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Optimal airfoil in an inverse problem of jet aerodynamics[1]

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The paper deals with a special inverse boundary problem, when the boundary of the domain is completely unknown and a singular integral equation for the velocity angle is obtained. For the model of free plane symmetric incompressible jet forked by an airfoil, the boundary equations and airfoil shape are "a posteriori" determined, while the velocity along them is "a priori" prescribed. With the aim to obtain minimum drag, in the present paper there is solved the optimization problem for airfoils, using the penalty method and the golden section method. In the case of optimum, numerical computations are performed and the airfoil design together with the drag coefficient are obtained.

Keywords: inviscid jets, inverse problems, singular integral equation, optimum airfoil, optimization

1. Introduction

The stationary potential plane flow of an inviscid fluid is considered in the absence of mass forces *(Hyp)*. Relating the velocity field $w(z) = u(x,y) + i\,v(x,y)$ to the xOy frame in the physical flow domain D_z, $z = x + iy$, then in the hypothesis *(Hyp)* formulated above, we have $\operatorname{rot}\nu = 0$, $(\nu = \operatorname{grad}\varphi(x,y))$, $\operatorname{div}\nu = 0$. The complex potential $f(z)$ and the complex velocity $w(z)$ are defined through the analytic functions:

$$f(z) = \varphi(x,y) + i\psi(x,y), \qquad \overline{w} = \frac{\mathrm{d}f}{\mathrm{d}z} = Ve^{-i\theta}, \qquad u = V\cos\theta \qquad v = V\sin\theta. \tag{1}$$

Here $\varphi = \varphi(x,y)$ is the velocity potential, $\psi = \psi(x,y)$ the stream function, $V = (u^2+v^2)^{\frac{1}{2}}$ and $\theta = \arctan(v/u)$, are the velocity magnitude and respectively its angle with the Ox axis. Passing to the hodographic plane (V,θ), i.e. $W = V + i\theta$, using (1), f is analytically generalized by W [3, 5, 9]:

$$\theta_\psi = -\frac{1}{V}V_\varphi, \qquad \theta_\varphi = \frac{1}{V}V_\psi, \qquad \varphi_\theta = V\psi_{_V}, \qquad \varphi_{_V} = -\frac{1}{V}\psi_\theta. \tag{2}$$

In the case of the direct problem the flow will be determined using the hodographic method [3, 4, 8] and $f(W)$, $z(W)$ will be obtained. In the case of a curvilinear domain D_z, it is generally difficult to obtain directly $f = f(z)$ and $w = w(z)$ by solving the Dirichlet or Volterra boundary problem, therefore a canonic auxiliary domain D_ζ, $\zeta = \xi + i\eta$, $\eta \geq 0$; [2, 5] should be introduced.

To the domain D_z, $y \geq 0$, of the plane symmetrical jets, it corresponds the domain D_f, $f = \varphi + i\psi$, $\varphi \in (-\infty,\infty)$, $0 \leq \psi \leq \frac{Q}{2} < \infty$, where Q is the total flow mass. We try to determine the analytic function $f = f(z)$ which realizes the conformal mapping $D_f \leftrightarrow D_\zeta$ with

$$\varphi_\xi = \psi_\eta\,; \qquad \varphi_\eta = -\psi_\xi\,; \qquad f_{\overline{\zeta}} = 0. \tag{3}$$

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To obtain the analyticity conditions for the velocity (V,θ) in (2) we introduce the Jukovski function ω, considering that along the stream lines (free lines), $V = V^0$. That is,

$$\omega = t + i\,\theta, \qquad \overline{w} = V^0 e^{-\omega}, \qquad t = \ln\frac{V^0}{V}, \qquad 0 \le V \le V^0; \tag{4}$$

$$\theta_\psi = t_\varphi, \quad \theta_\varphi = -t_\psi; \qquad \varphi_\theta = -\psi_t, \quad \varphi_t = \psi_\theta; \qquad \omega_{\overline{f}} = 0, \quad f_{\overline{\omega}} = 0. \tag{5}$$

In the case of free surface flow, the flow domain D_z is generally bounded by polygonal rigid walls, (curvilinear) obstacles and stream lines that diverge from the walls or the obstacle [6]. Along these free lines the velocity, pressure and density are $V^0, p^0, r^0 = \text{const.}$, respectively, and in the point whose velocity is zero $V \equiv 0$, $p = p_0$, $\rho = \rho_0 = \text{const.}$ Applying Bernoulli's law in the case of incompressible flow along a stream line $\psi = \text{const.}$ we obtain

$$\frac{1}{2}V^2 + \frac{p}{\rho} = \frac{1}{2}V^{0\,2} + \frac{p^0}{\rho^0}; \qquad p = p_0 - \frac{\rho V^2}{2}; \qquad p^0 = p_0 - \frac{\rho_0 V^{0\,2}}{2}. \tag{6}$$

Now we consider the Theorems 1 and 2 [5, 6].

Theorem 1. *In the hypothesis (Hyp), if there is a conformal mapping $f = f(\zeta)$, $f_{\overline{\zeta}} = 0$ with $D_f \leftrightarrow D_\zeta$ then $z = z(\zeta)$ is analytic (conformal), with $D_z \leftrightarrow D_\zeta$.*

That can be easily proved: f is analytic of z, $f_{\overline{z}} = 0$ and from *(Hyp)* z is analytic of f, $z_{\overline{f}} = 0$. If f is analytic of ζ then their composition $z = z(\zeta)$ is analytic, too. At this stage, we shall find $f = f(\zeta)$ so that the boundaries of the domains D_z, D_f correspond to the boundary of D_ζ, $\eta = 0$, $\xi \in (-\infty, \infty)$, on which we have the stream lines $\psi = \text{const.}$ As $x'Ox$ is the axis of symmetry, we shall prove that for the founded function $f = f(\zeta)$ the following conditions hold: $\eta = 0$, $\psi = \text{const.}$ and $\frac{\partial\varphi}{\partial\eta}\big|_{\eta=0} = 0$.

In this case, the passing relations (1) $\mathrm{d}z = \frac{1}{V}e^{i\theta}\mathrm{d}f$ become in D_ζ, on $\eta = 0$,

$$\mathrm{d}x + i\,\mathrm{d}y = \left(\frac{\partial\varphi}{\partial\xi}\right)\frac{1}{V}(\cos\theta + i\,\sin\theta)\,\mathrm{d}\xi. \tag{7}$$

Performing the separation of the real and imaginary parts, the geometrical equations of the boundary (the BOB' airfoil), are derived:

$$\frac{\mathrm{d}x}{\mathrm{d}\xi} = \frac{\cos\theta}{V}\varphi_\xi; \qquad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{\sin\theta}{V}\varphi_\xi; \tag{8}$$

whence

$$x(\xi) = \int_{\xi_0}^{\xi}\frac{\cos\theta}{V}\,\mathrm{d}\xi + x_0, \qquad y(\xi) = \int_{\xi_0}^{\xi}\frac{\sin\theta}{V}\,\mathrm{d}\xi + y_0.$$

In the case of the inverse problem, the boundary D_z is completely unknown and using these formulae, it will be determined by Theorem 2. We only need to know $w(\zeta)$, or $\omega(\zeta)$.

Theorem 2. *If in the hypothesis (Hyp) there is f analytic of ζ, and it is the conformal mapping between $D_f \leftrightarrow D_\zeta$ then $\omega = t + i\,\theta = \omega(\zeta)$ is analytic of ζ, $\omega_{\overline{\zeta}} = 0$ and it is the conformal mapping between $D_\omega \leftrightarrow D_\zeta$.*

As demonstrated above in Theorem 1, if f is analytic of ζ, and using (5), ω is analytic of f, then their composition $\omega = \omega(\zeta)$ will be analytic, too. Usually, in D_ζ, $\eta \ge 0$, for $\omega = \omega(\zeta)$ there is a mixed boundary problem. Solving it, we obtain $\omega(\zeta)$ and $w(\zeta)$ to be used in (8).

2. THE INTEGRAL EQUATION OF THE INVERSE PROBLEM

In the previously stated conditions, we consider the plane flow of a symmetrical free fluid jet, bounded by the free lines (AD) and $(A'D')$ along which the velocity is V^0. At infinite upstream, the jet width is $AA' = 2h$, and the velocity is $\mathbf{V}^0 = V^0 i$, while the total flow mass is $Q = 2hV^0$.

The jet encounters the symmetrical curvilinear obstacle BOB'. The stream lines (BC) and $(B'C')$, along which the velocity is V^0, emanate from B, B' and acquire an asymptotic direction at infinite downstream in $(CD), (C'D')$ at an angle $\pm\gamma\pi$ with the $x'Ox$ axis which is the symmetry axis A_0O of the figure. It is sufficient to study the flow in the halfplane D_z, $y \geq 0$, and therefore the domain boundary is (A_0OBCDA). The velocity angle in O is $\theta(0) = \alpha\pi$ and $\theta(B) = \beta\pi$ where $0 \leq \gamma < \beta < \alpha \leq \frac{1}{2}$ and the velocity downstream (CD) is $w = V^0 e^{i\gamma\pi}$ (Fig. 1). The direct boundary problem for analytic functions in D_z is to find an analytic function in D_z knowing the real (imaginary) part or mixed values along the boundary ∂D_z, which is also known. The inverse problem consists of determining an analytic function in D_z which will fulfil the same conditions on the boundary, but in this case the boundary is only partially known (or completely unknown, as is our case here) and it needs to be determined, too. Direct problems have been studied using these models — by Cisotti [1], Villat and Iacob [3].

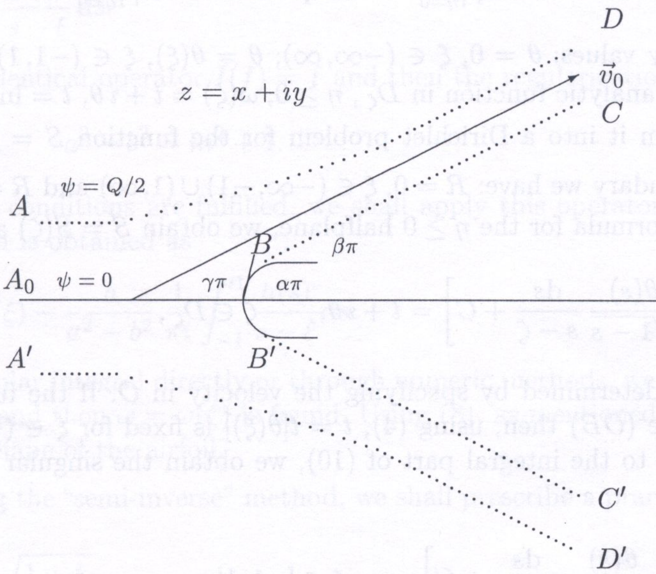

Fig. 1. Motion range in the physical plane D_z

For the problem stated above, the boundary $(A_0A'D'C'OBCDAA_0)$ is unknown, but the stream function ψ, the velocity V^0 are known on this boundary and the distribution $V = V(\theta)$ $(p = p(\theta))$ is given "a priori" on the $(B'OB)$ profile. Therefore, it is important to determine "a posteriori" the shape of this airfoil in the case.

Several papers, like [8] and [10], deal with the case of the flow with circulation. In order to solve the mentioned problem, for $\theta(\xi)$, an integral equation will be derived. First the Theorems 1 and 2 will be applied.

Let us consider the biunivocal correspondence between the domains D_z, D_f with the halfplane D_ζ, $\eta \geq 0$, so that the boundary $(A_0OBCDAA_0)$ be placed upon the $\eta = 0$ axis, $\xi \in (-\infty, \infty)$: $A_0(-\infty)$; $O(-1)$; $B(1)$; C; $D(a)$; $A(\infty)$ (Fig. 2). The parameter $a > 1$ will be determined.

We shall determine $f(\zeta) = \varphi + i\psi$ analytic in D_ζ, $\eta \geq 0$, such that $\Delta\psi = 0$ and with the boundary values $\psi = 0$ for $\xi \in (-\infty, a)$ and $\psi = \frac{Q}{2}$ for $\xi \in (a, \infty)$. The solution of the Dirichlet

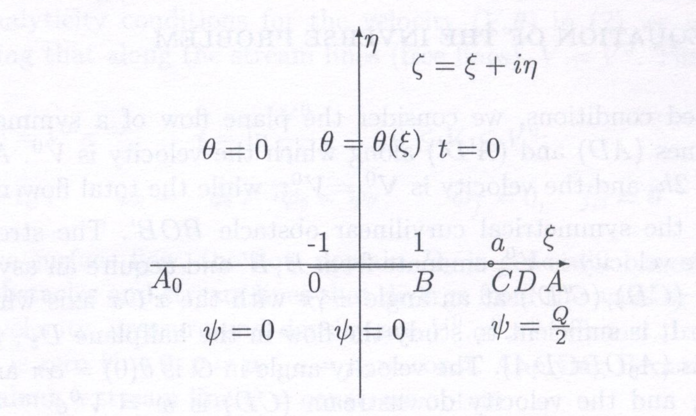

Fig. 2. Correspondence of fields plane D_f, D_ω with D_ζ

problem ($D_f \leftrightarrow D_\zeta$) when $\varphi \in (-\infty, \infty)$, $\psi \in [0, \frac{Q}{2}]$ is [3, 5]:

$$f(\zeta) = -\frac{Q}{2\pi}\ln(\zeta - a) + \frac{iQ}{2}\,; \qquad \left.\frac{\partial \varphi}{\partial \xi}\right|_{\eta=0} = -\frac{Q}{2\pi}\frac{1}{\xi - a}\,, \qquad \left.\frac{\partial \varphi}{\partial \eta}\right|_{\eta=0} = 0. \tag{9}$$

Knowing the boundary values: $\theta = 0$, $\xi \in (-\infty, \infty)$; $\theta = \theta(\xi)$, $\xi \in (-1, 1)$; $t = 0$, $\xi \in (1, a) \cup (a, \infty)$, we determine the analytic function in D_ζ, $\eta \geq 0, \omega(\zeta) = t + i\,\theta, t = \ln\frac{V^0}{V}$. This is a mixed problem and we transform it into a Dirichlet problem for the function $S = R + iT = \frac{\omega(\zeta)}{\sqrt{\zeta - 1}}$, in which case along the boundary we have: $R = 0$, $\xi \in (-\infty, -1) \cup (1, \infty)$ and $R = \frac{\theta(\xi)}{\sqrt{1-\xi}}$, $\xi \in (-1, 1)$. Using the Cisotti–Villat formula for the $\eta \geq 0$ halfplane, we obtain $S = S(\zeta)$ and

$$\omega(\zeta) = \frac{\sqrt{\zeta - 1}}{\pi i}\left[\int_{-1}^{1} \frac{\theta(s)}{\sqrt{1-s}}\frac{\mathrm{d}s}{s - \zeta} + C\right] = t + i\,\theta, \qquad \zeta \in D_\zeta\,, \tag{10}$$

where the constant C is determined by specifying the velocity in O. If the function $V = V(\theta)$ is prescribed on the obstacle (OB) then, using (4), $t = t[\theta(\xi)]$ is fixed for $\xi \in (-1, 1)$. Applying the Sohotski–Plemelj formula to the integral part of (10), we obtain the singular integral equation of the inverse problem [6]:

$$t[\theta(\xi)] = \frac{\sqrt{1 - \zeta}}{\pi i}\left[\int_{-1}^{1} \frac{\theta(s)}{\sqrt{1-s}}\frac{\mathrm{d}s}{s - \zeta} + C\right], \qquad \xi \in [-1, 1]. \tag{11}$$

This equation is a non-linear one and the singularity of the integral is to be taken in Cauchy's principal value sense. Solving this we obtain $\theta = \theta(\xi)$, then $t = t(\xi)$ and $V = V(\xi)$ for $\xi \in (-1, 1)$ and returning to (10), we find $\omega = \omega(\zeta)$ and $W = W(\zeta)$ which determine the flow. Using (8), we obtain the parameter equations of the boundary of domain D_z, $x = x(\xi)$ and $y = y(\xi)$ taking for the (OB) airfoil $\xi_0 = -1$, $\xi \in (-1, 1]$.

We shall study this equation in two important cases.

Case 1. — Along the (OB) airfoil we prescribe "a priori" the distribution:

$$V(\theta) = \left(\frac{V_0}{V}\right)^{\frac{\theta - \beta\pi}{\pi(\alpha - \beta)}} \Longleftrightarrow t = \frac{\theta - \beta\pi}{\pi(\alpha - \beta)}\ln\left(\frac{V_0}{V}\right) \equiv m\theta + n, \tag{12}$$

where: $V(0) = V_0$, $\theta(0) = \alpha\pi$, $t_0 = \frac{V^0}{V_0}$, $\xi = -1$ and $V(B) = V^0$, $\theta(B) = \beta\pi$, $t_B = 0$, $\xi = 1$, with $V_0 \ll V^0$, $\alpha > \beta$. The meaning of these conditions is that in the neighborhood of O, the velocity V_0 has a very little value, fact that may be experimentally verified. In this case, Eq. (11) is linear.

We impose in O the condition $t(\xi=-1)=\ln\frac{V^0}{V_0}$, and using (11) we obtain:

$$C=\frac{\pi}{\sqrt{2}}\ln\left(\frac{V^0}{V_0}\right)-\int_{-1}^{1}\frac{\theta(s)}{\sqrt{1-s}}\frac{\mathrm{d}s}{s+1}.$$

Next, considering again the linear equations (11) and (12), one gets:

$$m\theta+n=\frac{\sqrt{1-\xi}}{\pi}\int_{-1}^{\prime 1}\frac{\theta(s)}{\sqrt{1-s}}\frac{(1+\xi)\mathrm{d}s}{(s+1)(s-\xi)}+\sqrt{\frac{1-\xi}{2}}\ln\left(\frac{V^0}{V_0}\right); \tag{13}$$

$$m=\frac{1}{\pi(\alpha-\beta)}\ln\left(\frac{V^0}{V_0}\right),\qquad n=\beta\pi m.$$

Denoting $\theta(\xi)=(\xi+1)\sqrt{1-\xi}\,g(\xi)$, we obtain the following singular canonic integral equation:

$$ag(\xi)+\frac{b}{\pi}i\int_{-1}^{\prime 1}\frac{g(s)}{s-\xi}\,\mathrm{d}s=\frac{\beta\pi m}{\sqrt{1-\xi}(1+\xi)}+\frac{1}{\sqrt{2}(1+\xi)}\ln\left(\frac{V^0}{V_0}\right)\equiv h(\xi) \tag{14}$$

where: $a=m$, $b=-i$, and $h(\xi)$ is Hölderian. Since a and b are constant [6, 7], we can compute directly the solution. We introduce Schwarz's operator

$$S[f(\xi)]=\frac{1}{\pi i}\int_{-1}^{\prime 1}\frac{f(s)}{s-\xi}\,\mathrm{d}s$$

where $S^2=I$, the identical operator $I(f)=f$ and then the regularization operator will be

$$\frac{1}{a^2-b^2}[aI-bS],\qquad a^2-b^2=m^2+1.$$

Since the regularity conditions are fulfilled, we shall apply this operator to Eq. (14). Finally, like in [6, 7], the solution is obtained as

$$g(\xi)=\frac{a}{a^2-b^2}h(\xi)-\frac{b}{a^2-b^2}\frac{1}{\pi i}\int_{-1}^{\prime 1}\frac{h(s)}{s-\xi}\mathrm{d}s. \tag{15}$$

Computing the singular integral directly or through numeric methods, we obtain $g=g(\xi)$, $\theta=\theta(\xi)$, $t=t(\xi)$, $V=V(\xi)$ and then $\omega=\omega(\zeta)$ is found. Using (8), as mentioned above, we can obtain the boundary and the shape of the airfoil.

Case 2. — Using the "semi-inverse" method, we shall prescribe a practical model of the velocity angle

$$\theta(\xi)=\pi(\beta-\alpha)\sqrt{\frac{1+\xi}{2}}+\alpha\pi,\qquad \beta<\alpha;\qquad \xi\in[-1,1]. \tag{16}$$

$$\theta(0)=\theta(-1)=\alpha\pi;\qquad \theta(B)=\theta(1)=\beta\pi.$$

If we impose the condition $V(O)=0$, $t(\xi=-1)\to\infty$, then it follows that $C\equiv 0$ in (11) and the velocity is obtained through the computation of the Glauert singular integral [3, 5].

$$\begin{aligned}
I_1 &= \int_{-1}^{\prime 1}\sqrt{\frac{1+s}{1-s}}\frac{\mathrm{d}s}{s-\xi}=\pi;\\
I_2 &= \int_{-1}^{\prime 1}\frac{\mathrm{d}s}{\sqrt{1-s}(s-\xi)}=2\ln\left(\frac{\sqrt{2}+\sqrt{1-\xi}}{\sqrt{1+\xi}}\right),\quad \xi\in(-1,1)\\
t(\xi) &= \pi\sqrt{\frac{1-\xi}{2}}(\beta-\alpha)+2\alpha\sqrt{1-\xi}\ln\left(\frac{\sqrt{2}+\sqrt{1-\xi}}{\sqrt{1+\xi}}\right),\\
V(\xi) &= V^0\exp\left[\pi(\alpha-\beta)\frac{\sqrt{1-\xi}}{2}\right]\left[\frac{\sqrt{2}+\sqrt{1-\xi}}{\sqrt{1+\xi}}\right]^{2\alpha\sqrt{1-\xi}}
\end{aligned} \tag{17}$$

where the conditions $V(O) = V(\xi = -1) = 0$, $V(B) = V(\xi = 1) = V^0$ are fulfilled. It is obvious that eliminating ξ from (16) and (17), $V = V(\theta)$, $t = t(\theta)$ are obtained. In this case, the integral equation is a nonlinear one.

Returning to (10), we have:

$$\omega(\zeta) = t(\zeta) + i\theta(\zeta) = \frac{\sqrt{\zeta - 1}}{\pi i} \int_{-1}^{1} \frac{\theta(s)}{\sqrt{1-s}} \frac{\mathrm{d}s}{s - \zeta}. \tag{18}$$

Using (16) on the $\eta = 0$ axis and computing the integral, we shall obtain the distribution of the velocity along the boundary:

- along the free lines $(BCDA)$, $V = V^0$, $t \equiv 0$ from (18) for $\xi \in (1, a) \cup (a, \infty)$ we obtain:

$$\theta(\xi) = \frac{\pi(\beta - \alpha)}{\sqrt{2}} \left(\sqrt{\xi + 1} - \sqrt{\xi - 1}\right) + 2\alpha \arctan \sqrt{\frac{2}{\xi - 1}}; \tag{19}$$

$$u = V^0 \cos\theta \qquad v = V^0 \sin\theta.$$

We have used the quadrature formulas:

$$I_3 = \int_{-1}^{1} \sqrt{\frac{1+s}{1-s}} \frac{\mathrm{d}s}{s - \xi} = \pi \left(1 - \sqrt{\frac{\xi + 1}{\xi - 1}}\right),$$

$$I_4 = \int_{-1}^{1} \frac{\mathrm{d}s}{\sqrt{1-s}(s-\xi)} = -\frac{2}{\xi - 1} \arctan \sqrt{\frac{2}{\xi - 1}}.$$

- along (A_0O), $\xi \in (-\infty, -1)$, because $\theta = 0$, using (16) in (18) we obtain:

$$t(\xi) = \ln\left(\frac{V^0}{V}\right) = \frac{(\beta - \alpha)\pi}{\sqrt{2}} \left(\sqrt{1 - \xi} - \sqrt{-1 - \xi}\right) + \ln\left(\frac{\sqrt{1 - \xi} + \sqrt{2}}{\sqrt{1 - \xi} - \sqrt{2}}\right)^{\alpha}; \tag{20}$$

$$u = V = V^0 e^{-t(\xi)}, \qquad v \equiv 0.$$

The following results were used:

$$I_5 = \int_{-1}^{1} \sqrt{\frac{1+s}{1-s}} \frac{ds}{s - \xi} = \pi \left(1 - \sqrt{\frac{-1 - \xi}{1 - \xi}}\right),$$

$$I_6 = \int_{-1}^{1} \frac{\mathrm{d}s}{\sqrt{1-s}(s-\xi)} = \frac{1}{\sqrt{1-\xi}} \ln\left(\frac{\sqrt{1-\xi} + \sqrt{2}}{\sqrt{1-\xi} - \sqrt{2}}\right).$$

- along the airfoil OB; $u = V(\xi)\cos\theta(\xi)$, $v = V(\xi)\sin\theta(\xi)$, $\xi \in (-1, 1)$ where $\theta(\xi)$ and $V(\xi)$ are given by (16) and (17). It may be verified that $V(A_0) = V(A) = V^0$; $V(0) = 0$, $V(B) = V^0$ and from $V(C, D) = V^0$ when $\xi \to a$, $\theta(C, D) = \gamma\pi$, the parameter a may be computed. At the same time, using (6), the pressure along (A_0O) and (OB) can be determined.

3. Determining the shape and geometrical parameters of the airfoil

We study the problem in the 2-nd case. From the distribution (16), we note that along the airfoil

$$\theta'(\xi) < 0, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta > 0, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{V^2 \theta'(\xi)}{u \cos^2\theta} < 0$$

i.e. the arc (OB) is convex and symmetrical relative the Ox axis. From (8), (16) and (17) we obtain the coordinates of a point $P\ (x(\xi), y(\xi))$ describing the airfoil (OB):

$$x(\xi) = \frac{Q}{2\pi}\int_{-1}^{\xi} \frac{\cos\theta(s)}{V(s)}\,\frac{\mathrm{d}s}{(a-s)}, \qquad \xi \in (-1,1] \tag{21}$$

$$y(\xi) = \frac{Q}{2\pi}\int_{-1}^{\xi} \frac{\sin\theta(s)}{V(s)}\,\frac{\mathrm{d}s}{(a-s)} \tag{22}$$

The length of the elementary arc in xOy plane is $\mathrm{d}S = \frac{\varphi_\xi}{V(\xi)}\,\mathrm{d}\xi$, and therefore the length of the OB arc is:

$$l_{OB} = l = \frac{Q}{2\pi}\int_{-1}^{1} \frac{\mathrm{d}\xi}{V(\xi)(a-\xi)} = \frac{Q}{2\pi} I(a, V^0, \alpha, \beta). \tag{23}$$

If $Q = 2V^0 h$, $a = a(\alpha, \beta, V^0)$ are given, then l can be determined or conversely, if l is given, then a can be determined.

In (21), the coordinates functions $X(\xi) = \frac{x(\xi)}{l}$, $Y(\xi) = \frac{y(\xi)}{l}$ are normalized and the airfoil may be drown, see Table 1. Using (8), we compute the airfoil curvature. The airfoil curvature is

$$k(\xi) = \frac{1}{R(\xi)} = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} = \frac{\theta'(\xi)}{\varphi'(\xi)} V(\xi).$$

If one denotes

$$\overline{V} = \frac{R}{h} = \frac{2R}{Q} V^0,$$

using (16) and (9), we find:

$$\overline{k}(\xi) = \frac{(\alpha-\beta)(a-\xi)}{\sqrt{2(1+\xi)}}\left[\frac{\sqrt{1+\xi}}{\sqrt{2}+\sqrt{1-\xi}}\right]^{2\alpha\sqrt{1-\xi}} \exp\left[\frac{\pi}{\sqrt{2}}(\alpha-\beta)\sqrt{1-\xi}\right], \qquad \xi \in [-1,1]. \tag{24}$$

Introducing $\theta(\xi)$ in (19), the equations of the free lines $(BC) \cup (DA)$ are derived. On these free lines, the curvature has the expression $k = |\frac{\theta'}{\varphi'}| V^0$, i.e.

$$\overline{k}(BC, AD) = \frac{(\alpha-\beta)}{2\sqrt{2}(1+\xi)}|\xi - a|, \qquad \xi \in (1,a) \cup (a,\infty).$$

It may be observed that along the asymptotic direction $\xi \to a$, $\overline{K}(CD) \to 0$. Since the curvature sign does not change, the free line (AD) is convex, while $(OB) \cup (BC)$ is concave and there are no inflexion points along them.

Remark. If the direct problem is examined, i.e. the flow around a given curvilinear airfoil (BOB'), then the curvature $k(\xi)$ is known and from

$$t = \ln\left(\frac{V^0}{V}\right), \qquad V = k(\xi)\,\frac{\varphi'}{\theta'},$$

using (11) or (18) we obtain — for the first time in this form — the singular integral – differential equation:

$$\ln\frac{\pi\theta'(\xi - a)}{k(\xi)} = \frac{\sqrt{1-\xi}}{\pi}\int_{-1}^{1} \frac{\theta(s)}{\sqrt{1-s}}\,\frac{\mathrm{d}s}{s-\xi}. \tag{25}$$

From this equation, we find $\theta = \theta(\xi)$, then $V(\xi)$ and $\omega(\zeta)$ may be determined (in the particular case of a circle arc, $k(\xi) \equiv 1$).

Using (19) for $\xi = a$, we find the equation

$$\theta(B,C) = \gamma\pi = \frac{\pi(\beta-\alpha)}{\sqrt{2}}\left(\sqrt{a+1}-\sqrt{a-1}\right) + 2\alpha \arctan\sqrt{\frac{2}{a-1}}. \qquad (26)$$

Application for the case when the asymptotic direction γ is apriori prescribed. Usually, if we know α, β and the lenght l of the airfoil, from (23) we may find a and, then, from (26), γ.

Here, we shall study the case when α, β are given and the asymptotic direction γ is apriori prescribed with $0 \le \gamma < \beta < \alpha < \frac{1}{2}$. In this case the nonlinear equation (26) defines $a = a(\alpha, \beta, \gamma)$ and the geometrical parameters may be obtained from (21), (23), and (24); finally the shape of the airfoil $X = X(\xi)$, $Y = Y(\xi)$, is obtained.

As an example, for $\alpha = \frac{1}{3}$, $\beta = \frac{1}{4}$, $\gamma = \frac{1}{6}$, $V^0 = 1$ a is computed, than $\theta(\xi)$, $V(\xi)$, $k(\xi)$, $Y(\xi)$ for $\xi \in [-1, 1]$ (Table 1). In Table 2, the values of a are given for other values of α, β, γ.

Recently, in the case when the obstacle BOB' is in a symmetric channel with parallel walls AD and $A'D'$, a study for the inverse problem was done [6].

Table 1. The coordinates and geometrical parameters of the airfoil for $\alpha = \frac{1}{3}$, $\beta = \frac{1}{4}$, $\gamma = \frac{1}{6}$

ξ_i	X_i	Y_i	$\theta(\xi_i)$	V_i	k_i
−1	0	0	1.047	0	0
−0.9	0.098	0.158	0.989	0.174	0.487
−0.8	0.143	0.225	0.964	0.252	0.481
−0.7	0.179	0.277	0.946	0.316	0.476
−0.6	0.211	0.32	0.93	0.374	0.46
−0.5	0.24	0.358	0.916	0.428	0.464
−0.4	0.267	0.392	0.904	0.479	0.456
−0.3	0.292	0.425	0.892	0.529	0.447
−0.2	0.317	0.455	0.882	0.577	0.438
−0.1	0.341	0.484	0.872	0.623	0.427
0	0.365	0.511	0.862	0.669	0.415
0.1	0.388	0.538	0.853	0.713	0.402
0.2	0.411	0.565	0.844	0.757	0.389
0.3	0.435	0.591	0.836	0.799	0.374
0.4	0.458	0.617	0.828	0.84	0.358
0.5	0.482	0.643	0.82	0.88	0.342
0.6	0.507	0.67	0.813	0.918	0.324
0.7	0.533	0.696	0.806	0.953	0.305
0.8	0.559	0.724	0.799	0.985	0.285
0.9	0.587	0.752	0.792	1.011	0.263
1	0.617	0.783	0.785	1	0.233

Table 2. The values of the parameter a and the drag coefficient C_x

α	β	γ	a	C_x
1/4 (0.25)	1/6(0.167)	1/12(0.083)	4.483	0.325
1/3 (0.333)	1/4(0.25)	1/6 (0.167)	2.135	0.506
1/3 (0.333)	1/6(0.167)	1/12(0.083)	6.336	0.423
4/9 (0.444)	1/4(0.25)	1/12(0.083)	12.775	0.709
5/12(0.417)	1/4(0.25)	1/12(0.083)	11.843	0.673
4/9 (0.444)	1/6(0.167)	1/12(0.083)	9.298	0.598

4. THE AERODYNAMIC FORCES AND AIRFOIL OPTIMIZATION

Using Bernoulli's law (6), with

$$P = \int_{OB} (p - p^0)\, \mathrm{d}s,$$

along the symmetric airfoil $(B'OB)$, the resultant of the pressure forces is

$$P = \int_O^B \frac{\rho V^{0\,2}}{2}\left[1 - \left(\frac{V}{V^0}\right)^2\right] \mathrm{d}y$$

and passing on the $\xi \in [-1, 1]$ segment we obtain

$$P = \int_{-1}^{1} \frac{\rho V^{0\,2}}{2}\left[1 - \left(\frac{V(\xi)}{V^0}\right)^2\right] \frac{\partial y}{\partial \xi}\, \mathrm{d}\xi. \tag{27}$$

To compute P, the derivate $\frac{\partial y}{\partial \xi}$ results from (8), where $\theta(\xi)$ and $V(\xi)$ are defined through (16) and (17), respectively. Using (27) and (23) the expression of the drag coefficient C_x becomes:

$$C_x = \frac{Pl}{\frac{\rho V^{o2}}{2}} = \frac{\displaystyle\int_{-1}^{1}\left[1 - \left(\frac{V(\xi)}{V^0}\right)^2\right] \frac{\sin\theta(\xi)}{V(\xi)} \frac{\mathrm{d}\xi}{a - \xi}}{\displaystyle\int_{-1}^{1} \frac{\mathrm{d}\xi}{V(\xi)(a-\xi)}}\,. \tag{28}$$

For the previous application, using the relation (8), there are presented in Table 1 and Table 2 the computed values of C_x. These values agree with those obtained in the direct problem, by Iacob [3], when the airfoil $(B'OB)$ becomes dihedral $\alpha = \beta = \gamma$.

Futher, for a given angle α and an airfoil length l_{OB}, we shall obtain the parameter $\beta(a)$ and the asymptotic direction $\gamma(a)$, optimizing the airfoil shape BOB', imposing the condition of minimum drag, see (28).

Let then $\alpha \in (0, \pi]$, the airfoil length l and the upstream jet width $AA' = h = 1$, $V^0 = 1$ given. Then, the ratio $k = \frac{l}{h}$ is fixed and we are conducted to the problem of finding the parameter $a > 1$, which minimizes the function $P = P(a)$, see (28).

$$P(a) = \int_{-1}^{1} \frac{1 - V^2(s,\beta)}{V(s,\beta)} \sin\theta(s,\beta) \frac{\mathrm{d}s}{a - s} \tag{29}$$

where, from (16) and (17), we have

$$V(s,\beta(a)) = \exp\left[\pi(\alpha-\beta)\sqrt{\frac{1-s}{2}}\right]\left(\frac{\sqrt{1+s}}{\sqrt{2}+\sqrt{1-s}}\right)^{2\alpha\sqrt{1-s}}, \qquad s \in [-1, 1], \tag{30}$$

$$\theta(s,\beta(a)) = \pi(\beta-\alpha)\sqrt{\frac{1+s}{2}} + \pi\alpha, \qquad s \in [-1, 1]. \tag{31}$$

The value $\beta = \beta(a)$ is obtained from the relation (23), by setting the length l equal to k, or

$$R(a,\beta) = \int_{-1}^{1} \frac{\mathrm{d}s}{V(s,\beta(a))(a-s)} - k = 0 \tag{32}$$

and satisfies the inequalities

$$\max\{0, \gamma(\beta, a)\} \le \beta < \alpha \tag{33}$$

where, from (26),

$$\gamma(\beta(a),a)=\frac{\beta-\alpha}{\sqrt{2}}\left(\sqrt{a+1}-\sqrt{a-1}\right)+\frac{2\alpha}{\pi}\arctan\sqrt{\frac{2}{a-1}},\qquad a>1. \tag{34}$$

With the aim to prove the existence of the solution $\beta=\beta(a)$ of Eq. (32), we shall assure the verification of the implicit function theorem, specifying the monotony of the function $R=R(\beta)$. We analyse the restriction of the length of the airfoil (23), (32) and we shall obtain the bounds for the parameter a. Denoting

$$g(s)=\left(\frac{\sqrt{1+s}}{\sqrt{2}+\sqrt{1-s}}\right)^{2\alpha\sqrt{1-s}},\qquad s\in[-1,1],$$

from the inequalities $0<\beta<\alpha$ and using (30), we deduce

$$g(s)\le V(s,\beta)\le g(s)\exp\left[\pi\alpha\sqrt{\frac{1-s}{2}}\right]$$

and then the lower and upper bounds

$$r_i(a)\le\int_{-1}^{1}\frac{\mathrm{d}s}{V(s,\beta)(a-s)}\le r_s(a),$$

where

$$r_i(a)=\int_{-1}^{1}\frac{\mathrm{d}s}{(a-s)\exp\left[\pi\alpha\sqrt{\frac{1-s}{2}}\right]g(s)},\qquad r_s(a)=\int_{-1}^{1}\frac{\mathrm{d}s}{(a-s)g(s)}.$$

We can affirm then that the resolvability of Eq. (32) is provided if the constant k is between $r_i(a)$ and $r_s(a)$. In this manner, for a given k, the inequalities $r_i(a)\le k\le r_s(a)$ are verified for certain values of a, determining a set $A(k)$, which can be named "the minimized scale". One remarks that the functions $r_i(a)$, $r_s(a)$ are decreasing and the set $A(k)$ is then

$$A(k)=[a_i,\infty)\cap(1,a_s] \tag{35}$$

where a_i and a_s are the solutions of the equations $r_i(a)=k$, respectively $r_s(a)=k$.

From (30) we find

$$\frac{-\dfrac{\partial V}{\partial\beta}(s,\beta)}{V^2(s,\beta)}=\frac{\pi\sqrt{\dfrac{1-s}{2}}}{V(s,\beta)}$$

and consequently

$$\frac{\partial R}{\partial\beta}=\int_{-1}^{1}\frac{\pi\sqrt{\dfrac{1-s}{2}}}{V(s,\beta)(a-s)}\,\mathrm{d}s>0.$$

The conditions of the implicit theorem applied to the equation $R(a,\beta)=0$ are fulfilled and the existence of the function $\beta=\beta(a)$ satisfying $R(a,\beta(a))=0$ is proved.

If we derivate (32), relative to a, then we obtain

$$\beta'(a)=\frac{\displaystyle\int_{-1}^{1}\frac{\mathrm{d}s}{V(s,\beta)(a-s)^2}}{\displaystyle\int_{-1}^{1}\frac{\pi\sqrt{\frac{1-s}{2}}}{V(s,\beta)(a-s)}\,\mathrm{d}s}>0$$

Practically, using the bisection method [4], Eq. (32) may be solved efficiently.

We conclude that the function $P = P(a)$ with $a \in A(k)$ is well defined and the optimization problem becomes

$$\int_{-1}^{1} \frac{1 - V^2(s, \beta(a))}{V(s, \beta(a))} \sin\theta(s, \beta(a)) \frac{\mathrm{d}s}{a - s} \to \min! \qquad a \in A(k) \tag{36}$$

subject to the constraint

$$\gamma(\beta(a), a) \leq \beta(a). \tag{37}$$

Using the "penalty function method" [11], the optimization problem (36)–(37) is transformed into the following optimization problem without restrictions

$$\int_{-1}^{1} \frac{1 - V^2(s, \beta(a))}{V(s, \beta(a))} \sin\theta(s, \beta(a)) \frac{\mathrm{d}s}{a - s} + \Gamma \max\{0\,,\ \gamma(\beta(a), a) - \beta(a)\} \to \min! \qquad a \in A(k) \tag{38}$$

where Γ is a big number.

This optimization problem (38) is solved numerically using the "golden section method" [4].

Once $A(k)$ and a^* derived, one proceeds to compute from (32) $\beta(a^*)$. Finally, from (27), (28) we obtain $P_{\min} = P(a^*)$ and $C_x = C_x(a^*)$ and using (21)–(22), the optimum airfoil is derived $X = X(\xi, a^*)$, $Y = Y(\xi, a^*)$, $\xi \in [-1, 1]$.

4.1. Numerical results

For α and k specified below, with the penalty constant $\Gamma = 10$ [11] and with the imposed accuracy 10^{-5} the numerical results are presented in Table 3. In Table 3, β and γ mean $\beta(a^*), \gamma(\beta(a^*), a^*)$, respectively. The computation were performed in MathCAD 7, which offers the necessary programming facilities for our purposes. For $\alpha = 1/3$, $k = 0.7$ the design of the optimum airfoil is plotted in Fig. 3.

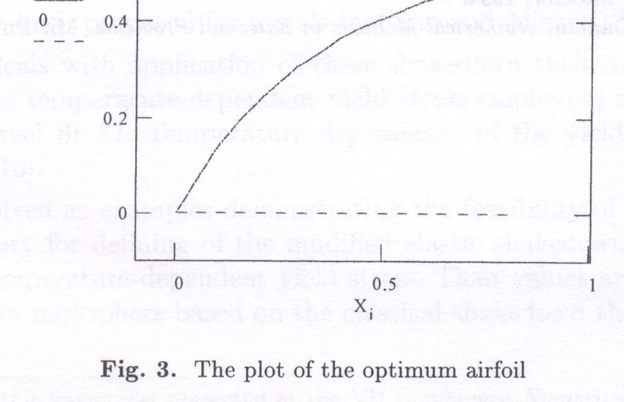

Fig. 3. The plot of the optimum airfoil

Table 3. The geometrical parameters for the optimum airfoil

α	k	$a_i(k)$	$a_s(k)$	a^*	β	γ	$P(a^*)$	C_x
1/3	0.7	3.3571	7.7004	3.714	0.043	0.043	0.1627	0.2325
1/3	1	2.359	5.3035	2.6462	0.0526	0.0526	0.2233	0.2233
1/3	1.3	1.8484	4.0167	2.0857	0.0609	0.0609	0.2771	0.2132
1/4	0.7	3.1421	5.5729	3.3821	0.0341	0.0341	0.09	0.1286
1/4	1	2.2368	3.8583	2.4276	0.0416	0.0416	0.1222	0.1222
1/4	1.3	1.7755	2.9494	1.9308	0.048	0.048	0.1501	0.1155

4.2. Conclusions

From the numerical results and the theoretical observations, the following conclusions can be listed [3, 6, 9]:

- If the width AA' of the upstream flow has a great value, the airfoil has a relatively little length and $0 < k < 1$ then $\beta(a^*) = \gamma(a^*) = 0$. In other words, the width has a negligible influence, as the aurfoil would be placed in unperturbed flow.
- For a little length of the segment AA' and a long length of the airfoil, $k > 1$, then $\beta(a^*) = \gamma(a^*)$, like as the airfoil would be placed in uniform unperturbed flow.
- The theoretical and numerical results obtained in this paper can be used as a first approximation for corresponding plane or axisymmetric problems, in the case of compressible fluids.

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