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Homogenization of sandwich panels

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The numerical modeling of plates with periodic corrugation requires some efforts to be made in terms of careful and precise discretization of the complicated structure. This automatically generates very computationally expensive models. One of the most popular methods of model simplification is analytical or numerical homogenization. The main goal of this paper is to present the homogenization techniques that can be used to effectively model sandwich panels such as corrugated plates in an elastic phase. Two methods of different complexity are described: homogenization through application of the classical laminated plate theory and homogenization through the deformation energy-equivalence method. The accuracy of these methods is compared with the literature data and the results of a structural sample in two basic tests, i.e., the four-point bending test and the uniaxial tensile test. The results show that each method provides similar effective parameters which proves the robustness of the presented methods.

Keywords: homogenization, finite element method, corrugated cardboard.

1. Introduction

Sandwich panels or laminated plates are widely used as bearing structures in many industries such as: packaging, structural engineering, shipbuilding, automotive, aerospace and many more.

Sandwich panels consist of a lightweight core material inserted in between two or more stiff plates. The core structure used depends on the application of the plate, and can be built from solid materials such as Styrofoam, PUR foams, rock wool or can be built from a curved thin material shaped in to triangles, hats or sinewaves, or even 3D trusses usually shaped in to pyramid-like units. Another important class of sandwich panels can also have a core composed of honeycomb. Such structures are characterized by great transverse load strength which depends on the stability of the core's walls. The greatest advantage of sandwich plates is their ratio of bending stiffness to weight of element, as it leads to economical efficiency. Important types of sandwich panels include: corrugated cardboard, used to build cardboard boxes (one of the most important container types in transportation and storage) and steel or aluminum sandwich panels with a corrugated core used for building ship hulls or aircraft. The polymer sandwich panels with a honeycomb core are used in such fields as Formula 1 cars due to their load resistance and primarily light self-weight. Sandwich panels with a thermal-insulating core are one of the most important prefabricated materials in structural engineering. They are used as walls and roofs and are typically applied on factory floors.

The complicated geometry of the core is usually a factor that prevents numerical modeling of large structures consisting of sandwich panels as it enforces a very fine mesh to be used, effectively making computations very expensive. In order to overcome this problem, many homogenization techniques have been developed. Homogenization is a process in which a heterogeneous body is replaced with a homogeneous material with a stiffness equal to average the stiffness of the heterogeneous one. This approach can lead to great problem-size reduction as standard shell elements are used instead, thus significantly fewer elements are used to mesh a domain.

Over the years, many homogenization techniques have been developed that are applicable to various problems including sandwich panels, continuum bodies with inclusions, or solids in which the micro-structure is known and ordered. Within the field of sandwich panels, multiple methods of varying complexity have been presented. Each method has its own range of application which depends both on the geometry of the core and the constitutive model of layers that can be captured. One of the simplest methods is the classical laminated plate theory (CLPT). This theory can be used to predict the effective stiffness of either panels consisting of flat layers such as carbon polymers of different orientations submerged in epoxy resins [18] or panels with a corrugated core that is periodic in one plane direction, such as cardboards and corrugated steel plates [2, 5, 19]. Another method is the standard Kirchhoff-Loves plate theory where the stiffness of a layered panel is obtained by integrating constitutive equations over a whole section of the plate. This approach limits its application to the elastic regime but due to its simplicity. It can be used without sophisticated finite element method (FEM) software. Another approach to homogenization was presented in [4, 16], where the effective elastic parameters of corrugated boards were obtained using the deformation energy-equivalence method (DEEM) of structural and homogeneous samples, which satisfy the Hill-Mandel condition. In this case, the FEM model of a structure must be built in order to obtain its global stiffness matrix. In return, some geometrical changes such as inclusions or discontinuities can be incorporated into the computation to some extent. A comprehensive description of all the methods mentioned above, with particular application to corrugated cardboards can be found in [8–11, 14].

In order to model panels with more complicated constitutive behavior (such as plasticity, damage through delamination, fracturing, etc.) or more complicated geometry (such as honeycomb or space trusses), other homogenization methods must be used. One of the most popular technique is called asymptotic homogenization (AH) which is based on the asymptotic expansion theory [6, 13, 15, 20]. Although this method can be used for panels of an arbitrary geometry and a constitutive formulation, it requires a great deal of effort to be implemented, such as modifying the FEM formulation thus making it a non-standard tool for engineering use. Another popular method that can include the non-linear mechanics of sandwich panels is called the multi-scale modeling, in particular the Arlequin method [12, 17], which enables macro and micro scales to be modeled separately.

In this work, two different homogenization techniques (based on the CLPT and the DEEM) are presented. These can be used for effective modeling of sandwich panels with a corrugated core in an elastic phase. The authors present two algorithms associated with each method and then test them against data available in the literature concerning the effective parameters of corrugated steel sandwich panels. Finally, the responses of the homogenized models are compared to the structural model responses for the selected cardboard structures in the four-point bending test and the uniaxial tensile test.

2. Mathematical formulation of plates

2.1. Shell stiffness

The plate is a structural element that carries transverse loads and bending moments, where one of its dimension is significantly lower than the others. If this element is allowed to be loaded also with in-plane forces then it is referred to as a shell element. In most of the applications, sandwich structures can be modeled with shell elements.

Many plate theories have been developed, though there are two that are most commonly used to model sandwich panels: the Kirchhoff-Loves plate theory and the Reissner-Mindlin's plate theory.

The former is used for the modeling of thin plates (whose ratio of thickness to smallest in-plane dimension is lower than 1/10), the latter is used for thick plates. According to the classical plate theory (the Kirchhoff-Loves theory), the strains over the thickness of an element can be disassembled into mid-plane straining and straining contributed by curvature which is linear over the thickness:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_m + z\boldsymbol{\kappa}. \tag{1}$$

In case of the linear elasticity, it is assumed that the shell elements undergo the plane stress condition, and stresses can be obtained using the Hooke's law:

$$\boldsymbol{\sigma} = \mathbf{Q}(\boldsymbol{\varepsilon}_m + z\boldsymbol{\kappa}), \tag{2}$$

where $\mathbf{Q}$ is the plane stress material stiffness matrix, formulated for an orthotropic material in the following way:

$$\mathbf{Q} = \mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}^{-1}. \tag{3}$$

In order to reduce the problem's dimensionality from 3D to 2D, the shell elements are usually formulated using internal forces instead of stresses [21]. The internal forces consist of the in-plane normal and shearing forces N_x, N_y, N_{xy} and bending moments M_x, M_y, M_{xy} which can be obtained using the following definitions:

$$\begin{aligned} \mathbf{N} &= \int_{-t/2}^{t/2} \sigma dz = \int_{-t/2}^{t/2} \mathbf{Q}(\varepsilon_m + z\kappa)dz, \\ \mathbf{M} &= \int_{-t/2}^{t/2} \sigma z dz = \int_{-t/2}^{t/2} \mathbf{Q}(\varepsilon_m + z\kappa)z dz. \end{aligned} \tag{4}$$

This system of equations can be written in an alternative form:

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix}, \tag{5}$$

where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ can be obtained from (4):

$$\begin{aligned} \mathbf{A} &= \int_{-t/2}^{t/2} \mathbf{Q}dz, \\ \mathbf{B} &= \int_{-t/2}^{t/2} \mathbf{Q}z dz, \\ \mathbf{D} &= \int_{-t/2}^{t/2} \mathbf{Q}z^2 dz. \end{aligned} \tag{6}$$

Assembling the above matrices we can obtain:

$$\mathbf{ABD} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix},$$

which is called the general shell stiffness matrix. The primary focus of homogenization is to identify this matrix for a given sandwich panel.

Provided that the **ABD** matrix is identified the effective parameters, important if one would like to use a commercial software to define shells, have to be recovered. The effective parameters can be obtained by using Eq. (6), which for a particular thickness of the homogenized plate t reduces to:

$$\begin{aligned} \mathbf{A} &= \mathbf{Q}\ t_{hom}, \\ \mathbf{B} &= 0, \\ \mathbf{D} &= \frac{\mathbf{Q}\ t_{hom}^3}{12}. \end{aligned} \tag{7}$$

The material stiffness matrix can be obtained with either the **A** or **D** matrix. The effective parameters are then recovered by inverting the **Q** matrix and reading the appropriate terms of the material compliance matrix. As sections of structural and homogenized panels are different, the distribution of the normal stresses along the thickness is not conserved in the process of homogenization. This leads to the conclusion that for an arbitrary thickness of a homogenized panel, only one of two sub-matrices will be recovered correctly. In order to overcome this problem, the effective thickness must be firstly approximated. Such thickness has a property that allows to formulate the **ABD** matrix with a set of effective elastic properties. In other words, both matrices **A** and **D** will be properly formulated. The effective thickness can be then approximated using the equation [1]:

$$t_{hom} = \sqrt{12 \frac{D_{11} + D_{22} + D_{33}}{A_{11} + A_{22} + A_{33}}}, \tag{8}$$

by making use of this thickness, one can recover two sets of parameters through definitions of **A** and **D**, and these sets will not differ significantly. One can decide, based on the application of the sandwich panel, to use either set from in-plane stiffness, bending stiffness or an average of those two.

3. Homogenization techniques

3.1. Homogenization based on the classical laminated plate theory

The CLPT is usually used for sandwich panels consisting of several flat layers with different material orientations. Such structure is commonly created with polymer composites with each layer differently oriented in its plane (the orientation of the polymer chains on a matrix provides orthotropy in each layer). Usually each layer is rotated against the neighboring layers by either 30, 45 or 90 degrees.

As shown in Fig. 1 the laminate sandwich panel consists of a few layers of different thicknesses, material types and material orientations. In order to calculate the general shell stiffness matrix, one must define the mid-surface, which lies in the geometric middle of the sandwich panel. Each i-th layer contributes to the stiffness of the sandwich panel. The distance from the top boundary to

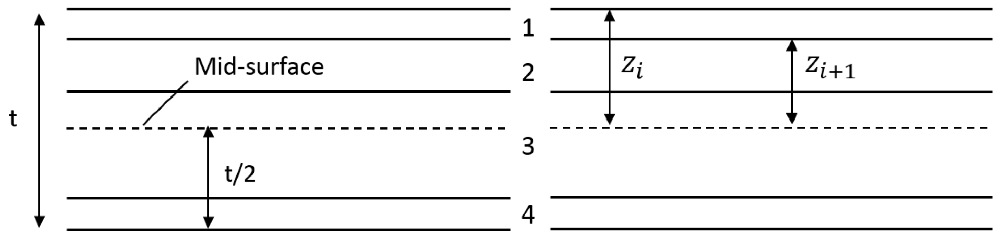

Fig. 1. Example of a four-layered laminate panel. Definition of mid-surface and convention of layer's boundary coordinate.

the mid-surface is defined by z_i while the distance from the bottom boundary by z_{i+1}. The general shell stiffness is then determined by using the following definition (6):

$$
\begin{aligned}
\mathbf{A} &= \int_{-t/2}^{t/2} \mathbf{Q}_i^* dz = \sum_{i=1}^{n} \mathbf{Q}_i^*(z_i - z_{i+1}),\\
\mathbf{B} &= \int_{-t/2}^{t/2} \mathbf{Q}_i^* z dz = \frac{1}{2}\sum_{i=1}^{n} \mathbf{Q}_i^*(z_i^2 - z_{i+1}^2),\\
\mathbf{D} &= \int_{-t/2}^{t/2} \mathbf{Q}_i^* z^2 dz = \frac{1}{3}\sum_{i=1}^{n} \mathbf{Q}_i^*(z_i^3 - z_{i+1}^3).
\end{aligned}
\tag{9}
$$

One must bear in mind that $\boldsymbol{Q}_i^*$ is built with material parameters' values according to each layer's orientation. The in-plane material stiffness matrix can be rotated by using transformation matrices:

$$
\begin{aligned}
\boldsymbol{Q}^* &= \boldsymbol{J}_{z,\sigma}\boldsymbol{Q}_{xyz}\boldsymbol{J}_{z,\varepsilon}^{-1},\\
\boldsymbol{J}_{z,\sigma} &= \begin{bmatrix}
c^2 & s^2 & 0 & 0 & 0 & 2cs\\
s^2 & c^2 & 0 & 0 & 0 & -2cs\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & c & s & 0\\
0 & 0 & 0 & -s & c & 0\\
-cs & cs & 0 & 0 & 0 & c^2 - s^2
\end{bmatrix},\\
\boldsymbol{J}_{z,\varepsilon} &= \begin{bmatrix}
c^2 & s^2 & 0 & 0 & 0 & cs\\
s^2 & c^2 & 0 & 0 & 0 & -cs\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & c & s & 0\\
0 & 0 & 0 & -s & c & 0\\
-\frac{1}{2}cs & \frac{1}{2}cs & 0 & 0 & 0 & c^2 - s^2
\end{bmatrix},
\end{aligned}
\tag{10}
$$

where c and s are the cosine and sine of the orientation angle (in-plane), respectively.

This method is very straightforward and widely used in composite modeling due to the fact that many commercial FEM systems make it possible to define a shell with a composite section. This means that the process of homogenization is automated. It can also be extended to panels in which not all layers are flat but can be, for example, corrugated. In other words, it can be adopted to calculate the stiffness of a sandwich panel whose core is made from a thin curved sheet, e.g., sine wave-like, semicircular, trapezoidal, triangular or elliptical arc-like core. The CLPT allows to calculate the **ABD** matrix by integrating of the cardboard section's stiffness according to the following formulas [2, 3, 19]:

$$\begin{aligned}
\mathbf{A}(x) &= \int \mathbf{Q}dz = t_{ls}\mathbf{Q}_{ls} + t_f\mathbf{Q}_f(\theta) + t_{li}\mathbf{Q}_{li},\\
\mathbf{B}(x) &= \int \mathbf{Q}zdz = t_{ls}\mathbf{Q}_{ls}z_{ls} + t_f\mathbf{Q}_f(\theta)z_f + t_{li}\mathbf{Q}_{li}z_{li},\\
\mathbf{D}(x) &= \int \mathbf{Q}z^2dz = \mathbf{Q}_{ls}\left(z_{ls}^2t_ls + \frac{t_{ls}^3}{12}\right) + \mathbf{Q}_f(\theta)\left(z_f^2t_{vf} + \frac{t_f^3}{12}\right) + \mathbf{Q}_{li}\left(z_{li}^2t_{li} + \frac{t_{li}^3}{12}\right),
\end{aligned} \tag{11}$$

where subscripts ls, li and f denote the superior liner (flat layer), the interior liner (flat layer) and the fluting (corrugated layer), respectively.

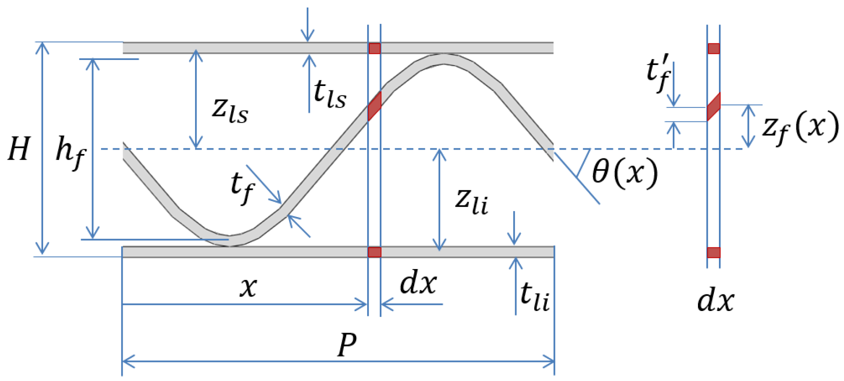

Fig. 2. Cardboard section with its characteristic CLPT properties.

It is vital to understand that due to the fluting's shape, the material parameters for each section differ in the global coordinate system (as the local system is constantly rotating). Thus for each segment dx, the fluting parameters are aligned with the rotated section. If the fluting's position is described with the sine function:

$$h(x) = \frac{h_f}{2}\sin\left(\frac{2\pi x}{P}\right),$$

where h_f is a distance between liners and P is a fluting's period, then the rotation angle is given by

$$\theta(x) = \arctan\left(\frac{dh(x)}{dx}\right). \tag{12}$$

The $\mathbf{Q}$ matrix can be obtained by an inversion of the compliance matrix in the global coordinate system. Unfortunately the compliance matrix must be built as a full 3D matrix (with nine parameters), rotated, reduced to a standard 2D matrix and then inverted. It is suggested when using this method that all out-of-plane parameters should be set as very small in comparison to in-plane parameters (even for the isotropic materials). The rotation of the compliance matrix can be accomplished by a matrix transformation:

$$\mathbf{C}_{xyz} = \mathbf{T}\ \mathbf{C}_{123}\ \mathbf{T}^T, \tag{13}$$

where xyz is a global coordinate system, 123 is the local coordinate system of a material and the rotation matrix $\mathbf{T}$ is defined as

$$\mathbf{T} = \begin{bmatrix} c^2 & 0 & s^2 & 0 & sc & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ s^2 & 0 & c^2 & 0 & -sc & 0 \\ 0 & 0 & 0 & c & 0 & -s \\ -2cs & 0 & 2cs & 0 & c^2 - s^2 & 0 \\ 0 & 0 & 0 & s & 0 & c \end{bmatrix}, \tag{14}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$. The transformation matrix here differs from the transformation matrix in (11) due to different rotation schemes.

The total stiffness of the section is now obtained by integrating the section's stiffness over an interval of the fluting's period:

$$\mathbf{A}_h = \frac{1}{P}\int_0^P \mathbf{A}(x)dx,$$
$$\mathbf{B}_h = \frac{1}{P}\int_0^P \mathbf{B}(x)dx, \tag{15}$$
$$\mathbf{D}_h = \frac{1}{P}\int_0^P \mathbf{D}(x)dx.$$

This method, though simple and straightforward to implement, provides good homogenization in the elastic region. It requires a numerical integration scheme, e.g., a mid-point rule with a fine integration step to achieve appropriate accuracy. Unfortunately, this method is suitable for the sandwich panels which are periodic in only one direction.

3.2. Homogenization using the deformation energy-equivalence method

The homogenization technique based on energy-equivalence was proposed in [4, 16]. It was suggested that the finite element (FE) model of the representative volume element (RVE) can be used to obtain the equivalent elastic properties of a corrugated cardboard [4]. The RVE is a minimal structure that periodically repeats itself in a whole sandwich structure. For example, in the corrugated cardboard the RVE would be an element whose length is equal to the fluting's period. Such an RVE can be modeled using shell elements to obtain the structure's stiffness. Provided none of the internal nodes are loaded, the following equation is fulfilled:

$$\overline{\mathbf{K}}\boldsymbol{u}_e = \mathbf{F}_e, \tag{16}$$

where $\overline{\mathbf{K}}$ is the stiffness matrix of external (boundary) degrees of freedom (DOFs) only, which is obtained using static condensation.

The overall stiffness matrix of a structure can be arranged as

$$\begin{bmatrix} \mathbf{K}_{ee} & \mathbf{K}_{ei} \\ \mathbf{K}_{ie} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_e \\ \boldsymbol{u}_i \end{bmatrix} = \begin{bmatrix} \mathbf{F}_e \\ \mathbf{0} \end{bmatrix}, \tag{17}$$

where subscripts e and i correspond to the external and internal degrees of freedom, respectively.

Transforming this equation to (16) leads to the following formula for $\overline{\mathbf{K}}$:

$$\overline{\mathbf{K}} = \mathbf{K}_{ee} - \mathbf{K}_{ei}\mathbf{K}_{ii}^{-1}\mathbf{K}_{ie}. \tag{18}$$

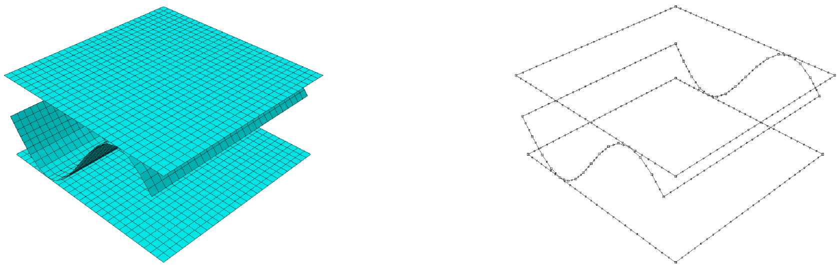

Fig. 3. An RVE and its external nodes used in the energy-equivalence homogenization technique.

Assuming that the displacement vector in the external nodes is known, the total energy stored in a system is given by

$$E = \frac{1}{2}\boldsymbol{u}_e^T\mathbf{F}_e. \tag{19}$$

Assuming the Kirchhoff-Loves plate theory and a constant strain distribution over an element, one can show that the displacement of the node is related to the strain of that node with the formula:

$$\begin{bmatrix} \vdots \\ \begin{bmatrix} u_x \\ u_y \\ u_z \\ \phi_x \\ \phi_y \end{bmatrix}_j \\ \vdots \end{bmatrix}^T = \begin{bmatrix} \vdots \\ [\boldsymbol{A}_e]_j \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}_j \\ \vdots \end{bmatrix}, \tag{20}$$

or simply:

$$\boldsymbol{u}_j^T = [\boldsymbol{A}_e]_j \boldsymbol{\kappa}_j, \tag{21}$$

where $[\mathbf{A}_e]_j$ represents the displacement distribution over an element given the strains, which for the shell elements are represented by

$$[\mathbf{A}_e]_j = \begin{bmatrix} x^j & 0 & \frac{y^j}{2} & x^j z^j & 0 & \frac{y^j z^j}{2} \\ 0 & y^j & \frac{x^j}{2} & 0 & y^j z^j & \frac{x^j z^j}{2} \\ 0 & 0 & 0 & -\frac{x^j x^j}{2} & -\frac{y^j y^j}{2} & -\frac{x^j y^j}{2} \\ 0 & 0 & 0 & 0 & -y^j & -\frac{x^j}{2} \\ 0 & 0 & 0 & x^j & 0 & \frac{y^j}{2} \end{bmatrix}. \tag{22}$$

Here x^j, y^j, z^j are coordinates of the j-th node of the FE model. Substituting (16) and (21) with (18) leads to:

$$E = \frac{1}{2}\boldsymbol{u}_e^T \overline{\mathbf{K}} \boldsymbol{u}_e = \frac{1}{2}\boldsymbol{\kappa}^T \mathbf{A}_e^T \overline{\mathbf{K}} \mathbf{A}_e \boldsymbol{\kappa}. \tag{23}$$

On the other hand, the internal energy of the shell subjected to bending can be represented by

$$E = \frac{1}{2}\boldsymbol{\kappa}^T [\mathbf{ABD}] \boldsymbol{\kappa} \{AREA\}. \tag{24}$$

Imposing equality on both of those energy representations leads to the well-known [**ABD**] matrix, which describes homogenized shell stiffness and can be used to obtain the effective elastic parameters:

$$[\mathbf{ABD}] = \frac{\mathbf{A}_e^T \overline{\mathbf{K}} \mathbf{A}_e}{\{AREA\}}. \tag{25}$$

This method requires the FE model of the RVE to be built and the assembled global stiffness matrix to be extracted. In fact, one does not have to define the forces or boundary conditions or to calculate a model, simply extracting the stiffness matrix and numbers of the external nodes is sufficient.

4. Results

In order to verify the proposed methods, a few tests are conducted here. Firstly, a sandwich panel of known constituents is homogenized, then the responses of the homogeneous and structural sample are compared in the simple bending and uniaxial tensile tests. Such a procedure shows the effectiveness of the homogenization process. As different methods are compared here, a similar outcome is expected to be found in all of them. Moreover, if the homogeneous and structural responses are similar, one can assume that a simplified model can give the desired results and can be used instead of a typical structural model. Furthermore, the accuracy of both the CLPT and the DEEM homogenization could be compared.

The panel under consideration is presented in Fig. 4. It is a symmetrical three-layered steel panel with an internal layer formed in circular shape. Its properties are given in Tables 1 and 2. The panel is homogenized in [5] by means of asymptotic homogenization AH which is the widely accepted method for numerous homogenization problems, thus our results can be compared with those given in the literature.

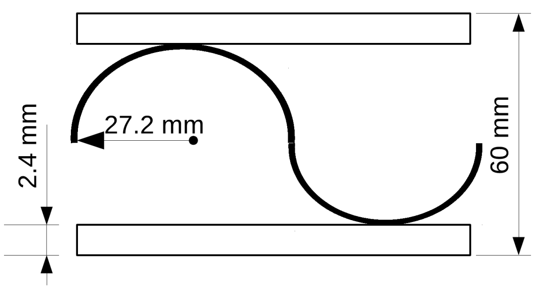

Fig. 4. Geometry of the analyzed sandwich panel.

In order to obtain the general stiffness matrix using the CLPT, a simple numerical integration scheme is adopted, the midpoint rule. The RVE's section is divided into 5000 intervals and then a simple numerical integration is carried out according to the algorithm presented in Subsec. 3.1.

Table 1. Properties of constituents of the sandwich panel used in the experiment.

	t [mm]	E [GPa]	ν [–]	G [GPa]
Top Liner	2.4	210	0.3	85
Fluting	0.8	210	0.3	85
Bottom Liner	2.4	210	0.3	85

Table 2. Geometric characteristics of the fluting.

Wavelength of fluting [mm]	Height of core [mm]
109.6	55.2

In order to implement the DEEM-based homogenization, the RVE is built in MATLAB Toolbox: FEMat [22] environment and meshed with the quadrilateral isoparametric shell elements (S4) whose size is approximately 5 mm. The global stiffness matrix and the coordinates of the external nodes are extracted and the general stiffness matrix is obtained according to the algorithm presented in Subsec. 3.2.

Table 3. The effective parameters of a steel sandwich panel obtained from the membrane stiffness matrices for the homogenized thickness of 5 mm.

Method	E_1 [GPa]	E_2 [GPa]	ν	G [GPa]
CLPT	205.59	255.94	0.241	81.64
DEEM	205.57	254.80	0.2426	90.65
Reference (AH)	205.4	254.4	0.242	85.76

Table 4. The effective parameters of a steel sandwich panel obtained from the bending stiffness matrices for the homogenized thickness of 5 mm.

Method	E_1 [TPa]	E_2 [TPa]	ν	G [TPa]
CLPT	81.21	91.30	0.267	32.52
DEEM	81.01	88.84	0.273	34.29
Reference (AH)	81.07	89.68	0.271	30.89

As seen in Table 4, the results obtained using both of presented methods have good agreement with the reference data. The biggest discrepancy is found in the in-plane shearing modulus G_{12}, differing by more than 10%. Additionally, it can be noted that even though the material is isotropic in terms of the constitutive relationship, the corrugated structure is not. This is due to the fact that the fluting does not carry much loading in the direction of corrugation while it gives additional stiffness for the other in-plane directions. The results provide evidence that for any effective stiffness, the equivalent parameters derived from the membrane and bending stiffness are different (here thickness was taken as 5 mm). Table 6 provides an evidence that Eq. (8) indeed gives a good estimation of effective thickness as the effective parameters acquired from **A** and **D** differ only by a factor of approximately 10%. Given this error, it is suggested to use an average from the **A** and **D** values.

Table 5. The effective parameters of a steel sandwich panel obtained from the membrane stiffness matrices for effective thickness.

Method	t [mm]	E_1 [GPa]	E_2 [GPa]	ν	G [GPa]
CLPT	97.46	10.55	13.13	0.241	4.19
DEEM	96.63	10.64	13.19	0.2426	4.69

Table 6. The effective parameters of a steel sandwich panel obtained from the bending stiffness matrices for effective thickness.

Method	t [mm]	E_1 [TPa]	E_2 [TPa]	ν	G [TPa]
CLPT	97.46	10.97	12.33	0.267	4.39
DEEM	96.63	11.23	12.31	0.273	4.75

The second part of the verification of the two methods consists of the standard bending and tension tests. Samples of sizes 300×100 mm prepared both in the machine direction (MD) and the cross direction (CD) (the two main in-plane orthotropic directions for corrugated cardboard) are tested in the three-point bending tests and the uniaxial tension tests. For the given type of cardboard (KLSKL595C [4]), the effective parameters are obtained using both methods. The homogeneous models are also checked in the standard tests in order to compare the characteristic displacements, which are presented in Table 9.

Table 7. Properties of constituents of the KLSKL595C corrugated cardboard used in the experiment.

	t [mm]	E_1 [MPa]	E_2 [MPa]	ν_{12} [–]	G_{12} [MPa]
Top Liner	0.29	3326	1694	0.34	859
Fluting	0.3	2614	1532	0.32	724
Bottom Liner	0.29	3326	1694	0.34	859

Table 8. Geometric characteristics of the fluting.

Wavelength of fluting [mm]	Height of core [mm]
8	3.8

Table 9. Difference of the response of the structural sample against the homogeneous plates.

Sample	Bending MD sample [mm]	Bending CD sample [mm]	Elongation MD sample [mm]	Elongation CD sample [mm]
Structural	6.06	9.82	0.101	0.122
CLPT	6.31	9.12	0.090	0.128
DEEM	6.30	9.13	0.092	0.130

5. Conclusions

In this work, two basic concepts of homogenization in the elastic regime were presented. The two techniques namely the CLPT and the DEEM were discussed with a reference to corrugated cardboard. It was shown that both methods provide reasonable approximation to the structural model, although their application is limited to the elastic regime. The implementation of both methods is straightforward for corrugated plates but the DEEM approach seems to be more tedious as it requires the FE model representation of the RVE to be built, while the CLPT relies only on a simple numerical integration scheme. Moreover, both methods are not limited to structures with a sine wave like core, in fact the core can have any shape (including a continuum core – e.g., made from foam).

The proposed methods were tested against structural response in the simple bending and the uniaxial tests in both MD and CD directions. The biggest error was noted for the tension test in MD, estimated as 10%. Moreover, both methods gave very similar results even though the acquired effective parameters differed by a factor of 5–10%. This is a satisfactory approximation given that the FE model is reduced by a factor of up to 100 times (depending on the sample's size).

In order to include more complicated mechanical properties such as perforations or plastic behavior of constituents, other methods must be used. Among the most prominent methods one might mention AH [5, 7, 15, 20] and multi-scale homogenization. Both of these methods can offer the full range of mechanical behaviors but they require much more sophisticated algorithms and much more computational power. This is the reason why, for simpler applications, the CLPT is recommended.

The main advantage of the proposed methods, compared to more sophisticated approaches, lays in their simplicity and robustness. Having a reliable tool that can automatically deliver a set of elastic orthotropic parameters is extremely useful if one would like to approximate a heterogeneous structural model with its computationally economical homogeneous version. The computational speed-up factor reaches 100, while the accuracy of the model response is kept within ±10%, which in our opinion is a good compromise between computational accuracy and calculation time.

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