Computer Assisted Mechanics and Engineering Sciences, 7: 239–245, 2000.

Shakedown of elastic-thermo-plastic structures[1]

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(Received March 8, 1999)

Recently formulated shakedown theorems for materials with temperature-dependent yield stress [1] are applied to evaluation of the elastic shakedown boundary. In order to simulate actual shakedown behavior of elastic-thermo-plastic structures resulting from experimental investigations, the material model of the German mild steel St 37 is considered. It is found that the obtained elastic shakedown boundaries are within the corresponding boundaries based on the classical shakedown theory.

Two examples are compared with the well-known solutions obtained for the neglected yield stress dependence on temperature.

1. Introduction

Two central theorems of the classical shakedown theory (Melan and Koiter theorems) provide criteria for determining whether the elastic shakedown occurs for a given structure under a specified variable load range or not. Melan and Koiter theorems were extended to the thermal loading of materials with the temperature-dependent yield stress by Prager [11] and König [6, 7]. These extended theorems preserve the formal simplicity of the classical theorems but have various undesirable theoretical and computational disadvantages in shakedown analysis. Namely, the extended kinematic theorem [6, 7] can be expressed as the upper bound theorem only for a simple case of thermally isotropic yield function and linear temperature-dependent yield stress. Neither of the extended theorems allows for the simplified computational procedure where only the vertices of the polyhedral thermo-mechanical load domain need to be considered, except when the yield function is convex in an enlarged stress–temperature space.

In their recent paper [1], Borino and Polizzotto have established the static and kinematic shakedown theorems for a class of materials with the temperature-dependent yield stress that avoid the described undesirable features. Formulation of these theorems is based on general thermo-plasticity theory that has recently been proposed for metals by Simo and Miehe [12].

The present paper deals with application of these shakedown theorems to an elastic perfectly plastic material with the temperature-dependent yield stress employing a realistic material model for the German mild steel St 37. Temperature dependence of the yield stress is experimentally determined by Szepan [13].

Two problems are solved as examples demonstrating the feasibility of this approach. These examples show the necessity for defining of the modified elastic shakedown multipliers for the class of materials with the temperature-dependent yield stress. Their values are lower than those of the corresponding shakedown multipliers based on the classical shakedown theory.

[1]Abbreviated version of this paper was presented at the VII Conference *Numerical Methods in Continuum Mechanics*, Stará Lesná, High Tatras, Slovakia, October 6–9, 1998, and published in its Proceedings.

2. Shakedown problem considering materials with temperature-dependent yield stress

Experimental data for a wide class of materials show that the yield stress is a concave function of temperature in the range 0–600 °C. Such materials can be modelled by using the yield function

$$F(\boldsymbol{\sigma},\theta) \equiv f(\boldsymbol{\sigma}) - \sigma_{\mathrm{y}}(\theta) \leq 0\,, \tag{1}$$

where $f(\boldsymbol{\sigma})$ is a degree-one homogeneous function of stresses convex and smooth by hypothesis and the yield stress $\sigma_{\mathrm{y}}(\theta)$ is a concave function of the process temperature θ in °C. Therefrom, the yield function $F(\boldsymbol{\sigma},\theta)$ is convex and smooth in the stress–temperature space. The yield surface shrinks homothetically with the increase in θ (isotropic softening process), that is $\frac{\partial F}{\partial \theta} > 0$ for all $\theta > 0$. For this class of materials, the associative thermoplastic flow laws are adopted [12]:

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{P}} = \dot{\lambda}\,\frac{\partial F}{\partial \boldsymbol{\sigma}}\,, \qquad \dot{\eta}^{\mathrm{P}} = \dot{\lambda}\,\frac{\partial F}{\partial \theta}\,, \tag{2}$$

$$F(\boldsymbol{\sigma},\theta) \leq 0\,, \qquad \dot{\lambda} \geq 0\,, \qquad \dot{\lambda} F(\boldsymbol{\sigma},\theta) = 0\,. \tag{3}$$

where $\dot{\boldsymbol{\varepsilon}}^{\mathrm{P}}$ denotes plastic strain rate, $\dot{\lambda}$ is the rate of plastic multiplier and $\dot{\eta}^{\mathrm{P}}$ denotes plastic entropy rate.

Let us consider a solid body composed of material defined by (1)–(3) which occupies the volume V referred to three orthogonal Cartesian axes $\mathbf{x} = (x_1, x_2, x_3)$, let it be bounded by the surface S which may be divided into a part S_u with constrained displacements and a part S_t on which given surface tractions occur ($S_u \cup S_t = S$). The body is loaded by body forces, imposed strains that are caused by the temperature field $\theta(\mathbf{x},t)$ in V, by tractions on S_t, all of which vary with time t in a quasistatic manner. All these thermo-mechanical actions, collectively denoted as $\mathbf{F}(\mathbf{x},t)$, can be represented as a path arbitrarily shaped within a given load domain Π of functional space, that is, any path within Π is an admissible load history. The domain Π is shaped as a convex hyperpolyhedron of n vertices. The vectors $\mathbf{F}_i(\mathbf{x})$, $i \in I(n) \equiv \{1, 2, \ldots, n\}$, that specify these vertices are referred to as the (thermo-mechanical) basic loads. Any load $\mathbf{F}(\mathbf{x},t) \in \Pi$ may be described as a linear convex combination of the basic loads,

$$\mathbf{F}(\mathbf{x},t) = \sum_{i=1}^{n} \beta_i(t)\,\mathbf{F}_i(\mathbf{x})\,, \tag{4}$$

where the coefficients β_i satisfy the admissibility condition,

$$\beta_i \geq 0 \qquad \text{all} \quad i \in I(n), \qquad \sum_{i=1}^{n} \beta_i = 1. \tag{5}$$

According to the shakedown theory (see, e.g. [3]), an elastic-thermo-plastic solid body subjected to the thermo-mechanical loads (4) whose values are below the elastic shakedown limit, may suffer plastic deformation of limited magnitude during the transient period; after that, it responds elastically without any further plastic deformation (steady-state response). During the stationary period, the thermoelastic stress response $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t)$ and temperature field $\theta(\mathbf{x},t)$ corresponding to $\mathbf{F}(\mathbf{x},t) \in \Pi$ have the forms

$$\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t) = \sum_{i=1}^{n} \beta_i(t)\,\boldsymbol{\sigma}_i^{\mathrm{E}}(\mathbf{x})\,, \tag{6}$$

$$\theta(\mathbf{x},t)) = \sum_{i=1}^{n} \beta_i(t)\,\theta_i(\mathbf{x})\,, \tag{7}$$

where, by definition, $\boldsymbol{\sigma}_i^{\mathrm{E}}(\mathbf{x})$ and $\theta_i(\mathbf{x})$ denote the analogous quantities corresponding to the basic loads $\mathbf{F}_i(\mathbf{x})$. Based on the classical Melan and Koiter theorems, Borino and Polizzotto [1] have reformulated the extended shakedown theorems in the discrete format [6, 7, 11] in a form suitable to the present thermo-plastic materials model.

The static shakedown theorem can be formulated as follows. A necessary and sufficient condition for elastic shakedown to occur in an elastic-thermo-plastic structure subjected to thermo-mechanical loading variable within a hyperpolyhedral domain Π, with basic loads $\mathbf{F}_i$, is the existence of a time independent self-stress field $\boldsymbol{\rho}$, such that the total stress fields $\boldsymbol{\sigma}_i(\mathbf{x}) = \boldsymbol{\sigma}_i^{\mathrm{E}}(\mathbf{x}) + \boldsymbol{\rho}(\mathbf{x})$ and temperature fields $\theta_i(\mathbf{x})$ nowhere violate the yield function, i.e.

$$F(\sigma_i\,,\,\theta_i) \quad \text{in } V\,, \quad \text{for all } i \in I(n)\,. \tag{8}$$

The kinematic shakedown theorem for elastic-thermo-plastic structure subjected to thermo-mechanical loading is formulated as follows. In order for the elastic shakedown to occur, the variable within a hyperpolyhedral domain Π, with basic loads $\mathbf{F}_i$, should satisfy the following inequality for arbitrary choices of plastic strain-entropy mechanisms

$$\sum_{i=1}^{n} \int_V D(\boldsymbol{\varepsilon}_i^{\mathrm{P}}\,,\,\eta_i^{\mathrm{P}})\,\mathrm{d}V \geq \sum_{i=1}^{n} \int_V (\boldsymbol{\sigma}_i^{\mathrm{E}} : \boldsymbol{\varepsilon}_i^{\mathrm{P}} + \theta_i \eta_i^{\mathrm{P}})\,\mathrm{d}V \qquad \text{in } V. \tag{9}$$

The plastic strain field $\boldsymbol{\varepsilon}^{\mathrm{P}}(\mathbf{x})$ related to the strain path has to be compatible with the displacements $\mathbf{u}(\mathbf{x})$ vanishing on S_u:

$$\boldsymbol{\varepsilon}^{\mathrm{P}}(\mathbf{x}) = \sum_{i=1}^{n} \boldsymbol{\varepsilon}_i^{\mathrm{P}}(\mathbf{x}) = \frac{1}{2}[\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^T] \qquad \text{in } V\,, \tag{10}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } S_u\,. \tag{11}$$

For the plastic entropy field $\eta^{\mathrm{P}}(\mathbf{x})$, the following relation holds:

$$\eta^{\mathrm{P}} = \sum_{i=1}^{n} \eta_i^{\mathrm{P}} \qquad \text{in } V\,. \tag{12}$$

The proof procedure for both theorems is similar to that of classical shakedown theory [1].

3. THE ELASTIC-THERMO-PLASTIC MATERIAL MODEL

The experimental investigations [13] of the German mild steel St 37 performed at temperatures 25 °C, 100 °C, 200 °C, 300 °C and 400 °C show that the yield stress depends strongly on the temperature. By interpolating a polynomial experimental data can be approximated. The yield stress

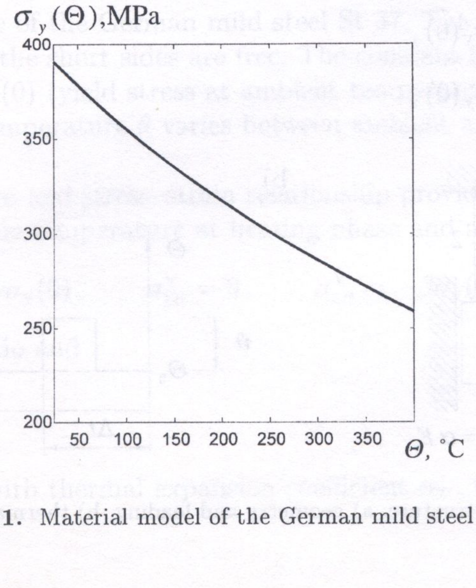

Fig. 1. Material model of the German mild steel St 37

of the German mild steel St 37 is a concave function of the process temperature within the range 25–400 °C, as shown in Fig. 1. These investigations also suggest that, under the assumption of isothermal material behavior, the deviation from the associativity of the flow can be neglected.

4. Examples

Two simple structures have been studied to illustrate the application of Borino and Polizzotto shakedown theorems [1], considering the realistic material model of the mild steel St 37.

4.1. Two bar structure

The two bars have equal length, cross-sections equal to $A_1 = A$, $A_2 = 4A$, respectively, and are connected with a rigid block (Fig. 2a). The bar 2 is kept at constant ambient temperature, whereas the bar 1 is subjected to temperature variation as shown in Fig. 2b. A steady mechanical load $F = \alpha F_y$, with $0 \le \alpha \le 1$ and $F_y = 5A\sigma_y(0)$ (plastic collapse limit load at ambient temperature), is applied on rigid block.

Elastic stress responses to constant load σ^C and variable temperature σ^V are obtained from the equilibrium equation, compatibility condition and stress–strain relationship:

$$\sigma_1^C = \sigma_2^C = \alpha\sigma_y(0)\,, \qquad \sigma_{1H}^V = -\beta\sigma_y(0)\,, \qquad \sigma_{2H}^V = \frac{1}{4}\beta\sigma_y(0)\,, \qquad \sigma_{1A}^V = \sigma_{2A}^V = 0\,,$$

where

$$\beta = \frac{\sigma_T}{\sigma_y(0)}$$

is the shakedown multiplier, $\sigma_y(0)$ is the yield stress at ambient temperature θ_0, $\sigma_T = \frac{4}{5}\alpha_T\vartheta E$ is the maximal thermoelastic stress, α_T is the thermal expansion coefficient, ϑ is the temperature step and E is the Young's modulus.

The subscripts H and A refer to the values at the heating phase and ambient temperature, respectively.

According to the static shakedown theorem for materials with the temperature-dependent yield stress, the stress in the bar 1 violates the yield surface at the heating phase and ambient temperature:

$$\alpha\sigma_y(0) - \beta\sigma_y(0) + \rho_1 = -\sigma_y(\vartheta)\,, \tag{13}$$

$$\alpha\sigma_y(0) + \rho_1 = \sigma_y(0) \tag{14}$$

or the bar 1 yields in tension at the ambient temperature, while the bar 2 yields in tension at the heating phase temperature:

$$\alpha\sigma_y(0) + \rho_1 = \sigma_y(0)\,, \tag{15}$$

$$\alpha\sigma_y(0) + \frac{1}{4}\beta\sigma_y(0) + \rho_2 = \sigma_y(0)\,, \tag{16}$$

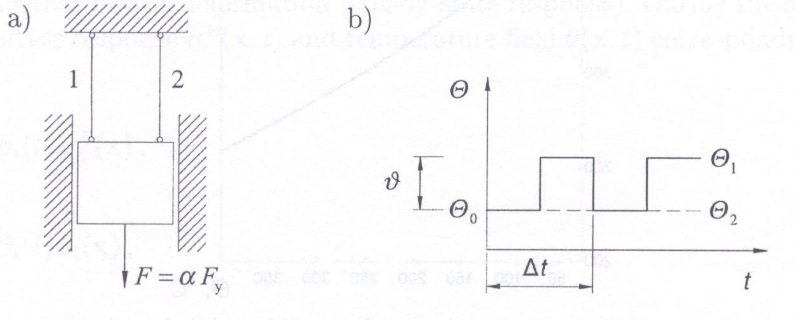

Fig. 2. Two-bar structure: a) geometry and loading, b) thermal loading history

where ρ_1 and ρ_2 denote the time-independent self stresses in bars and $\sigma_{\mathrm{y}}(\vartheta)$ is the yield stress at heating phase temperature $\theta = \theta_0 + \vartheta$.

The elastic shakedown boundary can be evaluated from (13), (14) and (15), (16), respectively:

$$\beta = 1 + \frac{\sigma_{\mathrm{y}}(\vartheta)}{\sigma_{\mathrm{y}}(0)}\,, \tag{17}$$

$$\beta = 5 - 5\alpha\,. \tag{18}$$

The yield stress at heating phase temperature $\sigma_{\mathrm{y}}(\vartheta)$ is a function of shakedown multiplier β. Considering material constants $E = 200$ GPa and $\alpha_{\mathrm{T}} = 12 \cdot 10^{-6}$ $1/°\mathrm{C}$ as well as the interpolation function for $\sigma_{\mathrm{y}}(\vartheta)$ of the German mild steel St 37 from Fig. 1, the shakedown multiplier (17) is:

$$\beta = 1.71\,. \tag{19}$$

Figure 3 presents the elastic shakedown boundary for the German mild steel St 37 compared with the solution obtained by the classical shakedown theory [8].

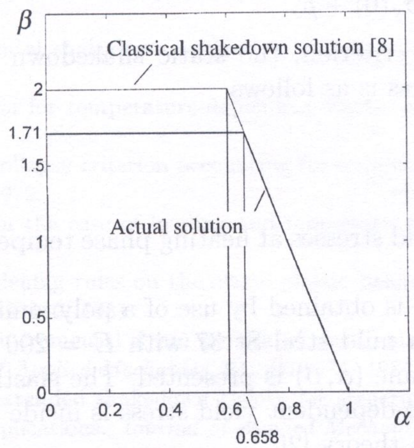

Fig. 3. Elastic shakedown boundary of two-bar structure for material model with temperature-dependent yield stress

4.2. Slab

Let us consider the slab made of the German mild steel St 37. The slab is constrained on the long sides by sliding hinges, while the short sides are free. The constant stress distribution $\sigma_1 = \alpha\sigma_{\mathrm{y}}(0)$, with parameter $\alpha \geq 0$ and $\sigma_{\mathrm{y}}(0)$ (yield stress at ambient temperature) is applied in direction 1, as shown in Fig. 4a. The slab temperature θ varies between ambient and heating phase temperatures (Fig. 4b).

The compatibility condition and stress–strain relationship provide the elastic stress components for constant stress and variable temperature at heating phase and ambient temperature:

$$\sigma_1^{\mathrm{C}} = \alpha\sigma_{\mathrm{y}}(0)\,, \qquad \sigma_2^{\mathrm{C}} = \alpha\nu\sigma_{\mathrm{y}}(0)\,, \qquad \sigma_{1\mathrm{H}}^{\mathrm{V}} = 0\,, \qquad \sigma_{2\mathrm{H}}^{\mathrm{V}} = -\beta\sigma_{\mathrm{y}}(0)\,, \qquad \sigma_{1\mathrm{A}}^{\mathrm{V}} = \sigma_{2\mathrm{A}}^{\mathrm{V}} = 0\,,$$

where ν denotes Poisson's ratio and

$$\beta = \frac{\alpha_{\mathrm{T}}\vartheta E}{\sigma_{\mathrm{y}}(0)}$$

is the shakedown multiplier with thermal expansion coefficient α_{T}, the temperature step ϑ and the Young's modulus E.

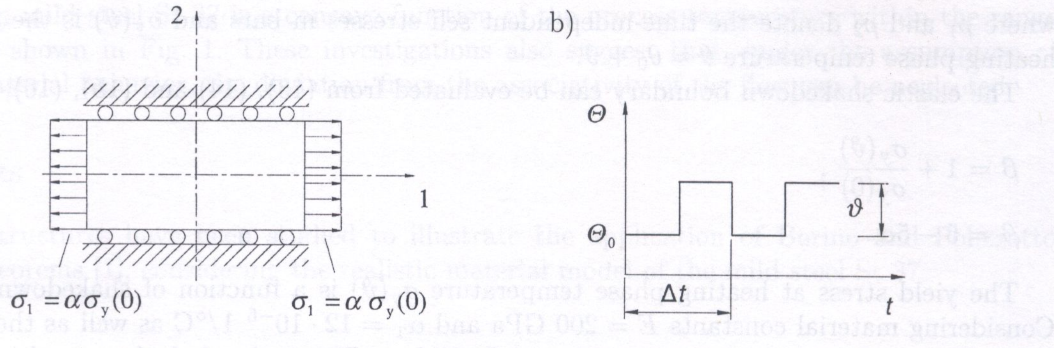

Fig. 4. Slab: a) structure scheme and loading, b) thermal cyclic load

Since $\rho_1 = 0$ and $\rho_2 = \rho$ the time-independent residual stress components, the stress components σ_1 and σ_2 in the slab can be written as

$$\sigma_{1\mathrm{H}} = \alpha\sigma_{\mathrm{y}}(0)\,, \qquad \sigma_{2\mathrm{H}} = \alpha\nu\sigma_{\mathrm{y}}(0) - \beta\sigma_{\mathrm{y}}(0) + \rho\,, \tag{20}$$

$$\sigma_{1\mathrm{A}} = \alpha\sigma_{\mathrm{y}}(0)\,, \qquad \sigma_{2\mathrm{A}} = \alpha\nu\sigma_{\mathrm{y}}(0) + \rho\,. \tag{21}$$

By adopting the Mises yield criterion, the static shakedown theorem for materials with the temperature-dependent yield stress is as follows,

$$\sigma_{1\mathrm{H}}^2 + \sigma_{2\mathrm{H}}^2 - \sigma_{1\mathrm{H}}\sigma_{2\mathrm{H}} \le \sigma_{\mathrm{y}}^2(\vartheta)\,, \tag{22}$$

$$\sigma_{1\mathrm{A}}^2 + \sigma_{2\mathrm{A}}^2 - \sigma_{1\mathrm{A}}\sigma_{2\mathrm{A}} \le \sigma_{\mathrm{y}}^2(0)\,, \tag{23}$$

where $\sigma_{\mathrm{y}}(\vartheta)$ and $\sigma_{\mathrm{y}}(0)$ are the yield stresses at heating phase temperature $\theta = \theta_0 + \vartheta$ and at ambient temperature θ_0, respectively.

An elastic shakedown solution is obtained by use of a polynomial interpolation based on experimental data [13] on the German mild steel St 37 with $E = 200$ GPa and $\alpha_{\mathrm{T}} = 12 \cdot 10^{-6}$ 1/°C. In Fig. 5, the Bree diagram in plane (α, β) is presented. The elastic shakedown boundary for class of materials with the temperature-dependent yield stress is inside the corresponding boundary obtained by the classical shakedown theory [2].

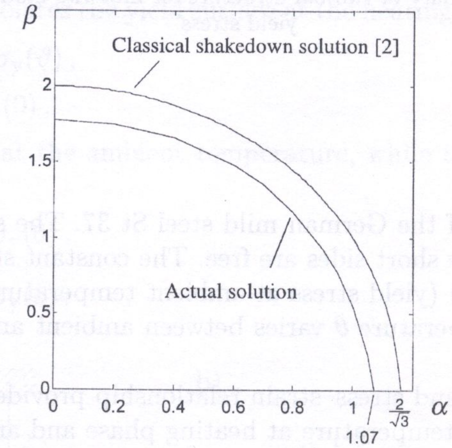

Fig. 5. Bree-diagram of slab for material model with temperature-dependent yield stress

5. Conclusion

For the elastic perfectly plastic materials with convex yield functions in the stress–temperature space, Borino and Polizzotto established the static and kinematic shakedown theorems which present a theoretical and computational improvement with respect to the classical shakedown theorems extended to the materials with temperature-dependent yield stress [6, 7, 11]. The application of Borino and Polizzotto theorems to the realistic material model of the German mild steel St 37, led us to the conclusion that resulting elastic shakedown boundary depends on the Young's modulus and thermal expansion coefficient as well as on the yield stress. Consequently, the elastic shakedown multipliers obtained by means of these theorems have to be different from those of the classical shakedown theory. Their values are generally lower than the corresponding multipliers obtained for the materials with constant yield stress.

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