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Determination of the field of constrain forces representing the effect of the impeller's blading

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The aim of this paper is to present a numerical method to determine a field of constrain forces which hydrodynamically represents the effect of the blading of the impeller of a radial-flow pump. The field of constrain forces are perpendicular to the stream surfaces of the relative velocity field and congruent to the blade surface of the impeller. The calculation of the constrain force field is based on the solution of the inverse problem of the hydrodynamic cascade theory. In the determination of the constrain force it is supposed that the frictionless and incompressible fluid flow is completely attached to the blade surfaces. The constrain force field can be calculated by the change of the moment of momentum in the absolute inviscid fluid flow which depends on the state of the pump. Knowing the constrain force field it is possible to calculate the distributions of the relative velocity, pressure and energy loss on the mean stream surface (F) of the impeller by solving the governing equations of the viscous relative flow. By calculating the energy loss belonging to different volume rates an approximate real head-discharge characteristic of the impeller also can be computed.

1. THE FIELD OF CONSTRAIN FORCES IN THE BLADED SPACE OF THE IMPELLER

The basic equations of the calculation method are formulated in cylindrical co-ordinate system rotated together with the blades, where the co-ordinates in cyclical order are r, φ, z. Figure 1 shows the meridional cross-section of the impeller, the shape of the blades viewed from the direction of the rotational axis of the pump, the directions of the co-ordinates and the vectors of the absolute velocity $\mathbf{c}$, the relative velocity $\mathbf{w}$, the peripheral velocity $\mathbf{u}$ and the specific constrain force $\mathbf{f}$ at any arbitrary point of the blade surface. In the middle of Fig. 1 the velocity triangles can be seen at inlet-, outlet- and any arbitrary sections of the impeller. The blade angle β which can be measured between the tangents of the co-ordinate line $r =$ constant and blade surface at the same point is also shown in Fig. 1. The blade angle β uniquely determines the normal unit vector $\mathbf{n}$ of the blade surface.

The tangent unit vector of the blade surface can be written as

$$\mathbf{t} = \sin\beta\, \mathbf{e}_r - \cos\beta\, \mathbf{e}_\varphi$$

Knowing the components of vector $\mathbf{t}$, the normal unit vector $\mathbf{n}$ of the blade surface can be calculated as follows:

$$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \sin\beta\, \mathbf{e}_\varphi + \cos\beta\, \mathbf{e}_r \tag{1}$$

Let us go through the significant assumptions in determining the field of specific constrain forces $\mathbf{f}$:

- The specific constrain force $\mathbf{f}$ – similarly to every mechanical constrain force – is a friction-proof effect, in this way the specific constrain force $\mathbf{f}$ is parallel to the normal unit vector $\mathbf{n}$ of the blade surface

$$\mathbf{f} \times \mathbf{n} = \mathbf{0}\,. \tag{2}$$

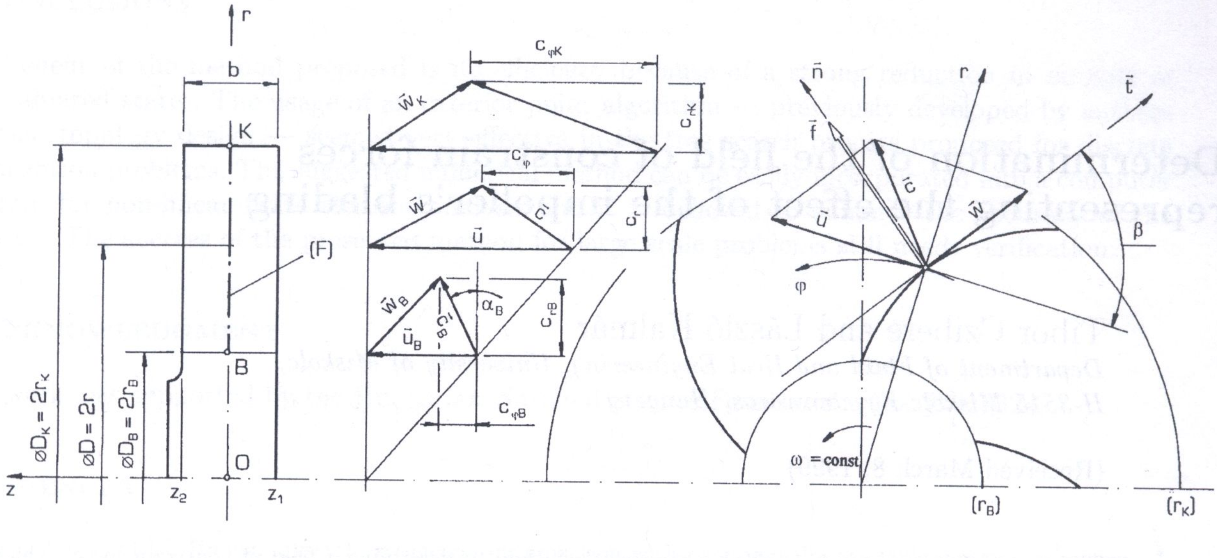

Fig. 1

Performing the vector multiplication in (2) yields

$$f_r\, n_\varphi = f_\varphi\, n_r\,. \tag{3}$$

- The frictional force is parallel to the wall near it, so the specific constrain force and the friction force are perpendicular to each other:

$$\mathbf{f} \cdot \sigma \cdot \mathbf{n} = 0 \tag{4}$$

where σ is the stress tensor. Carrying out the multiplications in Eq. (4), we get the following expression between the components of the specific constrain force, normal unit vector of the blade surface and the stress tensor:

$$f_r\, \tau_{r\varphi}\, n_\varphi + f_\varphi\, \tau_{\varphi r}\, n_r = 0\,. \tag{5}$$

- Since it is supposed that in the determination of the constrain force the fluid is inviscid, so the specific constrain force and the relative velocity vectors are perpendicular to each other

$$\mathbf{f} \cdot \mathbf{w} = 0. \tag{6}$$

- The mean surface (F) of the meridional channel is a stream surface of the relative flow consequently the relative velocity component of w_z is equal to zero. At the same time, along this stream surface, the stress vector (it is the third column of the stress tensor σ) and the relative velocity vector are parallel to each other. So we can write

$$\mathbf{w} \times \sigma_z = \mathbf{0}. \tag{7}$$

Performing the vector multiplication in Eq. (7), we get

$$\tau_{\varphi z}\, w_r = \tau_{rz}\, w_\varphi\,. \tag{8}$$

By using equations (1) and (3) we come to the expression between the different components of the specific constrain force as follows

$$f_r = f_\varphi \frac{n_r}{n_\varphi} = f_\varphi \cot\beta\,. \tag{9}$$

It is easy to realise that the component f_r of the constrain force can be determined by the geometrical data of the blades (β is the blade angle) from Eq. (9) if the component f_φ of the constrain force is known. The component f_φ of the constrain force is uniquely determined by the change of the moment of momentum in the absolute inviscid fluid flow.

Because the component f_z of the constrain force is equal to zero, thus by using Eq. (6), we can write

$$f_r\, w_r + f_\varphi\, w_\varphi = 0. \tag{10}$$

In view of Eqs. (3) and (5) and by using the fact that the stress tensor is a symmetrical tensor it is plausible that the component $\tau_{r\varphi}$ of the stress tensor has vanished.

Let us sum up the results obtained up to now. It is easy to see that along the co-ordinate surfaces determined by z = constant the unknown components w_r and w_φ of the relative velocity and the unknown components τ_{rz} and $\tau_{\varphi z}$ of the stress tensor depend on the co-ordinates r and z. The unknown components f_r and f_φ of the constrain force depend on the co-ordinate r.

The relative velocity vector and the magnitude of this vector on the main surface (F) can be expressed by the components of the relative velocity as follows

$$\mathbf{w} = w_r\, \mathbf{e}_r + w_\varphi\, \mathbf{e}_\varphi\,; \qquad w = \sqrt{w_r^2 + w_\varphi^2}\,. \tag{11}$$

The stress vector and the magnitude of this vector on the main surface (F) are calculated in a similar way

$$\sigma_z = \tau_{rz}\, \mathbf{e}_r + \tau_{\varphi z}\, \mathbf{e}_\varphi; \qquad \tau = \sqrt{\tau_{rz}^2 + \tau_{\varphi z}^2}\,, \tag{12}$$

where τ is the resultant shear stress.

The equation of the motion relating to the absolute inviscid fluid flow is as follows [5]

$$\operatorname{curl} \mathbf{c} \times \mathbf{w} = \mathbf{f} - \nabla \left(\frac{p}{\rho} + \frac{w^2}{2} - \frac{u^2}{2} \right).$$

Multiplying this equation by the co-ordinate unit vector $\mathbf{e}_\varphi$, we get the relationship for the calculation of the component f_φ of the constrain force:

$$f_\varphi = w_r\, (\operatorname{curl} \mathbf{c})_z = \frac{w_r}{r} \frac{\partial}{\partial r} (r\, c_\varphi)\,. \tag{13}$$

The product of $r\, c_\varphi$ in Eq. (13) means the change of the moment of momentum in absolute frictionless fluid flow. We should remark that the peripheral component f_φ of the constrain force is uniquely determined by the meridional velocity w_r and the change of the moment of momentum $r\, c_\varphi$ in the absolute frictionless fluid flow.

By assuming that the number of the blades are infinite, the energy transfer between the fluid and the blades is taking place continuously in the annulus domain determined by $r_B \le r \le r_K$. Thus the line integral of the absolute velocity for a closed curve in this domain, namely the circulation Γ_r belonging to the closed curve (L_r) at any arbitrary radius r can be calculated as follows (see Fig. 2):

$$\Gamma_r = \oint_{(L_r)} \mathbf{c} \cdot \mathbf{ds} = 2\pi\, (r\, c_\varphi - r_B\, c_{\varphi B})\,. \tag{14}$$

By using Eq. (14) it is possible to write down the expression to calculate the circulation difference $\Delta\Gamma$ belonging to the radius difference Δr by the subtraction of the line integrals for the closed curves of $(L_{r+\Delta r})$ and (L_r). Neglecting the terms which are small in second order yields

$$\Delta\Gamma = \oint_{(L_{r+\Delta r})} \mathbf{c} \cdot \mathbf{ds} - \oint_{(L_r)} \mathbf{c} \cdot \mathbf{ds} = 2\,\pi\, \Delta\, (r\, c_\varphi)\,.$$

The difference of the circulation $\Delta\Gamma$ can be expressed by the specific circulation γ as follows

$$\Delta\Gamma = \gamma\,\Delta r\,.$$

Comparing the last two equations, the change of the moment of momentum $r\,c_\varphi$ with respect to r is expressed by the following equation

$$\Delta\,(r\,c_\varphi) = \frac{\gamma}{2\,\pi}\,\Delta r\,. \tag{15}$$

By using the expressions developed above, the value of the circulation Γ_r at any arbitrary radius r in the interval of $r_B \le r \le r_K$ is calculated

$$\Gamma_r = 2\,\pi\,(r\,c_\varphi - r_B c_{\varphi B}) = \int_{r_B}^{r} \gamma\,\mathrm{d}r\,. \tag{16}$$

Of course, the integral of the specific circulation γ with respect to r between the inlet- and outlet cross-section of the impeller provides the total circulation Γ_{BK} of the impeller.

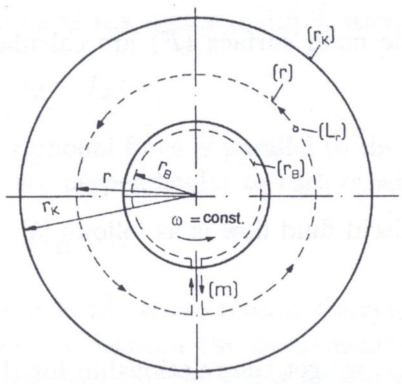

Fig. 2.

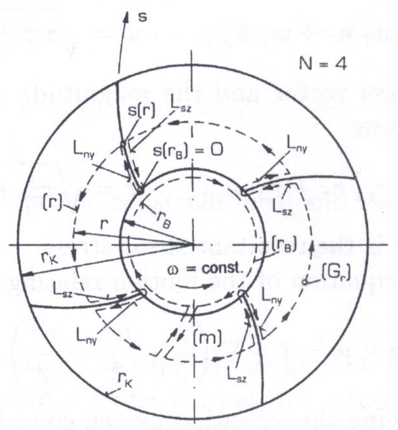

Fig. 3.

Considering Eqs. (13), (14), (15) and (16), we arrive at the relationship for the peripheral component f_φ of the constrain force

$$f_\varphi = \frac{w_r}{r}\,\frac{\gamma}{2\pi} \tag{17}$$

The meridional velocity component w_r and the specific circulation γ depend on the state of the operation and so consequently the constrain force also depends on the operating conditions of the pump.

Since the number of the blades N of a pump is finite, the energy transfer between the fluid and the blades is carried out only along the blades, but it is not taking place continuously in the annulus domain determined by $r_B \le r \le r_K$. It is easy to see that blade circulation comes from only the specific circulation placed on the blades and the absolute flow is vortex-free in the domain between any two neighbouring blades. Similarly to Eq. (14) the line integral of the absolute velocity for the closed curve (G_r) belonging to any arbitrary radius r (see Fig. 3) can be written down. By using Stokes' theorem yields

$$0 = \oint_{(G_r)} \mathbf{c}\cdot\mathrm{d}\mathbf{s} = \oint_{(r_B)} \mathbf{c}\cdot\mathrm{d}\mathbf{s} + \oint_{(r)} \mathbf{c}\cdot\mathrm{d}\mathbf{s} + N\left[\int_{(L_{ny})} \mathbf{c}\cdot\mathrm{d}\mathbf{s} - \int_{(L_{sz})} \mathbf{c}\cdot\mathrm{d}\mathbf{s}\right].$$

Comparing this equation with Eq. (14), it is evident that the circulation Γ_r can be determined by the line integrals of the absolute velocity along the curve sections (L_{sz}) and (L_{ny}) placed on the suction and pressure sides of the blades

$$\Gamma_r = N \left[\int\limits_{(L_{sz})} \mathbf{c} \cdot \mathrm{ds} - \int\limits_{(L_{ny})} \mathbf{c} \cdot \mathrm{ds} \right] = N \int\limits_{s(r_B)=0}^{s(r)} \left[\mathbf{c}|_{suc} - \mathbf{c}|_{pre} \right] \cdot \mathrm{ds}\,.$$

The circulation Γ_r can also be expressed by a specific vortex distribution γ_{blade} placed on the blade sections as it is well known in the field of the hydrodynamical cascade theory

$$\Gamma_r = N \int\limits_{s(r_B)=0}^{s(r)} \mathbf{\Delta c} \cdot \mathrm{ds} = N \int\limits_{s(r_B)=0}^{s(r)} \Delta c_t \,\mathrm{ds} = N \int\limits_{0}^{s(r)} \gamma_{\text{blade}} \,\mathrm{ds}\,. \tag{18}$$

By using Eqs. (15), (16) and (18) it is possible to develop the connection between the specific circulation γ belonging to infinite number of the blades and the vortex distribution γ_{blade} attached to the finite number N of the blades as follows

$$\gamma = N\,\gamma_{\text{blade}} \frac{\mathrm{d}s}{\mathrm{d}r} \tag{19}$$

The specific vortex distribution γ_{blade} is obtained from the solution of the second task of the hydrodynamical cascade theory relating to the absolute inviscid fluid flow [1].

For the 2-D flow on the mean surface (F) of the radial-flow impeller (see Fig. 1) the equation of continuity is

$$\frac{\partial}{\partial r}(r\, c_r) + \frac{\partial c_\varphi}{\partial \varphi} = 0\,. \tag{20}$$

The equation expressing the condition that the absolute flow is vortex-free can be written as follows

$$\frac{\partial}{\partial r}(r\, c_\varphi) - \frac{\partial c_r}{\partial \varphi} = 0\,. \tag{21}$$

To find the unknown distributions of the velocity components $c_r(r,\varphi)$ and $c_\varphi(r,\varphi)$ it is necessary to write down the boundary conditions belonging to the system of partial differential equations (20) and (20). By knowing the shape of the blades of the impeller these boundary conditions can be written at $r = r_B$ (at the inlet section of the impeller)

$$c_r = c_{rB}; \qquad c_\varphi = c_{\varphi B}$$

As it is well known, the specific vortex distribution γ_{blade}, which is hydrodynamically represented by the blade effect, can be determined by applying the following numerical procedure.

The 2-D flow on the mean surface (F) of the impeller can be mapped on to the one on a complex plane. Since the absolute flow remains vortex-free on this plane, the expressions for the determination of the velocity distribution can be obtained by the solution of a Poisson-like partial differential equation by using the theory of potentials. The kinematical condition of the relative flow can also be written down, which expresses that the blade section is a streamline of the relative flow. The specific vortex distribution γ_{blade} can be looked for as a form of a trigonometric series. By substituting this series into the governing equations and satisfying the kinematical conditions at discrete points along the blade sections the unknown coefficients of the series can be calculated by solving a system of linear equations. Knowing these coefficients, the current vortex distribution, the components of the constrain force, the velocity and pressure distribution along the blade sections and the bladed space of the impeller, the characteristics of optimal state of the impeller and the theoretical characteristics of the impeller can be calculated, belonging to different operating states of the impeller [1, 2, 7].

2. APPLICATION OF THE CALCULATION METHOD

To determine the specific vortex distribution attached to the finite number of the blades, and then using that to calculate the components of the constrain forces, is necessary for analysing the turbulent flow in the bladed space of the turbomachines. In our calculation method the effects of the blading, fluid friction and turbulence of the flow are taken into consideration separately. The effect of the blading of the impeller is represented hydrodynamically by the field of constrain forces [2].

Computerised solution as an application of the numerical method presented above is carried out for the impeller of a radial-flow pump designed by cylindrical blades with constant thickness. The first main step of this calculation method is to determine the change of the moment of momentum of the absolute inviscid fluid flow based on the solution of the second task of the hydrodynamic cascade theory. The vortex distribution attached to the finite number of the blades can be calculated by the results of the second task which is necessary to determine the components of the constrain force field.

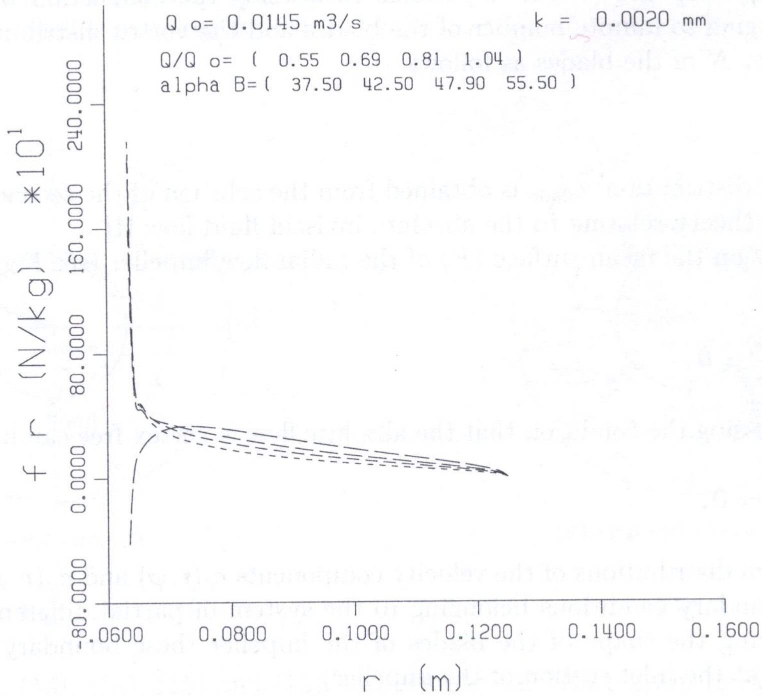

Fig. 4. Distribution of the radial component of the constrain force

The final result of this computation are presented here. Figures 4 and 5 show the distributions of the components of the constrain force field belonging to five different volume rates. The current volume rate is noted by Q and the volume rate at the optimal state of the pump by Q_o. In every case the ratio between the two volume rates are printed on the top of the figure. Under the title of the figures the current values of the roughness height k of the wetted walls of the pump and the angle α_B determined by the direction of the absolute velocity at inlet section of the impeller belonging to the different volume rates are also printed. In Fig. 4 the distributions of the radial component of the constrain force with respect to r can be seen. In Fig. 5 the distributions of the peripheral component of the constrain force with respect to the radius are plotted belonging to the same volume rates.

By knowing the components of the constrain force it is possible to solve a system of the ordinary differential equations based on the equations of the continuity, motion and energy of the viscous relative flow on the mean stream surface (F) of the impeller. to determine the distributions of the relative velocity, pressure and energy loss can be calculated [2, 4, 6, 7, 9].

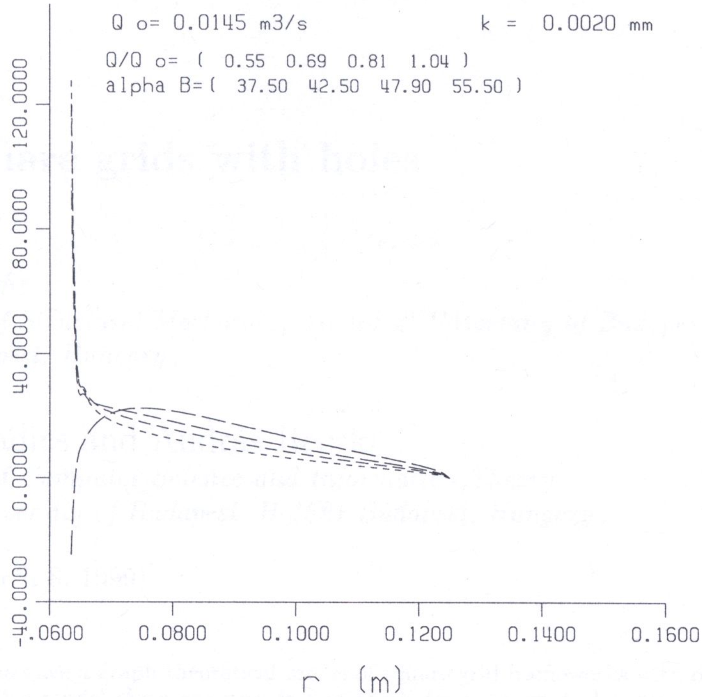

Fig. 5. Distribution of the peripheral component of the constrain force

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