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Discrete optimal weight design of geometrically nonlinear truss-structures

Anikó Csébfalvi
Janus Pannonius University, Boszorkány u. 2, H-7625 Pécs, Hungary

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In this paper, we introduce a new branch-and-bound type method, for discrete minimal weight design of geometrically nonlinear truss structure subject to constraints on member stresses and nodal displacements when the member cross-sectional areas are available from a discrete set of a given catalogue. The discrete optimization problem can be formulated as a tree search procedure. The initial — unfeasible — node of the searching tree is obtained by decreasing the relaxed cross-sectional areas to the closest discrete catalogue value. Each node of the branch-and-bound searching tree is characterized by the weight of the structure, the actual value of the infeasibility penalty function, and the minimal relaxed additional weight, that is needed to obtain a feasible structure from the given state. The proposed method involves an exterior point method to determine the relaxed solution of the minimum weight design problem. The algorithm seems computationally attractive and has been tested on a large number of examples. Numerical results are presented for a well-known test problem.

1. Introduction

In the truss-structure design the member cross section specification is a very important multicriteria optimization problem. This paper presents a new exact algorithm to solve the problem of weight minimization of a truss structure subject to constraints on member stresses and nodal displacements. The cross sections of the truss members are available from a discrete set of a catalogue. External nodal forces act at the nodal point of the truss structure and the deformation is described by the nodal displacements. In order to capture the effects of changes in the global geometry of the structure, a nonlinear, large displacement model has been adopted.

During the solution of optimal discrete sizing, beyond the problems arising from discrete design variables, we have to facing the following difficulties:

- The discrete optimal weight design of space trusses is a typically large, NP-class problem.
- The structural model is highly nonlinear. The usage of a linearized structural model might query the efficiency of the optimal results.
- The displacement and stresses of the truss members in the constraint functions are implicit functions of discrete design variables.
- The monotonicity of the displacement and stresses — in terms of discrete design variables — cannot be assumed.

A detailed review of literature for structural optimization problems having linked discrete design variables has been presented by Arora and Huang [1]. There are two main groups of methods for obtaining a discrete optimal solution:

- The existing exact methods of discrete design problems provide the optimal solution but the required CPU time is increased exponentially in case of large scale problems while we cannot determine the global optima without evaluation of all sorts of potential cases.

- The heuristic methods provide only the approximate values of optimal solution but the CPU time might be reduced greatly. The evaluating standard for the heuristic algorithm is the approximate degree of the approximate solution to the optimum one.

In the early sixties, the discrete optimal design of trusses subjected to stress and displacement constraints was formulated as an integer linear programming or a mixed integer-continuous variable LP problem using the Gomory's cutting plane method, branch and bound or heuristic methods. In order to reduce the number of evaluation steps a mixed, two phase approach has been developed and proposed by Huang and Arora [4] to obtain discrete optimal solution for large scale problems. The authors suggested three strategies that combine a continuous variable optimization method with a genetic algorithm, simulated annealing and an effective branch-and-bound type method to reduce the CPU time.

The methods published by Gutkowski and Zawidzka [3] and Kaliszky, Kirchner and Lógó [5], based on the knowledge and experiments of practical structural design turned out to be more effective from the point of view of the required CPU time. In order to overcome the difficulties arising from the increasing number of combinations, a sequential, less time consuming approximate solution was proposed by Gutkowski and Zawidzka [3] for discrete minimum weight design. A two phase method was presented by Kaliszky, Kirchner and Lógó [5] for discrete optimization of elasto-plastic trusses. The idea of this method is to solve first the continuous problem without deformation constraint and then, making use of results to solve the discrete problem with deformation using a segmental approach proposed by Templeman and Yates [6], and Groenwold and Stander [2].

In this paper a discrete minimum weight design problem will be considered, which is formulated as a tree search problem. The starting point of the searching tree is computed from a relaxed — continuous design variables assumed — result, such that each design variable will be decreased to the value of the closest discrete neighbour. The starting point of the process will be therefore an unfeasible solution. The following nodes of the searching tree as a "child state" from its "parent state" can be created by changing exactly one cross-sectional area to the next larger one in the catalogue. The pruning rules of the tree search process are based on the weight of the structure, actual value of the infeasibility penalty function, and the minimal relaxed additional weight that needed to obtain a feasible structure from the given state.

The efficiency of the method will be illustrated through a well-known test example. In the structural analysis a geometrically nonlinear, large displacement and only linear material model has been used. The procedure is programmed in MS Visual Basic for Windows 95, Version 5.0.

2. The discrete optimization problem

The geometrically nonlinear truss structure can be formulated as a large displacement model, using a total Lagrange representation. The length of the truss member after deformation is given by:

$$l = \left[\sum_{d=1}^{3} \left(l_d^0 + u_{2d} - u_{1d} \right)^2 \right]^{\frac{1}{2}} , \tag{1}$$

where:

l_d^0 – the projection of the undeformed length onto the coordinate axes,

u_{1d} – the displacements at the origin of truss member,

u_{2d} – the displacements at the end point of truss member.

The total potential function of the global truss structure can be formulated in the following way:

$$V\left(u_i(a_j), a_j\right) = \frac{E}{2} \sum_{j=1}^{e} \frac{a_j}{l_j^0} \left(l_j^0 - l_j\right)^2 - p_i u_i(a_j) , \qquad i = 1, 2, ..., n; \quad j = 1, 2, ..., e , \tag{2}$$

where:

u_i – the nodal displacement vector of space trusses,

a_j – the cross-section areas of truss members,

p_i – the external load vector at the nodal points,

l_j^0 – the undeformed length of truss member,

l_j – the length of the truss member after deformation,

E – the elastic modulus of material.

The discrete optimization problem of space trusses will be discussed in terms of above defined structural model:

The design variables a_j will be selected from a discrete set of the predetermined cross-sectional areas $a_j \in A = \{A_1, A_2, \ldots, A_J\}$ such that minimize the total weight of the structure

$$w = \rho l_j^0 a_j \to \min, \tag{3}$$

subject to the equality constraint in form of equilibrium equations

$$V_{,i}(u_i(a_j))|^{a_j} = 0, \tag{4}$$

and subject to the stress and displacement inequality constraints

$$\underline{u} \le u_i(a_j) \le \overline{u}, \tag{5}$$

$$\underline{s} \le s_j(u_i(a_j)) \le \overline{s}, \tag{6}$$

$$i = 1, 2, \ldots, n; \qquad j = 1, 2, \ldots, e$$

where:

$V_{,i}|^{a_j} = 0$ – the equilibrium equation system,

u_i – the nodal displacement vector of space trusses,

ρ – the density of the structural member,

s_j – the stress vector of the truss members.

w – the weight of the structure,

$V = V(u_i, a_j)$ – the total potential energy of the structure,

$V_{,i}|^{a_j} = 0$ – the equilibrium equation of the structure, at a fixed a_j point,

J – the number of catalogue elements,

$\overline{u}$ – the upper limit of the nodal displacements,

$\underline{u}$ – the lower limit of the nodal displacements,

$\overline{s}$ – the upper limit of the stresses in the truss elements,

$\underline{s}$ – the lower limit of the stresses in the truss elements.

The compatibility equation constraints between nodal displacements and truss member elongations will be fulfilled automatically because of the total Lagrange frame of reference. Assuming linear material model, the stresses of truss members can be computed on the following way

$$s_j = \frac{l_j^0 - l_j(u_i, a_j)}{l_j(u_i, a_j)} E \tag{7}$$

from the undeformed length l_j^0, and deformed length l_j of truss members.

3. Outline of the algorithm

The discrete optimization problem can be formulated as a tree search problem. During the optimization process, it is supposed that the $w^{\max}$ maximum weight — from the largest "off-the-shelf" elements constructed — structure is feasible, i.e. the constraint functions are fulfilled. If the structure, constructed from the smallest elements, is feasible, i.e. the equilibrium equation system (4), the nodal displacement constraints (5), and the stress constraints (6) are fulfilled, than this initial structure will be the optimal solution of the discrete minimal weight design problem. If the structure, constructed from the smallest parameters of the catalogue, is unfeasible, than — starting from an initial structure — a searching tree will be built up.

The initial structure will be created from a relaxed solution of the optimal design problem, such that each design variable will be decreased to the value of the closest discrete neighbour. The starting node of the searching tree is therefore an unfeasible point of the solution set, but — compared to the minimal weight structure — it is closer to the feasible set.

3.1. The tree search problem

In the construction of the branch-and-bound searching tree and of the originated nodes — as a "child state" from its "parent state" — the following variables are used.

1. The nodes of the branch-and-bound searching tree are numbered — starting from $k = 0$, and characterized by the values $\{a_j^{(k)}, w^{(k)}, p^{(k)}, r^{(k)}, v^{(k)}\}$, where:

   $a_j^{(k)}$ the member cross-sectional areas at the k^{th} node of the searching tree,

   $w^{(k)}$ the weight of the structure at the k^{th} node of the searching tree,

   $p^{(k)}$ the actual value of the penalty function at the k^{th} node,

   $r^{(k)}$ the minimal relaxed additional weight at the k^{th} node, that is needed to obtain a feasible structure from the given state, can be determined the following way:

$$w^{(k)} = \rho l_j^0 a_j^{(k)}, \tag{8}$$

$$p^{(k)} = \sqrt{\sum_{i=1}^{n} \max\left(|u_i| - \overline{u}, 0\right)^2 + \sum_{j=1}^{e} \max\left(|s_j| - \overline{s}, 0\right)^2}, \tag{9}$$

$$r^{(k)} = \rho l_j^0 \Delta a_j^{(k)} \to \min, \tag{10}$$

$$\Delta a_j^{(k)} \succeq 0, \tag{11}$$

$$p^{(k)}\left(a_j^{(k)} + \Delta a_j^{(k)}\right) = 0, \tag{12}$$

   $v^{(k)}$ the status variable of the k^{th} node, where

$$v^{(k)} = \begin{cases} 0, & \text{if the } k\text{-th node is an expandable,} \\ 1, & \text{if the } k\text{-th node is an expanded one.} \end{cases}$$

2. Let $w^{\min}$ be the minimal weight of a feasible structure — obtained from the results of the previous steps — which satisfies the expressions (4)–(6). The initial value of the minimal weight is $w^{\min} = w^{\max}$.

3. Let $k^{\max}$ be the greatest node number in the searching tree.

3.2. The stages of the searching algorithm:

1. At each step of the search process, we select the most promising state using the following searching conditions:

$$p^{(k^*)} \to \min, \tag{13}$$

$$w^{(k^*)} < w^{\min}, \tag{14}$$

$$r^{(k^*)} < w^{\min}, \tag{15}$$

$$v^{(k^*)} = 0, \tag{16}$$

$$k^* \in \{0, 1, ..., k^{\max}\}. \tag{17}$$

   If the search process is failed, i.e. there is no states satisfying the above searching conditions, the algorithm will be finished.

   If the search process results alternative states, which satisfy the above searching conditions, than the state with smallest serial number will be selected.

2. The status variable of the selected state will be changed to $v^{(k^*)} = 1$, and the selected state will be expanded. If there is at least one cross-sectional value in the selected state which does not reach the maximal value of the catalogue, than the selected state is expandable. The expanded state, as a child state will be created from its parent state by changing exactly one member cross-sectional area to the next larger one in the catalogue. Let c be the number of the expanded states. Let us determine the $\{a_j^{(k)}, w^{(k)}, p^{(k)}, r^{(k)}, v^{(k)}\}$, $k = k^{\max}+1,\ k^{\max}+2, \ldots, k^{\max}+c$ parameters of the new states. If it is found a state such that $p^{(k)} = 0$ and $w^{(k)} < w^{\min}$, $k \in \{k^{\max}+1,\ k^{\max}+2, \ldots, k^{\max}+c\}$, then $w^{\min} = w^{(k)}$ and $v^{(k)} = 1$. According to the number of new states, the value of $k^{\max}$ will be changed to $k^{\max} = k^{\max}+c$, and go back to the step 1.

3.3. The relaxed solution computation

Assuming continuous design variables, the relaxed solution will be used in the establishment of the initial structure and in a pruning rule of the tree searching process.

The proposed method is an exterior penalty function method. The optimal solution will be approached by a sequence of the linearized steps. Every step $r = 1, 2, \ldots$ can be characterized by the set of vectors $\{a_j^r, u_i^r, s_j^r\}$, where $a_j^r = a_j^{r-1} + \alpha \Delta a_j^r$. The procedure is terminated, when we reach the feasible region. The method is started from an infeasible $\{a_j^0, u_i^0, s_j^0\}$ structure, where every cross-section area is nearly zero, i.e. $a_j^0 = \xi$, $\xi \approx 0$.

The direction vector Δa_j^r, and the "distance of travel" α can be determined iteratively.

The direction vector Δa_j^r will be obtain from an LP problem, where the objective is the weight increment, subject to the linearized violated nodal displacement and member stress constraints. We solve the non-linear equilibrium equation system in the point $a_j^r = a_j^{r-1} + \alpha \Delta a_j^r$, which results the vector of exact displacements u_i^r, and the vector of exact stresses s_j^r. We set the step size α $(0 < \alpha \leq 1)$ by the limitation of the linearization error, i.e. by the limitation of the distance between the approximated and the exact nodal displacements and member stresses, and repeat the process until we reach an acceptable error level. A considerable computational advantage is that α changes very slowly from step to step.

The direction vector will be obtain from the following LP problem:

$$\rho l_j^0 \Delta a_j^r \to \min! \tag{18}$$

$$u_i^r + \widehat{\Delta u}_i^r \leq \overline{u_i} \qquad \text{if} \quad u_i^r \succ \overline{u_i}, \tag{19}$$

$$u_i^r + \widehat{\Delta u}_i^r \geq \underline{u_i} \qquad \text{if} \quad u_i^r \prec \overline{u_i}, \tag{20}$$

$$s_j^r + \widehat{\Delta} s_j^r \leq \overline{s_j} \qquad \text{if} \quad s_j^r \succ \overline{s_j}\,, \tag{21}$$

$$s_j^r + \widehat{\Delta} s_j^r \leq \underline{s_j} \qquad \text{if} \quad s_j^r \prec \overline{s_j}\,, \tag{22}$$

where:

$\widehat{\Delta} u_i^r$ – the approximated value of the incremental nodal displacements,

$\widehat{\Delta} s_j^r$ – the approximated value of the incremental stresses.

The approximated value of the incremental nodal displacements $\widehat{\Delta} u_i^r$ can be obtained from the following expression:

$$\widehat{\Delta} u_i^r = \alpha\, u_{i,j} \Delta a_j^r\,, \qquad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, e, \tag{23}$$

where $u_{i,j}$ can be computed by differentiation of the equilibrium equation system $V_{,k}(u_i) = 0$ of the structure,

$$V_{,kj} + V_{,ki}\, u_{i,j} = 0, \tag{24}$$

$$u_{i,j} = [V_{,ki}]^{-1}\, V_{,kj}\,. \tag{25}$$

The determination of the approximated value of the incremental stresses $\widehat{\Delta} s_j^r$ is similar:

$$\widehat{\Delta} s_j^r = \alpha\, s_{j,l}^r\, \Delta a_l^r\,, \qquad j = l = 1, 2, \ldots, e, \tag{26}$$

where:

$$s_{j,l}^r = s_{j,i}^r\, u_{i,l}. \tag{27}$$

Since

$$\widehat{\Delta} s_j^r = s_{j,i}^r\, \Delta u_i^r\,, \tag{28}$$

and substituting the expression (23), we obtain:

$$\widehat{\Delta} s_j^r = s_{j,i}^r\, u_{i,l}^r\, \Delta a_l^r\,. \tag{29}$$

4. NUMERICAL EXAMPLE

The proposed method has been successfully used for several problems appeared in the literature. The numerical results were compared with the results of the well-known ten bar truss (Fig. 1.) published by Gutkowski and Zawidzka [3]. The discrete minimum weight design of truss structure is formulated in terms of member cross-sectional areas, member stresses and nodal displacements. The truss is subject to a given $P = 10^5$ lb external nodal loads in nodes 2 and 4. The changes of the truss-geometry are described by the nodal displacements, using a geometrically non-linear, large deflection structural model. In this paper, only linear elastic material model has been considered. The modulus of elasticity is 10^7 psi, and the material density $\rho = 0.1\,\text{lb/in}^3$. The displacement limit is set to $u_{\max} = 2.0$ in along each degree of freedom. The stress constraint imposed on all members is $s_{\max} = 25000$ psi.

The design variables are cross-sectional areas A_i, $i = 1, 2, \ldots, 10$, selected from the following set of ten available values of the catalogue (in inches):

36.0 27.0 19.0 12,0 7.0 4.0 2.0 1.0 0.5 0.1 .

The starting values of the cross-sectional areas are obtained by converting down the relaxed solution to the value of the closest discrete neighbour. The resulted optimal structure is presented in Fig. 2. and in Table 1. Using the proposed method, the number of the evaluated states was 2469.

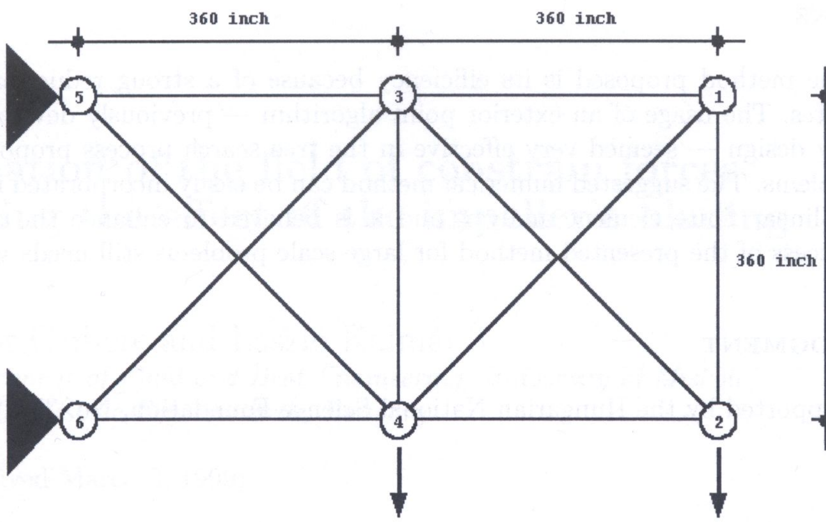

Fig. 1

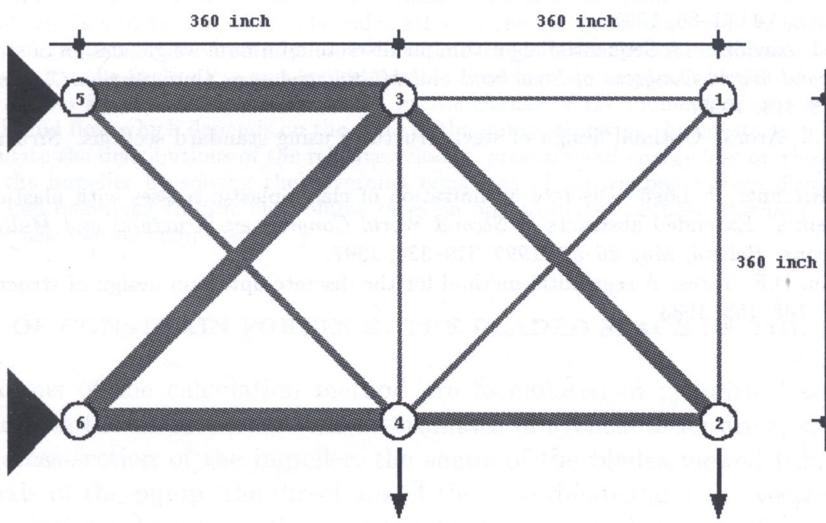

Fig. 2

Table 1. The optimum cross-sectional areas [in^2]

Present study:			Gutkowski and Zawidzka [3]:		
bars	relaxed m.	proposed m.	relaxed m.	seq. m.	enum. m.
1–2	0.6623	0.1	0.5665	0.5	2.0
1–3	0.1	0.1	0.1	0.5	0.5
1–4	0.1	0.1	0.1	0.5	0.1
2–3	21.5409	19.0	21.618	19.0	19.0
2–4	15.2160	19.0	15.286	12.0	19.0
3–4	0.1	0.5	0.1	0.5	0.5
3–5	30.4015	36.0	30.031	36.0	36.0
3–6	20.9631	19.0	21.198	27.0	19.0
4–5	7.5049	7.0	7.4683	7.0	7.0
4–6	23.1041	27.0	23.274	27.0	27.0
weight:	5056.15	5273.32	5061.6	5477.77	5356.18

5. Conclusions

The benefit of the method proposed is its efficiency because of a strong reduction in number of the evaluated states. The usage of an exterior point algorithm — previously developed by authors for truss topology design — seemed very effective in the tree search process proposed for discrete optimization problems. The suggested numerical method can be easily incorporated into a computer program for non-linear finite element analysis and it is believed to enhance the accuracy of the program. The success of the presented method for large scale problems still needs verification.

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