Efficient free vibration analysis of large structures with close or multiple natural frequencies. Part II: Damped structures

In-Won Lee, Hyung-Jo Jung and Man-Cheol Kim

Department of Civil Engineering
Korea Advanced Institute of Science and Technology
Taejon 305-701, Korea

An efficient solution method is presented to solve the eigenvalue problem arising in the dynamic analysis of nonproportionally damped structural systems with close or multiple eigenvalues. The proposed method is obtained by applying the modified Newton–Raphson technique and the orthonormal condition of the eigenvectors to the linear eigenproblem format through matrix augmentation of the quadratic eigenvalue problem. In the iteration methods such as the inverse iteration method and the subspace iteration method, singularity may be occurred during the factorizing process when the shift value is close to an eigenvalue of the system. However, even if the shift value is an eigenvalue of the system, the proposed method guarantees nonsingularity, which is analytically proved. The initial values of the proposed method can be taken as the intermediate results of iteration methods or results of approximate methods. Two numerical examples are also presented to demonstrate the effectiveness of the proposed method and the results are compared with those of the well-known subspace iteration method and the Lanczos method.

Keywords: quadratic eigenproblem, eigenvalue, non-proportional damped system

1. INTRODUCTION

In the analysis of dynamic response of structural systems, the equation of motion of damped systems can be written as

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = f(t) \tag{1}$$

where M, K and C are the (n by n) mass, stiffness and nonproportional damping matrices, respectively, and $\ddot{u}(t)$, $\dot{u}(t)$ and $u(t)$ are the (n by 1) acceleration, velocity and displacement vectors, respectively. To find the solution of free vibration of the system, we consider the following quadratic eigenproblem

$$\lambda^2 M\phi + \lambda C\phi + K\phi = 0 \tag{2}$$

in which λ and ϕ are the eigenvalue and eigenvector of the system. There are $2n$ eigenvalues for the system with n degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or underdamped modes.

The common practice is to reformulate the quadratic system of equation to a linear one by doubling the order of the system such as

$$A\psi = \lambda B\psi \tag{3}$$

where

$$A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \qquad B = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \quad \text{and} \quad \psi = \left\{ \begin{array}{c} \phi \\ \lambda\phi \end{array} \right\}. \tag{4}$$

The classical inverse iteration method [3] is commonly used to solve for only a small number of desired modes. The subspace iteration method [5] is a more efficient alternative than the inverse iteration method. However, in the iteration methods such as the inverse iteration method and the subspace iteration method, a large number of complex arithmetic operations are required. Furthermore, when the shift value becomes close to an eigenvalue of the system, singularity may be encountered during triangularization process.

The Lanczos algorithm for the computation of eigenvalues and eigenvectors of a real symmetric matrix was proposed by Lanczos in 1950 and improved by numerous researchers [6]. The Lanczos algorithm to solve the eigenvalue problem of nonproportionally damped system is developed in References [7] (two sided-Lanczos algorithm) and [2] (symmetric Lanczos algorithm). Although only real arithmetic is solved during the solution process, in contrast to the case of real symmetric eigenproblems, there will be a possibility of serious breakdown and the accuracy of the solutions obtained is low [8].

In this paper, the method to solve an eigenproblem with guaranteed nonsingularity for a nonproportionally damped structural system with multiple eigenvalues is developed.

2. Method of analysis

2.1. Problem definition

We consider an eigenproblem of which the m eigenvalues, λ_i, are close or multiple. For simplicity let us assume that the first m eigenvalues are close or multiple.

$$\lambda \equiv \lambda_1 \cong \lambda_2 \cong \cdots\cdots \cong \lambda_m \tag{5}$$

Then Eq. (3) can be presented in a matrix form for the m eigenvalues as follows

$$A\Psi = B\Psi\Lambda \tag{6}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ and $\Psi = [\psi_1 \ldots \psi_m]$ is a (n by m) matrix satisfying the orthonormal condition with respect to matrix B such as

$$\Psi^T B \Psi = I_m \tag{7}$$

where I_m is an unitary matrix of order m.

Let $X = [x_1, \ldots, x_m]$ be the vectors in the subspace Ψ, and X be the orthonormal with respect to matrix B. Then

$$\Psi = XZ, \tag{8}$$
$$X^T B X = I_m, \tag{9}$$

where Z is the unknown rotation matrix of order m. Introducing Eq. (8) into Eq. (6), we get

$$AXZ = BXZ\Lambda. \tag{10}$$

Let

$$DZ = Z\Lambda, \tag{11}$$

where $D = [d_1, d_2, \ldots, d_m] = X^T A X$ and symmetric. Then,

$$AXZ = BXDZ \tag{12}$$

and

$$AX = BXD \tag{13}$$

or

$$Ax_i = BXd_i, \qquad (i = i, \ldots, m).$$ (14)

We obtain the m close or multiple eigenvalues and associated eigenvectors from Eqs. (8), (11) and (14).

Note when $\lambda_1 \cong \ldots \cong \lambda_m$, from Eq. (13)

$$D = \Lambda$$ (15)

$$X = \Psi.$$ (16)

The objective is to develop an efficient solution method with guaranteed non-singularity for an eigenproblem described by Eq. (14).

2.2. Proposed method

Let us assume that initial approximate solutions of Eq. (14), $d_i^{(0)}$ and $x_i^{(0)}$, are known. Denoting the approximate eigenvalues and the associated eigenvectors after k iterations by $d_i^{(k)}$ and $x_i^{(k)}$, we can get

$$r_i^{(k)} = Ax_i^{(k)} - BX^{(k)}d_i^{(k)}$$ (17)

and

$$(X^{(k)})^T BX^{(k)} = I_m$$ (18)

where the residual vector, $r_i^{(k)}$, denotes the error for each eigenpair, and is not generally zero because of substitution of approximate values into Eq. (14).

In order to get the solutions converged to the close or multiple eigenvalues and the associated eigenvectors of the system, the residual vectors should be removed. For the purpose of that, the Newton–Raphson technique is applied such as

$$r_i^{(k+1)} = Ax_i^{(k+1)} - BX^{(k)}d_i^{(k+1)} = 0$$ (19)

and

$$(X^{(k+1)})^T BX^{(k+1)} = I_m$$ (20)

where

$$d_i^{(k+1)} = d_i^{(k)} + \Delta d_i^{(k)},$$ (21)

$$x_i^{(k+1)} = x_i^{(k)} + \Delta x_i^{(k)},$$ (22)

$$X^{(k+1)} = [x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_m^{(k+1)}].$$ (23)

$\Delta d_i^{(k)}$ and $\Delta x_i^{(k)}$ are unknown incremental values of $d_i^{(k)}$ and $x_i^{(k)}$.

Substituting Eqs. (17), (18), (21) and (22) into Eqs. (19) and (20) and neglecting the nonlinear terms, we can get the linear simultaneous equations for unknown incremental values, $\Delta d_i^{(k)}$ and $\Delta x_i^{(k)}$, as follows;

$$A\Delta x_i^{(k)} - B\Delta X^{(k)}d_i^{(k)} - BX^{(k)}\Delta d_i^{(k)} = -r^{(k)}$$ (24)

and

$$(X^{(k)})^T B\Delta X^{(k)} = 0.$$ (25)

Since the eigenvalue is close or multiple, the off-diagonal elements of D are zero or very small compared with its diagonal at kth iteration step, and the diagonal elements very close. Thus, the second term in left side of Eq. (23) may be approximated by $d_{ii}^{(k)} B \Delta x_i^{(k)}$, which yields

$$A\Delta x_i^{(k)} - d_{ii}^{(k)} B\Delta x^{(k)} - BX^{(k)}\Delta d_i^{(k)} = -r^{(k)}. \tag{26}$$

Writing Eqs. (24) and (25) in matrix form, we can get

$$\begin{bmatrix} (A - d_{ii}^{(k)}B) & -BX^{(k)} \\ (-BX^{(k)})^T & 0 \end{bmatrix} \left\{ \begin{array}{c} \Delta x_i^{(k)} \\ \Delta d_i^{(k)} \end{array} \right\} = - \left\{ \begin{array}{c} r_i^{(k)} \\ 0 \end{array} \right\}. \tag{27}$$

Because the new coefficient matrix should be reformed and refactorized in each iteration step, the above method adopting the Newton–Raphson technique and a side condition, despite of its rapid convergence, is not efficient.

These blemishes may be overcome by applying the modified Newton–Raphson technique to Eq. (26) such as

$$\begin{bmatrix} (A - d_{ii}^{(0)}B) & -BX^{(k)} \\ (-BX^{(k)})^T & 0 \end{bmatrix} \left\{ \begin{array}{c} \Delta x_i^{(k)} \\ \Delta d_i^{(k)} \end{array} \right\} = - \left\{ \begin{array}{c} r_i^{(k)} \\ 0 \end{array} \right\}. \tag{28}$$

The symmetric coefficient matrix of Eq. (27) is of order $(2n + m)$. While singularity occurs in factorization process of the iteration methods such as the inverse iteration method and the subspace iteration method when the shift is close to an eigenvalue of the system, nonsingularity of the proposed method is always guaranteed [4] This is the main difference compared with the iteration method with shift.

Initial values of the proposed method can be obtained as the intermediate results of the iteration methods or results of approximate methods. In this paper, the starting values are taken as the results of the symmetric Lanczos method [2] with selectively reorthogonalization process because the method does not need complex arithmetic in the Lanczos recursive process, and because the multiplicity of the desired eigenvalues can be checked by the results of the $4p$ Lanczos vectors (p: the number of desired eigenvalues).

3. NUMERICAL EXAMPLES

In this section two test problems with close or multiple eigenvalues are used to assess the performance of the proposed method for generalized eigenproblems. The CPU time spent for the first twelve eigenvalues and the associated eigenvectors ($p = 12$) and the variation of the error norm to each iteration step of the proposed method are compared with those of the subspace iteration method [5]. The least subspace dimension to effectively calculate required eigenpairs is $2p$ (24 eigenpairs). Each method is stopped when the error norms are reduced by the factor of 10^{-6}, which yields a stable eigensolution and sufficient accuracy in the calculated eigenvalues and eigenvectors for practical analysis [1]. The physical error norm [1] is defined as

$$\varepsilon_i^{(k)} = \frac{\left\| Ax_i^{(k)} - BX^{(k)}d_i^{(k)} \right\|_2}{\left\| Ax_i^{(k)} \right\|_2}. \tag{29}$$

3.1. Plane frame structure with lumped dampers (multiple case)

The finite element model of a plane frame is used as the first example. The dimensionless values of the geometric configuration and material properties are shown in Fig. 1. The model is discretized in 200 beam elements resulting in the system of dynamic equations with a total of 590 degrees
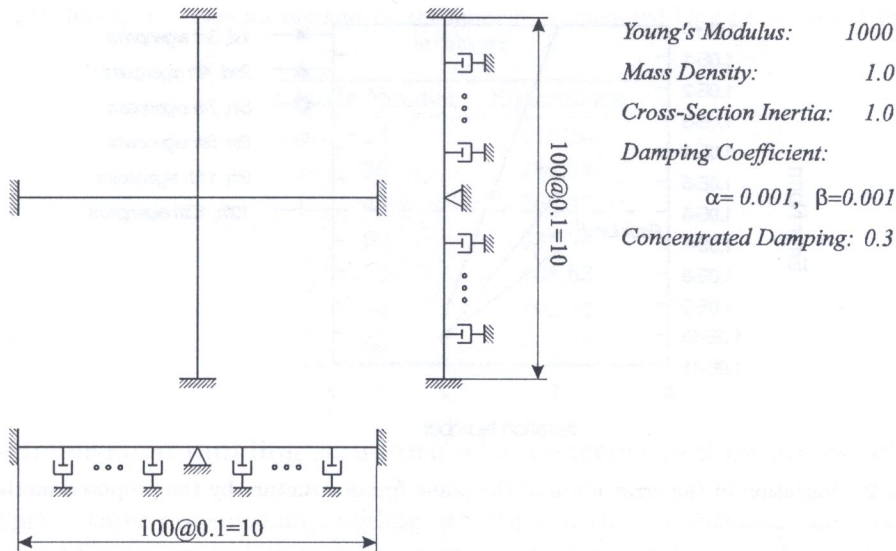
Young's Modulus: *1000*

Mass Density: *1.0*

Cross-Section Inertia: *1.0*

Damping Coefficient:

         α= 0.001, β=0.001

Concentrated Damping: 0.3

Fig. 1. Plane frame structure with lumped dampers

of freedom. Thus, the order of the associated eigenproblem is 1,180. The consistent mass matrix is used for M. Its damping matrix is derived from the proportional damping expression given by $C = \alpha M + \beta K$ and concentrated dampers.

The eigenvalues of the model are shown in Table 1. All the eigenvalues of the model are multiple. The variations of the error norms to the iteration step are shown in Figs. 2 to 4. The error norms of the initial values obtained by using the $4p$ (= 48) Lanczos vectors are about 0.7 to 10^{-7}. Using the results of Lanczos method, the multiplicity of the desired eigenvalues can be checked. The number of iterations for the proposed method applied to the initial values that do not satisfy the error norm 10^{-6} is only one. The results in Figs. 2 to 3 indicate that the convergence of the proposed method is much better than that of the subspace iteration method. The CPU time for the proposed method is compared with that of the subspace iteration method in Table 2. If we let the solution time for the proposed method be 1, it takes 3.55 times for the subspace iteration method. In Table 3, the CPU time for the Lanczos method is summarized. Because the method does not need complex operations, the less solution time is required. However, the some results of the Lanczos method as shown in Fig. 4 are not improved in spite of the increase of the number of the Lanczos vectors.

Table 1. Eigenvalues of the plane frame structure with lumped dampers

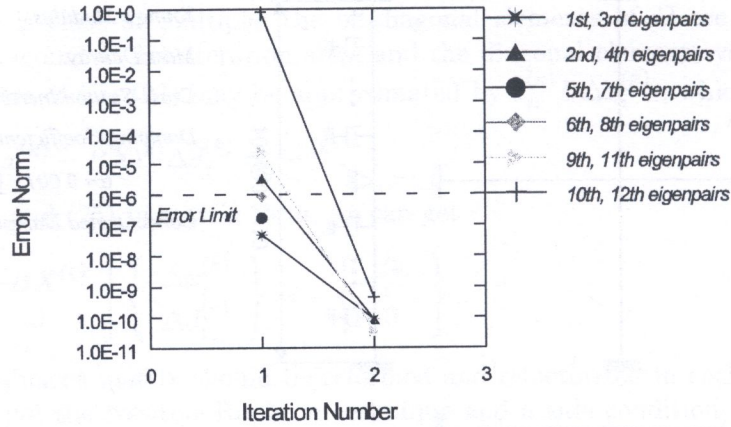| Mode Number	Eigenvalues
1	$-0.09590 + j8.66792$
2	$-0.09590 + j8.66792$
3	$-0.09590 - j8.66792$
4	$-0.09590 - j8.66792$
5	$-0.60556 + j15.5371$
6	$-0.60556 + j15.5371$
7	$-0.60556 - j15.5371$
8	$-0.60556 - j15.5371$
9	$-0.57725 + j20.7299$
10	$-0.57725 + j20.7299$
11	$-0.57725 - j20.7299$
12	$-0.57725 - j20.7299$

Fig. 2. Variation of the error norm of the plane frame structure by the proposed method
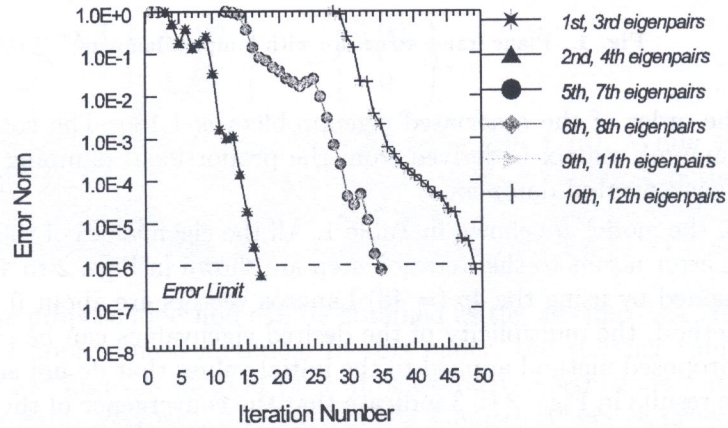

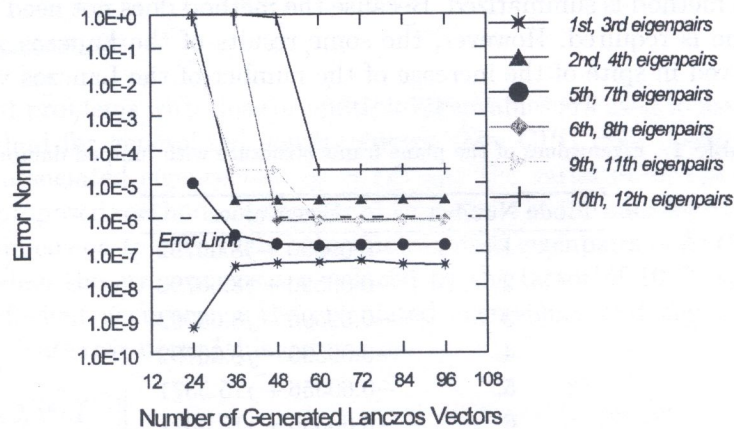**Fig. 3.** Variation of the error norm of the plane frame structure by the subspace iteration method

Fig. 4. Variation of the error norm of the plane frame structure by the Lanczos method

Table 2. CPU time spent for the first twelve eigenvalues of the plane frame structure with lumped dampers

Method	CPU time in seconds(Ratio)
Proposed method	872.69(1.00)
Subspace iteration method	3,096.62(3.55)

Table 3. CPU time for the Lanczos method vs. the number of generated Lanczos vectors of the plane frame structure

Mode Number	Eigenvalues
24	116.20
36	185.54
48	260.37
60	332.90
72	408.63
84	492.83
96	664.27

3.2. Three-dimensional building structure with concentrated dampers (close case)

In this example a three-dimensional building structure with concentrated dampers is presented. The geometric configuration and material properties are shown in Fig. 5. The model is divided into 436 beam elements and has 1 128 degrees of freedom. The order of the associated eigenproblem is 2 256. The consistent mass matrix is used to define M. The damping matrix consists of the Rayleigh damping and concentrated dampers.

The results of the proposed method are summarized in Table 4. The first and third eigenvalue are clustered, and also the ninth and eleventh eigenvalues and their conjugate eigenvalues clustered. The variations of the error norms to the iteration step are shown in Figs. 6 to 8. The first step of the proposed method denotes the results of the Lanczos algorithm. The error norms of the initial values
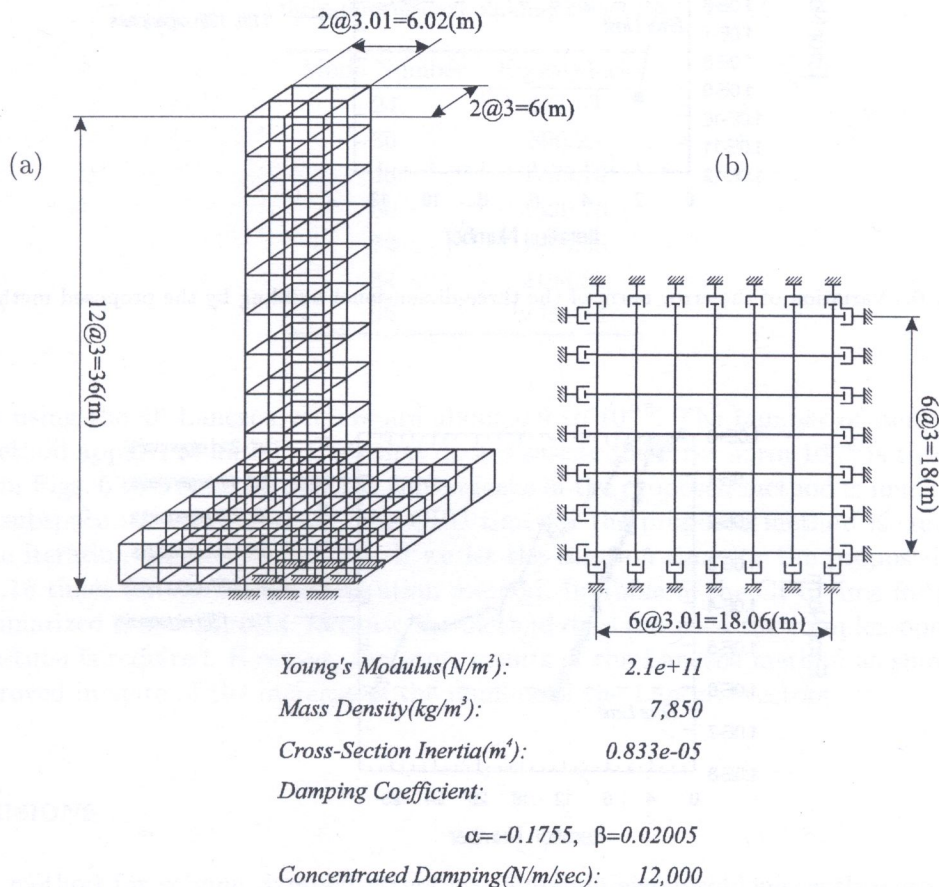

Young's Modulus(N/m²):	2.1e+11
Mass Density(kg/m³):	7,850
Cross-Section Inertia(m⁴):	0.833e-05
Damping Coefficient:	

$$\alpha = -0.1755, \quad \beta = 0.02005$$

Concentrated Damping(N/m/sec):	12,000

Fig. 5. (a) Three-dimensional building structure, (b) Damping from two-layer foundation

Table 4. Eigenvalues of the three-dimensional building structure with concentrated dampers

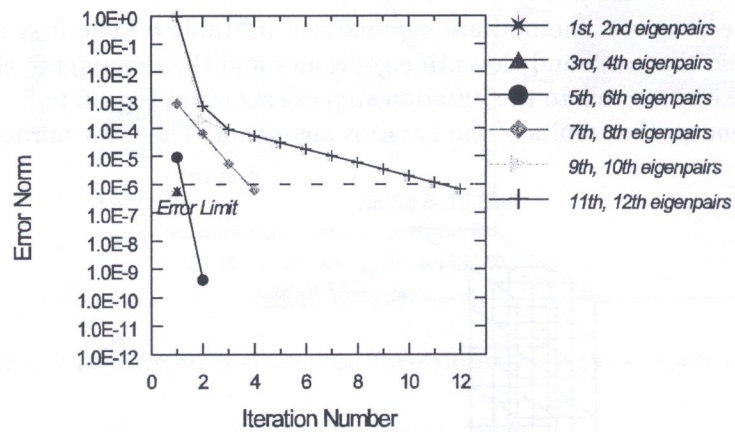| Mode Number	Eigenvalues
1	$-0.13763 + j3.08907$
2	$-0.13763 + j3.08907$
3	$-0.13763 - j3.08907$
4	$-0.13763 - j3.08907$
5	$-3.52574 + j2.20649$
6	$-3.52574 - j2.20649$
7	$-0.24236 + j4.16556$
8	$-0.24236 - j4.16556$
9	$-1.64294 + j7.02958$
10	$-1.64294 - j7.02958$
11	$-1.65070 + j7.03590$
12	$-1.65070 - j7.03590$

Fig. 6. Variation of the error norm of the three-dimensional building by the proposed method
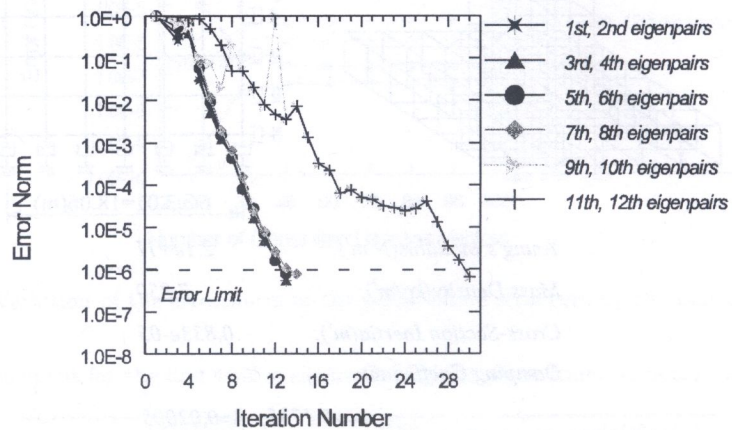

Fig. 7. Variation of the error norm of the three-dimensional building by the subspace iteration method
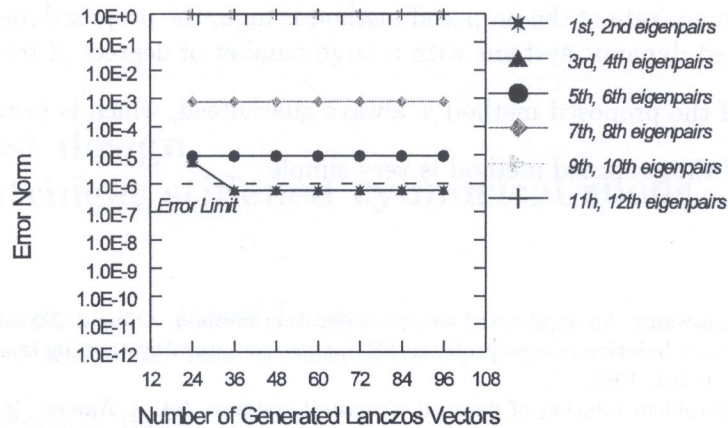
Fig. 8. Variation of the error norm of the three-dimensional building by the Lanczos method

Table 5. CPU time spent for the first twelve eigenvalues of the three-dimensional building structure with concentrated dampers

Method	CPU time in seconds(Ratio)
Proposed method	8,335.19(1.00)
Subspace iteration method	9,644.75(1.16)

Table 6. CPU time for the Lanczos method vs. the number of generated Lanczos vectors of three-dimensional building structure

Mode Number	Eigenvalues
24	574.54
36	896.23
48	1209.19
60	1529.75
72	1969.55
84	2194.85
96	2551.14

obtained by using the 48 Lanczos vectors are about 0.9 to 10^{-6}. The number of iterations for the proposed method applied to initial values that do not satisfy the error norm 10^{-6} is three or eleven. The results in Figs. 6 to 8 indicate that the convergence of the proposed method is much better than that of the subspace iteration method. The CPU time for the proposed method is compared with the subspace iteration method in Table 5. If we let the solution time for the proposed method be 1, it takes 1.16 times for the subspace iteration method. In Table 6, the CPU time for the Lanczos method summarized is summarized. Because the method does not need the complex operations, the less solution time is required. However, the some results of the Lanczos method as shown in Fig. 8 are not improved in spite of the increase of the number of the Lanczos vectors.

4. CONCLUSIONS

An efficient method for solving damped structural dynamic eigenproblems with close or multiple eigenvalues is presented. Characteristics of the proposed method identified by the numerical results from a test problem are identified as follows:

1. Since the convergence rate of the proposed method is high, the proposed method is very effective for solving damped dynamic systems with a large number of degrees of freedom.

2. Nonsingularity of the proposed method is always guaranteed, which is proved analytically.

3. The algorithm of the proposed method is very simple.

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