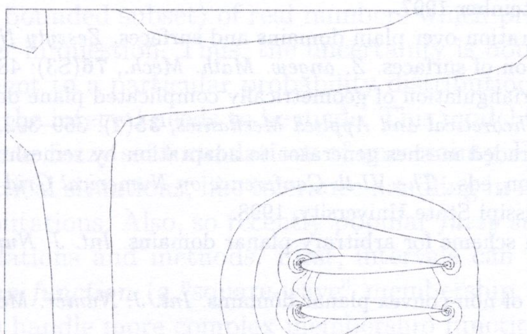

Fig. 13. Modulus of gradient of the solution after 3 steps

In Figs. 8–10 and 11–13 there are initial, final mesh after 3 steps of adaptation and line contours for modulus of gradient of the solution for shapes, where B-spline approximation of the boundary is used, with $\mu = 0.5$ and $\lambda = 1.0$.

In all the examples the values of error indicator were less than 0.008 excluding the points of the singularities.

8. Final remarks

It is concluded, that based upon mesh density function improvement method of adaptation leads to refined meshes giving meaningful reduction of the error indicator. The proposed method was convergent in all the presented examples. At every step of adaptation nice meshes were obtained.

Further approach of the method will be connected with the extension of the area of application and taking into account more sophisticated error indicators.

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References

[1] M.B. Bieterman, J.E. Bussoletti, C.L. Hilmes, F.T. Johnson, R.G. Melvin, D.P. Young. An adaptive grid method for analysis of 3D aircraft configuration. *Computer Methods in Applid Mechanics and Engineering*, **101**: 225–249, 1992.

[2] B. Delaunay. Sur la Sphere Vide. *Bulletin De L'Academie des Sciences de L'URSS, Classe des Sciences Mathematiques et Naturelles*, 793–800, 1934.

[3] J.T. Oden, L. Demkowicz, W. Rachowicz, T.A. Westermann. Towards a Universal h-p finite element strategy. *Comput. Methods Appl. Mech. Engrg.*, **77**: 113–180, 1989.

[4] J. Kucwaj. *Unstructured Grid Generation Package.* Cracow University of Technology, Applied Mathematics Section, Report No 1/1997, September 1997.

[5] J. Kucwaj. Automatic grid generation over plain domains and surfaces. *Zeszyty Naukowe P.K.*, **5**: 61–82, 1995.

[6] J. Kucwaj. Delaunay triangulation of surfaces. *Z. angew. Math. Mech.*, **76**(S3): 487–488, 1996.

[7] J. Kucwaj. The modelling and triangulation of geometrically complicated plane domains and some applications to fluid mechanics. *Journal of Theoretical and Applied Mechanics*, **35**(2): 369–392, 1997.

[8] J. Kucwaj. The application of graded meshes generator to adaptation by remeshing. In: N.M. Weatherill, P.R. Eisman, J. Häuser, J.F. Thompson, eds., *The VI-th Conference on Numerical Grid Generation in Computational Field Simulation*, 421–430. Mississipi State University, 1998.

[9] S.H. Lo. A new mesh generation scheme for arbitrary planar domains. *Int. J. Numer. Meth. Engrg.*, **21**: 1403–1426, 1985.

[10] S.H. Lo. Delaunay triangulation of non-convex planar domains. *Int. J. Numer. Meth. in Engrg.*, **28**: 2695–2707, 1989.

[11] S.H. Lo. Automatic mesh generation and adaptation by using contours. *Int. J. Numer. Meth. in Engrg.*, **31**: 689–707, 1991.

[12] S.H. Lo. Generation of high-quality gradation finite element mesh. *Engineering Fracture Mechanics*, **41**(2): 2695–2707, 1992.

[13] J.F. Thompson, B.K. Soni, N.P. Weatherwill. *Handbook of Grid Generation.* CRC Press, Boca Raton, London–New York–Washington D.C. 1999.

[14] O.C. Zienkiewicz, J.Z. Zhu. Adaptavity and mesh generation. *Int. J. Numer. Meth. in Engrg.*, **32**: 783–810, 1991.