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A new computational method for structural vibrations in the medium-frequency range[1]

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In the paper a new approach for the computation of slightly damped elastic structural vibrations over the medium frequency range is proposed. The effective quantities (deformation energy, vibrational intensity, etc...) are evaluated after resolution of a small system of equations that does not in any way result from a fine "finite element" discretisation of the structure.

1. Introduction

The modelling and computation of the vibrational response of elastic structures represents, without a doubt, one of the key issues raised with respect to the design of structures. The low-frequency range no longer poses major difficulties, even for complex structures, at least for the modelling and computation (see Fig. 1).

Concerning high frequencies, computational tools do exist which are quite distinct for those utilized for low frequencies, and in particular the SEA method, where the spatial aspect disappear almost entirely (see Figs. 2, 3 and references [7, 8]).

In contrast, the modelling and computation of "medium frequency" vibration, the focus of this paper, still raise certain problems. Structure shape appear; the difficulty lies in the length of variation of the phenomena under study, which remains very small in comparison with the structure characteristic dimension. It would follow that, by extending the "low frequency" methods, the finite element computation to be conducted would require an unreasonable number of degrees of freedom, beyond the serious numerical obstacles presented. Difficulty would also be experienced when extending the SEA method appropriate for high frequencies (see Fig. 4).

Ohayon [10] provide an assessment of the primary paths which have already been explored. With the exception of the theory introduced by Belov et al. [1] and Buvailov–Ionov [2], these methods are in fact not true "medium frequency" methods, as the phenomena with small-length variations, which detail is not highly significant, do nevertheless remain present. Put in another way, these methods

Degrees of freedom:	172122
Eigenmodes:	216
Memory required:	440 Mb
Disk space required:	1.3 Gb
Total Time:	1.5 h CPU

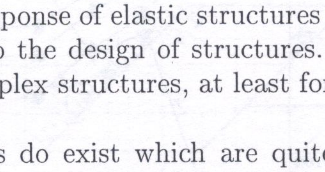

Fig. 1. Standard finite element calculation for frequency extraction (using a high performance, single CPU station)

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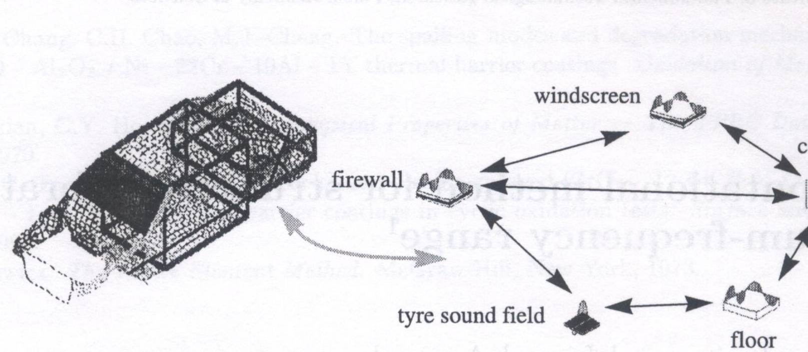

Fig. 2. Part of a SEA model of a car

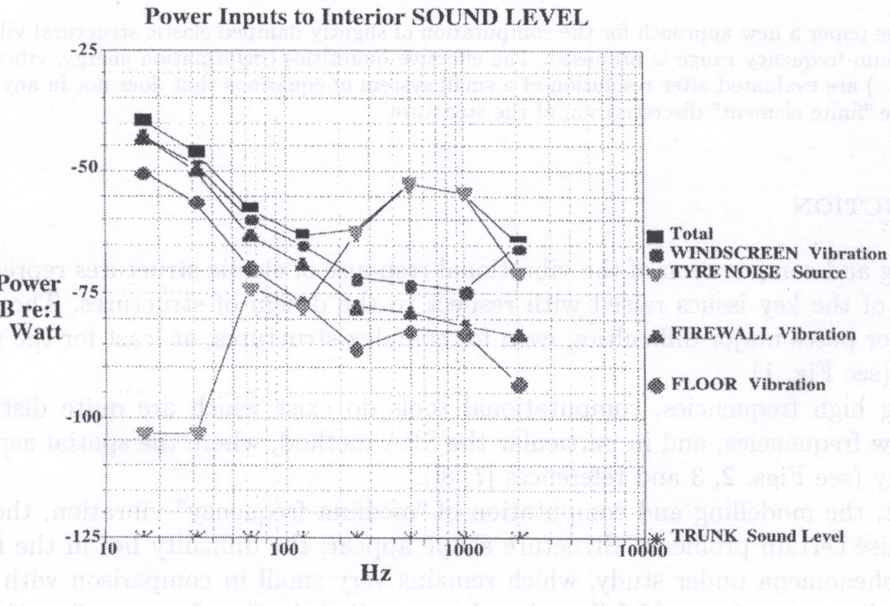

Fig. 3. Global results given by SEA model

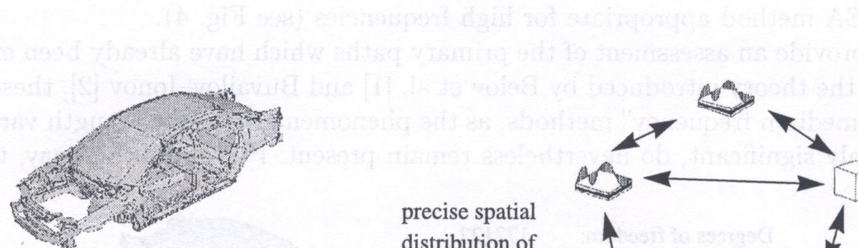

Fig. 4. Extending low frequency and high frequency methods

do not strictly involve the "effective" quantities for the time and space scales under consideration and therefore give results very sensitive to data errors. The theory initiated by Belov et al. [1] and Buvailov–Ionov [2] is built upon the notion of "effective energy density" and of "effective vibration energy". This heuristic theory is extremely attractive; however, despite the improvements forwarded, notably in France by Jézéquel [5], Guyader [3] and their respective teams, this theory still contains some obstacles that are difficult to circumvent.

The approach described here is thus a true "medium-frequency" method, proposed by Ladevèze [6]. In order to simplify its presentation, we shall consider only the basic problem, which consists in computing forced vibrations for a given frequency. The main limitation is that the structure must be decomposable into a relatively small number of homogeneous structures.

Here we described the basic features of this computational method and the first numerical examples.

2. NEW APPROACH

2.1. Referring problem

We consider here only the forced harmonic flexural vibration of thin isotropic Kichhoff–Love plates (see Fig. 5, Eqs. (1)–(5)).

Find:

$$\left(w_{(\underline{X})}\,,\,\mathbb{M}_{(\underline{X})}\right) \in \mathbf{U}\times\mathbf{S} \qquad \text{with } \underline{X}\in\Sigma$$

$$\left(w'_{(\underline{X})}\,,\,\mathbb{M}'_{(\underline{X})}\right) \in \mathbf{U}'\times\mathbf{S}' \qquad \text{with } \underline{X}\in\Sigma'$$

verifying Eqs. (1)–(5).

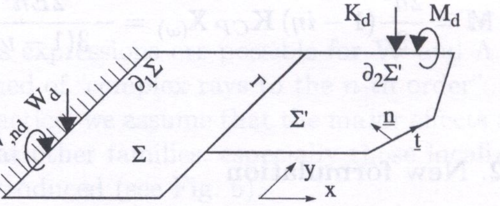

Fig. 5. Referring problem

Equilibrium equations (on $\partial_2\Sigma$ and $\partial_2\Sigma'$):

$$\begin{cases} \underline{n}\,\underline{\mathrm{div}}\,[\mathbb{M}] + (\underline{t}\,\mathbb{M}\,\underline{n})_{,t} = -K_d & \text{on } \partial_2\Sigma \\ \underline{n}'\,\underline{\mathrm{div}}\,[\mathbb{M}'] + (\underline{t}'\,\mathbb{M}'\,\underline{n}')_{,t'} = -K'_d & \text{on } \partial_2\Sigma' \\ \underline{n}\,\mathbb{M}\,\underline{n} = \mathbb{M}_d & \text{on } \partial_2\Sigma \\ \underline{n}'\,\mathbb{M}'\,\underline{n}' = \mathbb{M}'_d & \text{on } \partial_2\Sigma' \\ [\![\underline{t}\,\mathbb{M}\,\underline{n}]\!] = 0 & \text{at angular points of } \partial_2\Sigma \\ [\![\underline{t}'\,\mathbb{M}'\,\underline{n}']\!] = 0 & \text{at angular points of } \partial_2\Sigma' \end{cases} \tag{1}$$

Equilibrium equations (on Σ and Σ'):

$$\begin{cases} \mathrm{div}\,[\underline{\mathrm{div}}\,[\mathbb{M}]] = -2\rho h\,\omega^2 w & \text{on } \Sigma \\ \mathrm{div}\,[\underline{\mathrm{div}}\,[\mathbb{M}']] = -2\rho' h'\,\omega^2 w' & \text{on } \Sigma' \end{cases} \tag{2}$$

Material behaviour:

$$\begin{cases} \mathbb{M} = \dfrac{2h^3}{3}(1+i\eta)\,\mathbf{K}_{CP}\,\mathbb{X}_{(\omega)} & \text{on } \Sigma \\ \mathbb{M}' = \dfrac{2h'^3}{3}(1+i\eta')\,\mathbf{K}'_{CP}\,\mathbb{X}_{(\omega')} & \text{on } \Sigma' \end{cases} \tag{3}$$

Usual boundary conditions:

$$\begin{cases} w = w_d & \text{on } \partial_1\Sigma \\ w_{,n} = w_{,nd} & \text{on } \partial_1\Sigma \\ w' = w'_d & \text{on } \partial_1\Sigma' \\ w'_{,n} = w'_{nd} & \text{on } \partial_1\Sigma' \end{cases} \tag{4}$$

Transmission conditions:

$$\begin{cases} w = w' & \text{on } \Gamma \\ w_{,n} + w'_{,n'} = 0 & \text{on } \Gamma \\ \underline{n}\,\underline{\operatorname{div}}\,[\mathbb{M}] + (\underline{t}\,\mathbb{M}\underline{n})_{,t} + \underline{n}'\,\underline{\operatorname{div}}\,[\mathbb{M}'] + (\underline{t}'\,\mathbb{M}'\,\underline{n}')_{,t'} = 0 & \text{on } \Gamma \\ \underline{n}\,\mathbb{M}\underline{n} + \underline{n}'\,\mathbb{M}'\,\underline{n}' = 0 & \text{on } \Gamma \end{cases} \tag{5}$$

Nomenclature:

$w_{(\underline{X})}$:	displacement of the plate
E :	Young's modulus
ν :	Poisso's ratio
η	damping coefficient
$2h$:	thickness of the plate

$$\mathbb{M} = \frac{2h^3}{3}(1 - i\eta)\,\mathbf{K}_{CP}\,\mathbb{X}_{(\omega)} = -\frac{2Eh^3}{3(1-\nu^2)}\begin{bmatrix} \left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right) & (1-\nu)\,\frac{\partial^2 w}{\partial x \partial y} \\ (1-\nu)\,\frac{\partial^2 w}{\partial x \partial y} & \left(\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}\right) \end{bmatrix}. \tag{6}$$

2.2. New formulation

The new formulation (Eqs. (7)–(10)) is strictly equivalent to the referring problem (Eqs. (1)–(5)):

$$(w, \mathbb{M}) \in \mathbf{S}_{ad} \quad \text{verifying} \quad \mathbf{a}\left(\begin{pmatrix} \delta w \\ \delta\mathbb{M} \\ \delta w' \\ \delta\mathbb{M}' \end{pmatrix}, \begin{pmatrix} w \\ \mathbb{M} \\ w' \\ \mathbb{M}' \end{pmatrix}\right) = \mathbf{L}\begin{pmatrix} \delta w \\ \delta\mathbb{M} \\ \delta w' \\ \delta\mathbb{M}' \end{pmatrix}, \quad \forall \begin{pmatrix} \delta w \\ \delta\mathbb{M} \\ \delta w' \\ \delta\mathbb{M}' \end{pmatrix} \in \mathbf{S}^0_{ad}, \tag{7}$$

i.e., in extended form,

$$(\text{let:} \quad -K_n = \underline{n}\,\underline{\operatorname{div}}\,[\mathbb{M}] + (\underline{t}\,\mathbb{M}\underline{n})_{,t} \quad \text{and} \quad -K'_n = \underline{n}'\,\underline{\operatorname{div}}\,[\mathbb{M}'] + (\underline{t}'\,\mathbb{M}'\,\underline{n}')_{,t'}) \tag{8}$$

$$\begin{aligned} \operatorname{Re}\Bigg\{ i\omega \Bigg[& \int_{\partial_1\Sigma} [\delta\underline{n}\,\mathbb{M}\underline{n}(w_{,n} - w_{,nd})^* - \delta K_n(w - w_d)^*]\,\mathrm{d}L \\ & + \int_{\partial_2\Sigma} [(\underline{n}\,\mathbb{M}\underline{n} - M_d)\delta w^*_{,n} - (K_n - K_d)\delta w^*]\,\mathrm{d}L \\ & + \int_{\partial_1\Sigma'} [\delta\underline{n}'\,\mathbb{M}'\,\underline{n}'(w'_{,n'} - w'_{,n'd})^* - \delta K'_{n'}(w' - w'_d)^*]\,\mathrm{d}L \\ & + \int_{\partial_2\Sigma'} [(\underline{n}'\,\mathbb{M}'\,\underline{n}' - M'_d)\delta w'^*_{,n'} - (K'_{n'} - K'_d)\delta w'^*]\,\mathrm{d}L \\ & - \sum_{\substack{\text{angular points} \\ \text{of } \partial_2\Sigma}} [\![\underline{t}\,\mathbb{M}\underline{n}]\!]\,(\delta w^*) - \sum_{\substack{\text{angular points} \\ \text{of } \partial_2\Sigma'}} [\![\underline{t}'\,\mathbb{M}'\,\underline{n}']\!]\,(\delta w'^*) \\ & + \int_\Gamma \frac{1}{2}\,[(\underline{n}\,\mathbb{M}\underline{n} - \underline{n}'\,\mathbb{M}'\,\underline{n}')\,\delta(w_{,n} - w'_{,n'}) - (K_n + K'_{n'})\,\delta(w + w')^*]\,\mathrm{d}S \Bigg]\Bigg\} = 0 \end{aligned} \tag{9}$$

with $\mathbf{S}_{ad}$ and $\mathbf{S}'_{ad}$ that define $(w, \mathbb{M})$ and $(w', \mathbb{M}')$ such as

$$\begin{cases} \mathbb{M} = \dfrac{2h^3}{3}(1+i\eta)\,\mathbf{K}_{CP}\,\mathbb{X}_{(\omega)}\,, \\ \Delta\Delta w \dfrac{E(1+i\eta)h^3}{3(1-\nu^2)} = \rho h \omega^2 w, \end{cases} \quad \text{and} \quad \begin{cases} \mathbb{M}' = \dfrac{2h'^3}{3}(1+i\eta')\,\mathbf{K}'_{CP}\,\mathbb{X}_{(\omega')} \\ \Delta\Delta w' \dfrac{E(1+i\eta')h'^3}{3(1-\nu^2)} = \rho' h' \omega^2 w'. \end{cases} \tag{10}$$

The initial element serving to characterize this approach is the utilization of a new variational formulation of the problem to be resolved, which we have developed so as to authorize a priori independent approximations within the substructures, or, in other terms, those that does not necessarily satisfy a priori the transmission condition for the interfaces existing between the substructures, both with respect to displacement as well to stress. These conditions are incorporated within the variational formulation.

The second element defining this approach is the introduction of approximations with strong mechanical content: the solution is supposed to be well described locally in the neighbourhood of a point X, as for the superposing of a infinite number of local vibration modes. Theses basics modes satisfy the law of dynamics. It is proved that each mode is associated with a wave vector $\underline{P}$ that, for plate problem, belongs to one material characteristic curve C. For a low value of the damping coefficient η ($\eta \ll 1$), the basic mode (displacement-stress) associated with $\underline{P}$ can be written as:

$$\begin{aligned} w_{(\underline{X},\underline{P})} &= \underline{a}_{(\underline{P})}\mathbf{W}_{(\underline{X},\underline{P})} \exp(i\omega \underline{P}\cdot\underline{X}) \exp\Big(\frac{\eta}{2}\omega \underline{P}\cdot\underline{X}\Big), \\ \mathbb{M}_{(\underline{X},\underline{P})} &= \underline{a}_{(\underline{P})}\Lambda_{(\underline{X},\underline{P})} \exp(i\omega \underline{P}\cdot\underline{X}) \exp\Big(\frac{\eta}{2}\omega \underline{P}\cdot\underline{X}\Big), \end{aligned} \tag{11}$$

where $a_{(\underline{P})}$ is called the "generalized amplitude". Various expressions are possible for $\mathbf{W}$ and Λ. A significant and well-known family of basic modes is formed of "complex rays to the n-th order": $\mathbf{W}$ is a polynomial expression to the n-th degree in $\underline{X}$. In practice, we assume that the major effects are well described by this family. It goes without saying that other families, especially those localized in the neighbourhood of the boundaries, may also be introduced (see Fig. 6).

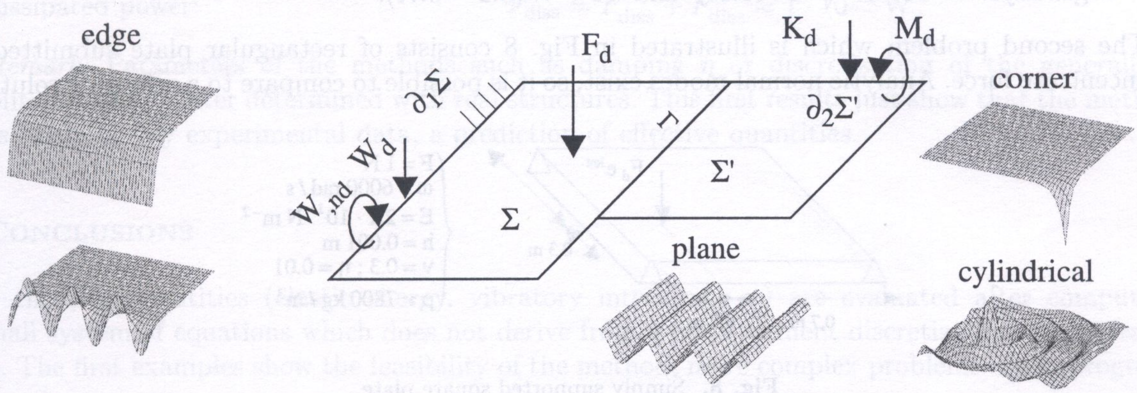

Fig. 6. Description of the solution with the complex rays family

The solution is being sought therefore in the following integral form where small-length part are explicitly described:

$$\begin{aligned} w_{(\underline{X})} &= \int_C a_{(\underline{P})}\mathbf{W}_{(\underline{X},\underline{P})} \exp(i\omega \underline{P}\cdot\underline{X}) \exp\Big(\frac{\eta}{2}\omega \underline{P}\cdot\underline{X}\Big)\,\mathrm{d}s_{(\underline{P})}\,, \\ \mathbb{M}_{(\underline{X})} &= (1+i\eta)\frac{2}{3}h^3\,\mathbf{K}_{CP}\mathbb{X}_{(w(\underline{X}))} \end{aligned} \tag{12}$$

$a_{(\underline{P})}$ designates the generalized amplitude of the basic mode associated with the wave vector $\underline{P}$ belonging to the C curve. In practice the generalized amplitude $a_{(\underline{P})}$ is discretized by finite elements a_i.

The discrete generalized amplitudes a_i do not depend on $\underline{X}$ and therefore are large-wave quantities. These approximations within the substructures, along with the variational formulation, lead to a small system of equations.

Lastly, computation are performed on a certain number of test points for the pertinent effective quantities with respect to: kinetic energy, deformation energy, dissipation and vibrational intensity.

3. Results and illustrations

The first case has an analytical solution. It consists of a rectangular plate with Poisson ratio $\nu = 0$, clamped at one edge and subjected to a harmonic normal force $F_{(x=L)} = \exp(i\omega t)$ at the opposite edge (see Fig. 7). This case is accurately equivalent to modelling of a clamped bar, which has an analytical solution.

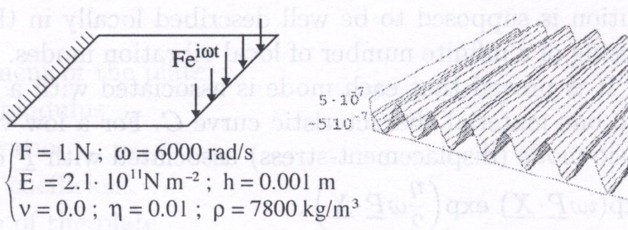

Fig. 7. A beam-like plate

The results for this case – as any flexural beam problem – is exactly the analytical solution, because the exact solution is a combination of the complex rays introduced.

The solution is described by four generalized complex amplitudes a_i :

– interior rays: $10^{-7} \times (-0.8 + 2i)$ and $10^{-7} \times (-2 - 3i)$,

– edges rays: $10^{-7} \times (-5 - 0.5i)$ and $10^{-7} \times (0.2 - 0.7i)$.

The second problem which is illustrated in Fig. 8 consists of rectangular plate submitted to a concentrate force. Analytic normal modes exist, so it is possible to compare to a reference solution.

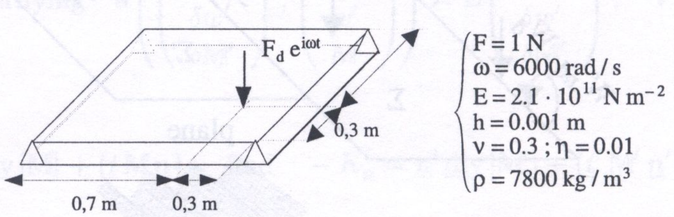

Fig. 8. Simply supported square plate

It is possible to represent the generalized amplitudes $a_{(\underline{P})}$ (or to compute effective quantities such as effective displacement):

$$\begin{aligned} &\text{exact:} && \sqrt{\langle ww^* \rangle_{x,y}} \approx 2 \cdot 10^{-7}\ \mathrm{m}, \\ &\text{approximate:} && \sqrt{\langle ww^* \rangle_{x,y}} \approx 1 \cdot 10^{-7}\ \mathrm{m}. \end{aligned} \tag{13}$$

Also, solution can be zoomed (see Fig. 9). If needed, the phenomena with small-length variations is post-processor computed. The detail is not highly significant but global effective quantities may be insufficient for particular applications.

The third problem which is illustrated in Fig. 10 consists of an assembly of two plates with different material properties.

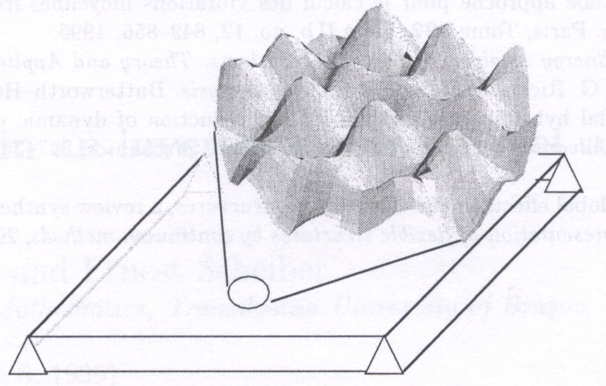

Fig. 9. Small length variations

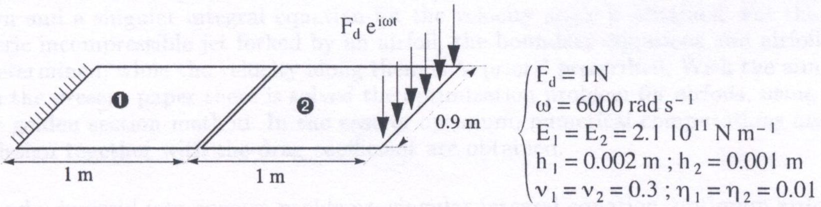

Fig. 10. Two plates

The results are:

$$\begin{aligned}
&\text{effective displacement in substructure 1:} && \langle ww^*\rangle_{x,y}^{(1)} \approx 5\cdot 10^{-8}\,\mathrm{m}\,,\\
&\text{effective displacement in substructure 2:} && \langle ww^*\rangle_{x,y}^{(2)} \approx 1\cdot 10^{-7}\,\mathrm{m}\,,\\
&\text{dissipated power:} && P_{\mathrm{diss}} = P_{\mathrm{diss}}^{(1)} + P_{\mathrm{diss}}^{(2)} \approx 1\cdot 10^{-4}\,\mathrm{W}\,.
\end{aligned} \tag{14}$$

Remark: Parameters of the methods such as damping η or discretisation of the generalized amplitudes will be later determined with real structures. This first results just show that the method gives, without any experimental data, a prediction of effective quantities.

4. Conclusions

The effective quantities (elastic energy, vibratory intensity, ...) are evaluated after computing a small system of equations which does not derive from a finite element discretization of the structure. The first examples show the feasibility of the method; more complex problems are in progress.

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