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Application of a Hopfield type neural network to the analysis of elastic problems with unilateral constraints

Zenon Waszczyszyn and Ewa Pabisek

Institute of Computer Methods in Civil Engineering, Cracow University of Technology, ul. Warszawska 24, 31-155 Kraków, Poland

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On the base of Hopfield–Tank neural network the Panagiotopoulos approach is briefly discussed. The approach is associated with the analysis of quadratic programming problem with unilateral constraints. Then modifications of this approach are proposed. The original Panagiotopoulos approach is illustrated by the analysis of crack detachment in an elastic body [11]. Efficiency of the proposed modifications is shown on a numerical example of an angular plate. Finally some special conclusions are expressed.

1. Introduction

In recent years artificial neural networks (ANNs) have been successfully applied to the analysis of numerous problems of mechanics, cf. [13]. ANNs are a basis to formulate new algorithms also for analyzing problems related to mathematical programming. This especially concerns the Hopfield–Tank neural network [5], designed for the analysis of Quadratic Programming (QP) problems. In this type of neural network Hopfield's ideas were generalized on continuous variables and founded a theoretical background in theory and methods of solution of ordinary differential equations.

The above mentioned ideas were developed by P.G. Panagiotopoulos and his associates for the analysis of many problems of solid mechanics, related especially to fracture mechanics and plasticity [13]. In papers [7, 11] it was proved that QP problems with unilateral constraints can be solved by the HT analogues with a corresponding amplifier (activation function). In this way the Panagiotopoulos approach was proposed, i.e. the mechanical problem was formulated as a QP problem with unilateral constraints

$$\min\left\{\frac{1}{2}\mathbf{Q}^T\mathbf{K}\mathbf{Q} - \mathbf{P}^T\mathbf{Q} \mid \mathbf{Q} \geq \mathbf{0}\right\} \tag{1}$$

and solved (1) by means of HT network.

The Panagiotopoulos approach opens the door to the efficient analysis of many direct and inverse problems of mechanics, cf. [10], but the algorithms sketched by him, need more precise development. This paper proposes some improvements of the Panagiotopoulos algorithm.

Similarly as in [7, 11] we omit the storage phase of the HT network, taking the stiffness matrix $\mathbf{Q}$ and the vector of nodal forces $\mathbf{P}$ in (1) from the linear FE method. The numerical analysis is restricted to the plane stress problem. In the frame of these assumptions it is shown that besides unilateral constraints a non-incremental FE analysis is performed. This is possible due to change of algebraic boundary-value FE problem to an initial-value problem associated with the evolutionary equations of the HT analogue.

2. Hopfield type neural networks

In Fig. 1a scheme of the Hopfield neural network is shown with notation corresponding to that in the papers by Panagiotopoulos. The network is of recurrent type and we are interested only in the retrieval phase of the network operation. This means that all weights T_{ij} and biases I_i are known for $i, j = 1, \ldots, N$. The iteration process of computation of variables $V_i(t)$ is initiated by introduction of initial values $V_i(0)$. The iteration process is then continued recurrently due to feed back connections shown in Fig. 1.

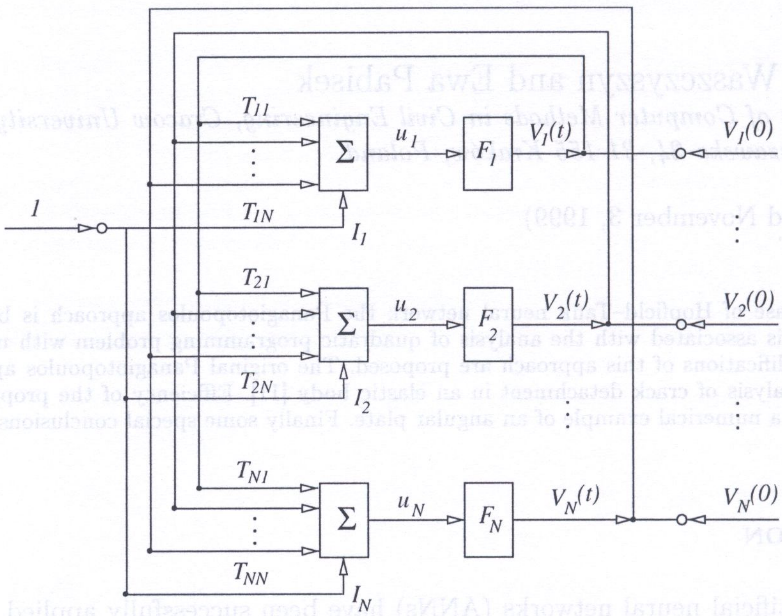


Fig. 1. Scheme of Hopfield neural network

The first Hopfield networks were formulated for discrete (binary or bipolar) variables, cf. [3]. Then the Hopfield–Tank network was designed for continuous variables [5] and continuous, real time type independent variable t. This approach corresponds to the circuit analogue of the HT network in which I_i and $V_i(t)$ are currents and voltages [4, 9].

In order to assure stable convergence of the iteration process the self feedback of neurons is usually cancelled and symmetry of weight matrix is assumed [4],

$$T_{ii} = 0, \quad T_{ij} = T_{ji} \quad \text{for } i, j = 1, \ldots, N. \tag{2}$$

The network dynamics can be described by means of mechanical type variables. Hopfield defined the network energy function $E(t)$ which in case of the HT network takes the form

$$E = -\frac{1}{2} \sum_i \sum_j T_{ij} V_i(t) V_j(t) + \sum_i \frac{1}{R_i} \int_0^{V_i(t)} F_i^{-1}(V_i)\, dV - \sum_i I_i V_i(t), \tag{3}$$

where summing up is for $i, j = 1, \ldots, N$ and R_i are resistance type parameters and F_i are activation functions corresponding to the following formulae, cf. Fig. 1,

$$V_i(t) = F_i(u_i(t)), \qquad u_i = \sum_j T_{ij} V_j(t) + I_i. \tag{4}$$

The equilibrium solutions can be computed each time by means of the evolutionary equations [4, 6]

$$C_i \left(\frac{\mathrm{d}u_i}{\mathrm{d}t} \right) = -\frac{\partial E}{\partial V_i}\bigg|_t, \tag{5}$$

where C_i are capacity type parameters. After substitution (3) to (5) the following form of evolutionary equations is obtained,

$$C_i \frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{1}{R_i}\, u_i(t) + \sum_j T_{ij}\, V_j(t) + I_i\,, \tag{6}$$

$$V_j(t) \;\; = F_j(u_j(t)) \qquad \text{for} \quad i = 1, \ldots, N\,. \tag{7}$$

The iteration process ends at the stable, equilibrium state of the network. This state is defined by the following criterion,

$$\min{}_t E(t) \quad \Longleftrightarrow \quad \frac{\mathrm{d}u_i}{\mathrm{d}t} = 0\,, \quad \forall_i\,. \tag{8}$$

In computations time t is discretized and criterion (7) takes the form

$$\Delta E = 0 \quad \Longleftrightarrow \quad u_i(t_{s+1}) \approx u_i(t_s)\,, \quad \forall_i\,, \tag{9}$$

where s is the number of iteration step.

3. Panagiotopoulos approach

In papers [7, 11] the following substitutions were proposed

$$T_{ij} = \begin{cases} -K_{ij} & \text{for} \quad i \neq j\;\;, \\ -K_{ij} + \dfrac{1}{R_i} & \text{for} \quad i = j\;\;, \end{cases} \tag{10}$$
$$V_i = Q_i\,, \qquad I_i = P_i \qquad \text{for} \quad i,j = 1, \ldots, N\,.$$

Relations (9) were completed by two assumptions:

1. $C_i = 1\,, \quad R_i = 1\,,$
2. selection of activation functions:
   - identity function for bilateral constraints

$$V_s = Q_s = u_s\,, \tag{11a}$$

   - bilinear function for unilateral constraints

$$V_r = Q_r = F_r(u_r) = \begin{cases} u_r & \text{for} \;\; u_r > 0\,, \\ 0 & \text{for} \;\; u_r \leq 0\,. \end{cases} \tag{11b}$$

After the above substitutions and assumptions are taken into account evolutionary Eq. (6) takes the form

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\sum_j K_{ij}\, Q_j + P_i\,, \tag{12}$$

where u_i depends on type of constraints:

- bilateral constraints

$$Q_s = u_s\,, \tag{13a}$$

- unilateral constraints

$$Q = \begin{cases} u_r & \text{for} \;\; u_r > 0 \\ 0 & \text{for} \;\; u_r \leq 0\,. \end{cases} \tag{13b}$$

It was proved in [7, 11] that the iteration process is convergent for both the bilateral constraints (13a) and unilateral constraints (13b). Condition (7) implies the equilibrium state of the FE system

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{R} \equiv -\mathbf{K}\mathbf{Q} + \mathbf{P} = \mathbf{0}\,, \tag{14}$$

where $\mathbf{R} \in \mathcal{R}^N$ is the vector of residual forces.

The Panagiotopoulos approach described above is in fact a change of the algebraic boundary-value problem (14) into an initial value problem, corresponding to the set of ordinary differential equations (12). These equations describe the retrieval phase of the HT network operation. The storage phase of the network, cf. [4], is associated with correspondence of the network parameters T_{ij} and I_i to values of the FE stiffnesses K_{ij} and nodal forces P_i. These values can be computed by a FE code, satisfying boundary conditions related to the bilateral constraints.

It was shown in [7, 11] that the speed of iteration process increases if the unilateral constraints are active. That is why it is worthwhile to eliminate bilateral degrees of freedom. In what follows a condensed FE system is considered and instead of Eqs. (12) the following equations are analyzed,

$$\frac{\mathrm{d}u_l}{\mathrm{d}t} = r_l \equiv -\sum_j k_{lj}\, q_j + p_l \qquad \text{for} \quad j,\, l = 1, \ldots, n\,, \tag{15}$$

where: k_{lj}, p_l – parameters of a FE condensed system of $n < N$ degrees of freedom, $q_j \in \mathcal{R}^n$ – nodal displacement corresponding to a great number of unilateral DOFs.

4. Improvements of Panagiotopoulos approach

Equations (15) correspond to the simplest HT analogue for the solution of the system of algebraic equation. This is related to the steepest descent gradient method, cf. [9]. It is much more efficient to apply the conjugate gradient method which leads to the following equations

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\mathbf{r}_t + \beta_t\, \mathbf{r}_{t-1} \tag{16}$$

for

$$\mathbf{r}_t = \mathbf{k}\,\mathbf{q}_t - \mathbf{p}\,, \qquad \beta_t = \frac{\mathbf{r}_t^T\,\mathbf{r}_t}{\mathbf{r}_{t-1}^T\,\mathbf{r}_{t-1}}\,, \tag{17}$$

where: $\mathbf{u}, \mathbf{r}_t, \mathbf{r}_{t-1} \in \mathcal{R}^n$.

The next improvement corresponds to a more general formulation of unilateral constraints,

$$q_l = \begin{cases} u_l & \text{for} \ \ (\mathbf{u}^T\mathbf{n})_l \le a_l\,, \\ a_l & \text{for} \ \ (\mathbf{u}^T\mathbf{n})_l > a_l\,, \end{cases} \tag{18}$$

where $\mathbf{n}_s = \{n_n, n_t\}_s$ – vector normal to boundary of body at the node with normal displacement q_l.

Equations (15) were analyzed in [7, 11] by means of the 4-th order Runge–Kutta method. In our computations we have used the Runge–Kutta–Fehlberg formulae of the 5-th and 6-th order with automatic step control, cf. [2]. This approach seems to be superior for the analysis of linear elastic problems, i.e. for matrices $\mathbf{k}$ and $\mathbf{p}$ independent of iteration time t.

5. Numerical analysis

5.1. Crack detachment in elastic body

The analysis of crack detachment was carried out in [11]. This problem is discussed briefly below as an example of application of the original Panagiotopoulos approach, described in Section 3.

In this example all the bilateral degrees of freedom (node displacements) were eliminated and stiffness matrix $\mathbf{K}$ was condensed to matrix $\mathbf{k}$ of dimension (36×36). This corresponds to 18 pairs of the contact (unilateral) nodes of the crack interface, cf. Fig. 2b. Equations (15) were numerically integrated by the 4-th order Runge–Kutta method assuming $\Delta t = 0.5$. The stable results $q_i(s+1) = q_i(s)$ for $i = 1, \dots 36$, were achieved after 250 steps for several random choices of the initial values $q_i(0)$. Displacement and major stress fields, corresponding to $s = 250$ are shown in Figs. 2c,d.

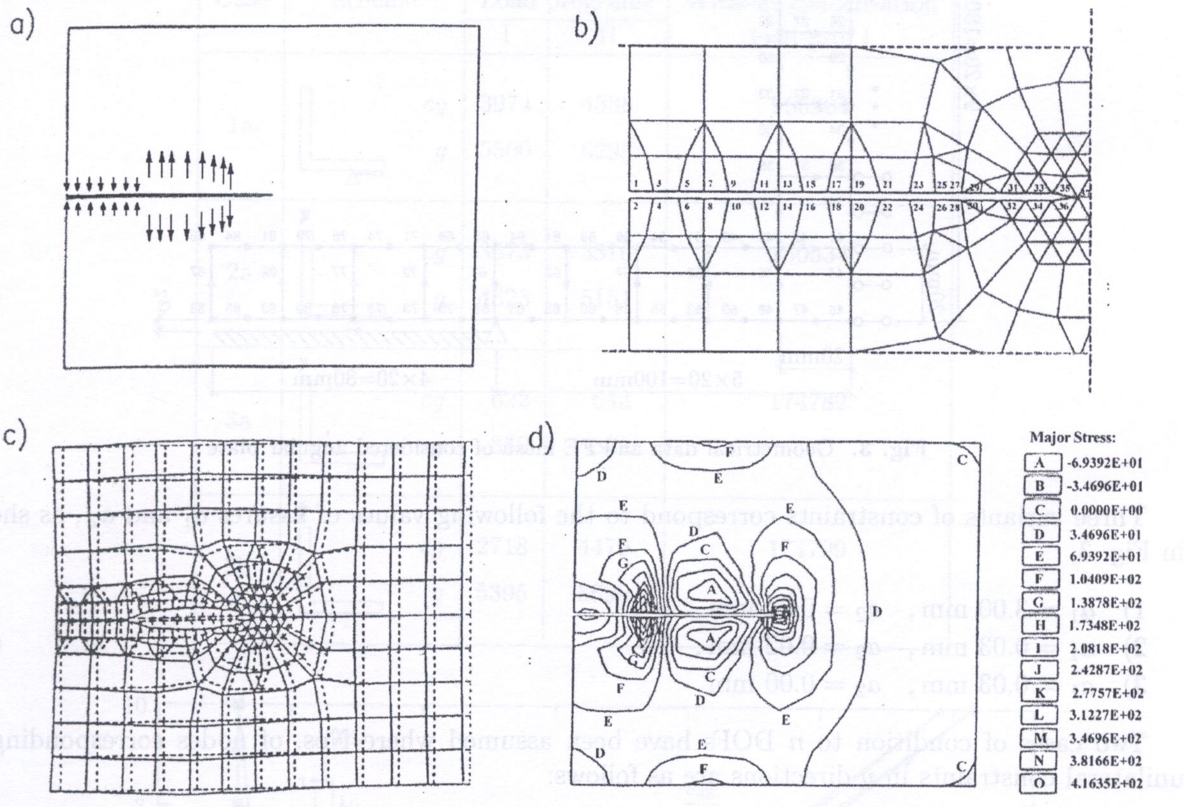


Fig. 2. a) Geometry and loads of elastic body, b) FE mesh in the crack vicinity, c,d) displacement and major stress fields

5.2. Angle plate in plane stress state

When considering the analysis of an angle plate problem we have made an extended numerical analysis in order to examine the improvements to Panagiotopoulos' approach as well as discuss details of numerical analysis which were not mentioned in papers [7, 11].

Two load programs have been considered, corresponding to the increase and then decrease of the load parameter within 11 load increments:

$$\begin{aligned} &\text{I)} \quad 0 \rightarrow \lambda_1 \longrightarrow 10 * (-\lambda_1/10)\,, \\ &\text{II)} \quad 0 \longrightarrow (\lambda_1/10) \longrightarrow (-\lambda_1)\,. \end{aligned} \tag{19}$$

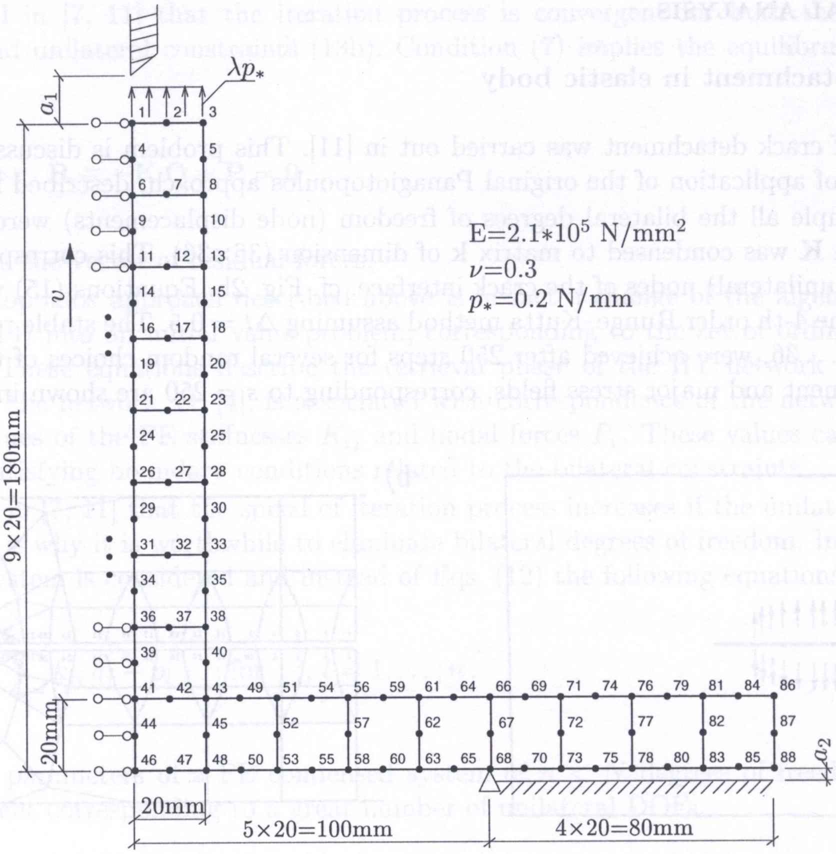


Fig. 3. Geometrical data and FE mesh of considered angular plate

Three variants of constraints correspond to the following values of fissures a_1 and a_2, as shown in Fig. 3,

$$\begin{aligned} &1)\quad a_1 = 3.00 \text{ mm}, \quad a_2 = 0.02 \text{ mm}, \\ &2)\quad a_1 = 0.03 \text{ mm}, \quad a_2 = 0.02 \text{ mm}, \\ &3)\quad a_1 = 0.03 \text{ mm}, \quad a_2 = 0.00 \text{ mm}. \end{aligned} \tag{20}$$

Two cases of condition to n DOFs have been assumed where Nos. of nodes corresponding to unilateral constraints in v directions are as follows:

$$\begin{aligned} &\text{a)}\quad n = 9, \quad \text{Nos} = 1, 70, \ldots, 88, \\ &\text{b)}\quad n = 20, \quad \text{Nos} = 1, 46, 47, \ldots, 88. \end{aligned} \tag{21}$$

In case b) the constraints can be active at node 1 and at all lower boundaries of the plate (cf. Nos. of FE nodes in Fig. 3). In case a) the constraints can act at node 1 and the part 68–88 of the lower boundary.

The iteration procedure has been continued until the following error condition was fulfilled,

$$\max_i \{ | u_i(s) - u_i(s-1) | , \; v_i(s) - v_i(s-1) | \} < \varepsilon . \tag{22}$$

The value of admissible error ε was estimated numerically to have closed loops of displacement trajectories for all nodes i after the load programs (19) were computed. In the considered plate it was evaluated to be $\varepsilon = 1 \cdot 10^{-9}$.

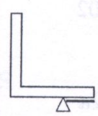
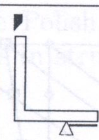
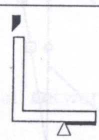
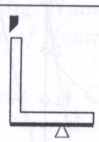
The computations were performed by a special computer program based on the ANKA code [14], using 8-node isoparametric, plane stress FEs. The iteration process started from the initial values $u_i(0) = 0$ for $i = 1, \ldots, n$.

In Table 1 the number of iterations is put together for different cases associated with selected combinations of (20) and (21). Symbols *cg* and *g* correspond to the conjugate-gradient and gradient formulae (16) and (15), respectively.

It is evident that the conditions of bilateral DOFs and application of *cg* method significantly decrease iteration. The same concerns the influence of the unilateral constraints, i.e. the bigger number of those constraints the lower the number of iterations. These conclusions fully agree with those in [7, 11].

The equilibrium paths for selected nodes of the plate are shown in subsequent Figs. 4–6. The figures correspond to cases listed in Table 1. Graphics for cases 4a and 4b are the same since the unilateral constraints do not occur at nodes 46–65.

Table 1. Number of iterations for different cases

Case	Scheme		Load programs I	Load programs II	Without condensation load pro. I
1a		*cg*	3974	4538	966904
		g	5500	6298	
2a		*cg*	3573	3816	160534
		g	4823	5151	
3a		*cg*	622	643	174782
		g	850	871	
3b		*cg*	2718	4479	174799
		g	5395	5690	

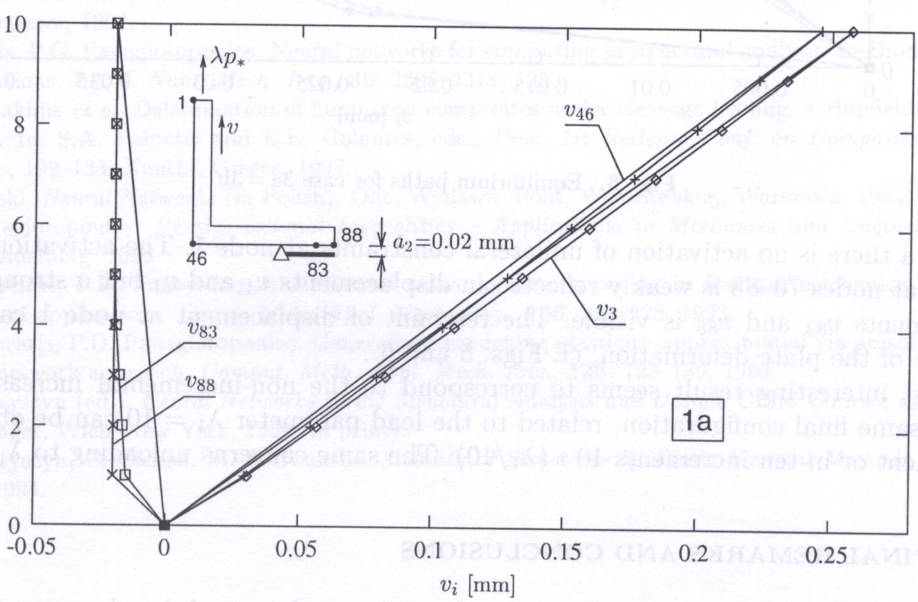


Fig. 4. Equilibrium paths for case 1a

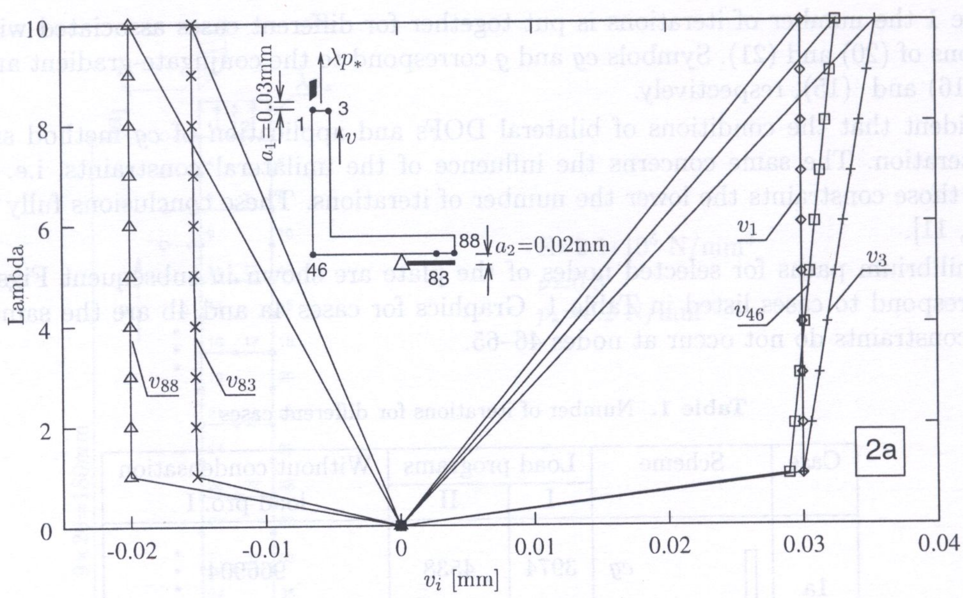


Fig. 5. Equilibrium paths for case 2a

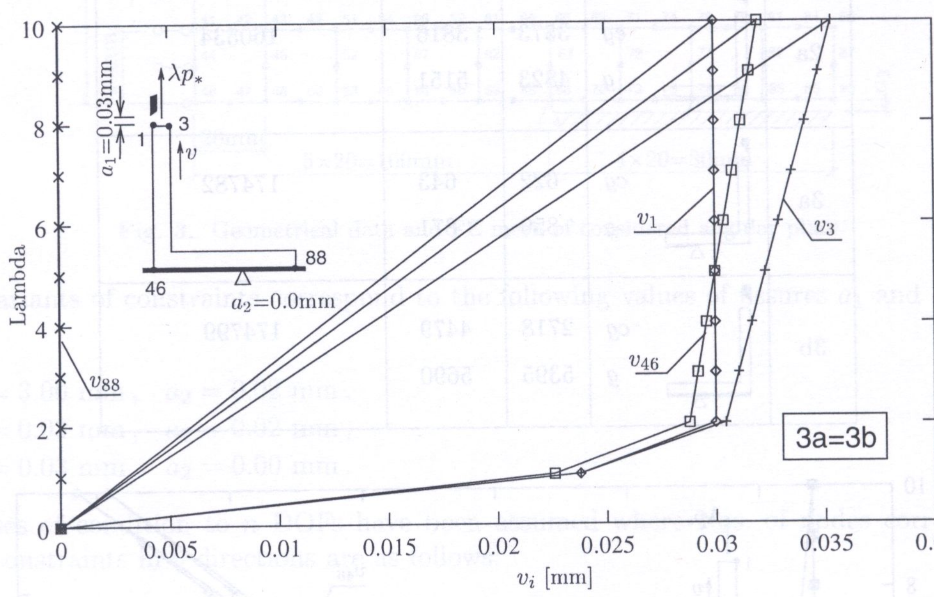


Fig. 6. Equilibrium paths for case 3a = 3b

In case 1a there is no activation of unilateral constraints at node 1. The activation of unilateral constraints at nodes 70–88 is weakly reflected in displacements v_1 and v_3 but a strong nonlinearity of displacements v_{83} and v_{88} is visible. The restraint of displacement at node 1 causes a strong nonlinearity of the plate deformation, cf. Figs. 5 and 6.

The most interesting result seems to correspond to the non-incremental increase/decrease of loads. The same final configuration, related to the load parameter $\lambda_1 = 10$ can be obtained in one load increment or in ten increments $10 * (\lambda_1/10)$. The same concerns unloading to $\lambda_1 = 0$.

6. Some final remarks and conclusions

A number of conclusions from the papers by Panagiotopoulos *et al.* has been confirmed in this paper. The main conclusions are:

i. The higher the number of unilateral constraints the lower the number of iterations is needed to find a stable solution of evolutionary equations;

ii. Condensation of bilateral DOFs increases significantly the efficiency of iteration algorithm;

iii. In case of linear state equations the neural analogue enables us to achieve the top value of load parameter in one load step (without load incremental steps), i.e. the iteration process adopts itself to the activation of unilateral constraints;

iv. A change of simplest gradient algorithm into the conjugate gradient algorithm shortens the iteration process of about 1.27–1.98 times.

The Panagiotopoulos approach was used successfully also to the analysis of elastoplastic structures [1], theory of plasticity [12] and delamination problems [8]. In all the papers by Panagiotopoulos and his associates it was stated that the HT analogue application is much superior to the application of various mathematical programming algorithms, especially for problems with variational inequalities [10].

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