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An application of a decomposed ortho-diagonal method to discrete polyoptimization of a hall with spatial grid structure

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The paper deals with application of the ortho-diagonal (O-D) method of finding the nondominated sets of solutions and evaluations for discrete polyoptimization problems. First, the (O-D) method was modified for finding minimum of a scalar function. The monotonicity property of a vector objective function is used by the (O-D) method for consecutive finding of jth-criteria partial nondominated sets, $j \in \{1, 2, \ldots, J\}$. To find the nondominated evaluations sets the discrete neighbourhoods S of the point $\mathbf{x}_i$ are investigated starting from the solution $\mathbf{x}_1^*$ which minimizes the first objective function. In this way the consecutive nondominated solutions $\mathbf{x}_{ND}^k$ are determined. An accuracy of solution and CPU computing time depend on the way the discrete neighbourhoods S of the point $\mathbf{x}_i$ in the design space are defined. The algorithm of the (O-D) method is applied to solve the discrete polyoptimization of a hall with spatial grid structure.

1. Introduction

The ortho-diagonal (O-D) method belongs to the group of direct methods of direction improvement without gradient calculations [2, 8, 14]. The method was originally used for searching an extremum of partial objectives inside a discrete feasible domain [15]. Actually, the method is modified and used for searching the partial nondominated solutions and evaluations of multicriteria optimization problems. The method is matched to the requirements of discrete optimization problems, where the computing time for constraints checking is almost negligible as compared with the value determination of objectives [8]. For scalar optimization, the method is combined with an expert system [12] and implemented as part of the OPTIM system [14]. The (O-D) method has been applied to discrete multicriteria optimization of space truss structure used for covering a large span hall. The sets of nondominated solutions and evaluations as well as determination of a preferred solution have been found.

2. Formulation of polyoptimization problem

An optimization problem is formulated when design variables, optimization criteria and constraints are determined. The *design variables* describe the considered object, e.g. a structure or its element and they are varying during the optimization process. The *optimization parameters* describe the considered object and are fixed earlier on the basis of the design assumptions, an initial analysis or a sensitivity analysis. Each object parameter can be considered as a design variable or an optimization parameter, depending on material, structural, economical or calculation possibilities. The design variables constitute a vector of the following form:

$$\mathbf{x} = [x_1, x_2, \ldots, x_n, \ldots, x_N]^T, \qquad n \in N = \overline{1, N} \equiv \{1, 2, \ldots, N\}, \tag{1}$$

in an N-dimensional design variable space $A \subset R^N$. Each point $\mathbf{x}$ in the design variable space A corresponds with the object described by the N-dimensional vector of the design variables. The design variables can belong to continuous or finite sets. They are called the *continuous* or *discrete* design variables, respectively. In the following, the discrete design variables are usually assumed. Each design variable x_n can take M_n values, as follows

$$x_n = \{x_{n,1}, \ldots, x_{n,m_n}, \ldots, x_{n,M_n}\}, \qquad m_n \in M_n = \overline{1, M_n} \equiv \{1, 2, \ldots, m_n, \ldots, M_n\}. \tag{2}$$

A *domain of feasible solutions* X (or shortly a *feasible domain*) contains a part of the design variable space A, i.e., $X \subset A$. The boundaries of the feasible domain are determined by the constraints imposed on the design variables. The constraints have the inequality or equality forms, as follows

$$\begin{aligned} &g_k(\mathbf{x}) \le 0, \quad k \in K = \{1, 2, \ldots, K\}, \quad g_k : X \subset A, \\ &g_l(\mathbf{x}) = 0, \quad l \in L = \{1, 2, \ldots, L\}, \quad g_l : X \subset A. \end{aligned} \tag{3}$$

The constraints describe certain requirements for the optimization object. They are imposed directly on the design variables or indirectly by constraining certain quantity dependent on the design variables. The design (code), technological and computational constraints are distinguished in structural design. The *design constraints* occurring in design codes for a particular type of structure contain stress, displacement, stability and vibration constraints. The *technological constraints* concern methods of realization of a structure, material possibilities, typification of elements, and object exploitation. The *computational constraints* concern memory size and computing speed efficiency of optimization algorithms and human design groups. The technological and computational constraints influence the size of the feasible domain X as well as its dimension N.

The constraints can be considered as side and behavioural, active and passive, essential and inessential. The *side constraints* occur in explicit form and they are imposed on the particular design variables. They determine the explicit part of the feasible domain $X_J \subset X$ (Fig. I). The *behavioural constraints* occur in implicit form, i.e. they are imposed on certain relations, e.g. stresses, displacements, critical forces and frequencies of vibration. The constraint is *active* (passive) for a given feasible solution $\mathbf{x}_d$, if it is satisfied for $\mathbf{x} = \mathbf{x}_d$ as equality (sharp inequality). The constraint is *essential* (inessential) for a given problem, if its omission changes (does not change) the set of feasible solutions X (Fig. I). Neglecting the inessential constraints can accelerate the solution of the optimization problem. In the N-dimensional design variables space A, the constraints create a hypersurface containing the points satisfying the constraints in an equality form.

The objective function (optimization criterion, quality index) $f(\mathbf{x})$ describes certain property of the considered object that maps the particular realization $\mathbf{x} \in X$ into a real number axis with a prescribed unit of measure. The objective function is an explicit or implicit function of the design variables and can be expressed in mathematical or algorithmic form.

The optimization criterion can be satisfied by finding minimum or maximum of the objective function $f(\mathbf{x})$ in the set of feasible solutions X. In the following, the consideration will be limited to finding minimum of the function $f(\mathbf{x})$, but maximization of the function can be transformed to minimization using the substitution as follows

$$\operatorname{Max}\{f(\mathbf{x}) : \mathbf{x} \in X\} = -\operatorname{Min}\{-f(\mathbf{x}) : \mathbf{x} \in X\}, \tag{4}$$

$\operatorname{Min}\{\cdot\}$ denotes the problem to be solved, and $\min\{\cdot\}$ denotes the minimal value of the function.

If there is a need to evaluate an object according to $J > 1$ criteria, then the problem is called a *multicriteria* (multiobjective, vector) optimization or polyoptimization problem, with the objective function denoted as follows

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), \ldots, f_j(\mathbf{x}), \ldots, f_J(\mathbf{x})]^T, \qquad j \in J = \{1,\ 2, \ldots, j, \ldots, J\} = \overline{1, J}. \tag{5}$$

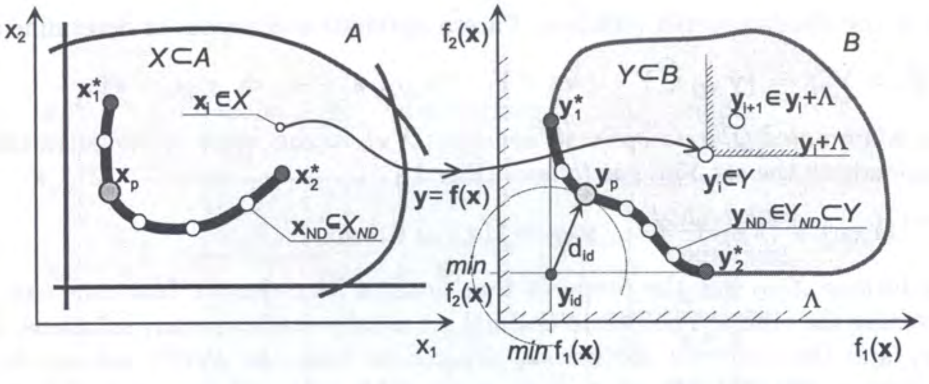

Fig. 1. Graphical illustration of a polyoptimization problem

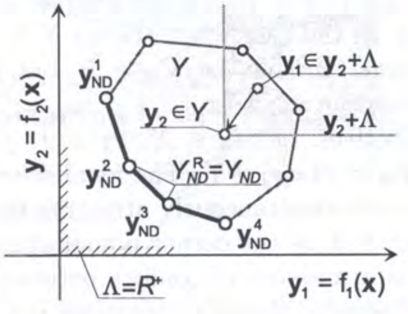

Fig. 2. Set of the nondominated evaluations with cone ordering relations

The values of the vector function $\mathbf{f}(\mathbf{x})$ represent points in the J-dimensional *objective space* B, which is called also a space of evaluation of the solutions (or shortly evaluation space) — Fig. 1. The objective space B is contained in a J-dimensional Cartesian space R^J. The *set of evaluations of solutions* (objective domain, evaluation domain) Y is a part of the objective space B, containing all values of the objectives which correspond to the feasible values of the design variables. The set of evaluations of solutions in discrete polyoptimization problem is a finite set. The objective space B is the Cartesian product $B = \overline{Y}_1 \times \ldots \times \overline{Y}_j \times \ldots \times \overline{Y}_J$, where $\overline{Y_j}$ is the hull of the set Y_j of all values of the j-th objective function determined over the set X

$$Y_j = \{f_j(\mathbf{x}) \in R^1 : \quad \mathbf{x} \in X, \quad j \in J = \{1,\ 2, \ldots, J\}\}. \tag{6}$$

The vector objective function describes a mapping of the design variable space A into the objective space B. Each evaluation point $y \in Y$ corresponds to one or more solutions $\mathbf{x} \in X$. Therefore, this mapping is a surjection

$$f: X \overset{on}{\rightarrow} Y : \mathbf{x} \rightarrow \mathbf{y} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in X \subset A \subset R^N, \qquad \mathbf{y} \in Y \subset B \subset R^J. \tag{7}$$

The evaluations of the solutions $\mathbf{y}$ contained in the set of evaluations Y are subjected to mutual comparison on the basis of a relation $\mathcal{R}$ ordering the set Y. This relation should make it possible to compare two arbitrary evaluations and to decide which one dominates the other. In the case of indistinguishability, this relation should make it possible to decide that these evaluations are nondominated. The set of nondominated (efficient) evaluations contained in the set Y subjected to the relation $\mathcal{R}$ is denoted by $Y_{ND}^{\mathcal{R}}$, and the set of dominated evaluations is denoted by $Y_D^{\mathcal{R}}$ (Fig. 2). In the structural optimization problems, usually $Y_D^{\mathcal{R}}$ is an empty set, but if $Y_D^{\mathcal{R}} \neq \emptyset$, then it contains only one element.

If the ordering relation is a cone relation $\prec_\Lambda$ with the cone determined by positive quadrant R_0^+ in the space R^J as follows

$$\Lambda = \left\{(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_J) \in R^J : \quad \lambda_j \geq 0, \quad j \in \overline{1, J}\right\} = R_0^+ \subset R^J, \tag{8}$$

then the set of the nondominated (efficient, Pareto optimal) evaluations is determined as follows

$$Y_{ND} = Y_{ND}^{\mathcal{R}} = Y_{ND}^{\prec_\Lambda} = \{\mathbf{y}_{ND} \in Y : \quad \forall \mathbf{z} \in Y, \quad (\mathbf{y}_{ND}, \mathbf{z}) \in \prec_\Lambda \Rightarrow \mathbf{y}_{ND} = \mathbf{z}\}. \tag{9}$$

The set of *nondominated (Pareto optimal) solutions* X_{ND} in the space of design variables (solution space) corresponds to the set Y_{ND}, as follows (Fig. 1)

$$X_{ND} = \mathbf{f}^{-1}(Y_{ND}) = \{\mathbf{x}_{ND} \in X : \quad \mathbf{y}_{ND} = \mathbf{f}(\mathbf{x}_{ND}) \in Y_{ND}\}. \tag{10}$$

The set of solutions X_{ND} has the property that none of its objective functions can be improved without worsening the others. The Pareto optimal set usually contains many solutions. The *preferred evaluation* $\mathbf{y}_p$ and the *preferred solution* $\mathbf{x}_p$ are chosen from the Pareto set on the basis of an additional criterion (Fig. II). If each optimization problem $\text{Min}\{f_j(\mathbf{x}) : \mathbf{x} \in X\}$, $j \in J$ has the optimal solution $\mathbf{x}_j^* \in X$ (j-th optimal solution), then the vector $\mathbf{y}_{id} = [y_{id,1}, \ldots, y_{id,J}]^T \in B$ with elements $y_{id,j} = \min\{f_j(\mathbf{x}) : \mathbf{x} \in X\}$, $j \in J$, is called an ideal point (ideal evaluation). The *ideal point* $\mathbf{y}_{id}$ may not exist (e.g. in the case when the set Y is not constrained), and if it exists, then it is unique, but it can be unattainable, i.e. $\mathbf{y}_{id} \in B$, but $\mathbf{y}_{id} \notin Y$ (Fig. II). If $\mathbf{y} \in Y$, then $\{\mathbf{y}_{id}\} = \{\mathbf{y}_D\} = Y_D$. The *ideal solution* $\mathbf{x}_{id} = [x_{id,1}, \ldots, x_{id,N}]^T$ of the polyoptimization problem minimizes simultaneously all objectives, i.e. $\forall \mathbf{x} \in X$, $\forall j \in J$, $f_j(\mathbf{x}_{id}) \leq f_j(\mathbf{x})$. Usually, the ideal solution does not exist because $\mathbf{y}_{id} = f(\mathbf{x}_{id}) \notin Y$. The ideal solution exists if $\mathbf{y}_{id}$ is the dominated element in the set Y. The objectives simultaneously attaining the minimum are called *cooperative functions.* The space angle determined by the corner points y_j^*, $j \in J$ of the set of nondominated evaluations Y_{ND} and the origin is a measure of conflict between the objectives and it is called a *cooperative angle* β. The *antiideal point* in the objective space is a vector $\mathbf{y}_{ai} \in B$ with the elements

$$y_{ai,j} = \max\{f_j(\mathbf{x}) : \mathbf{x} \in X\}, \quad j \in J. \tag{11a}$$

The *Nadir point* is a vector $\mathbf{y}_{nad} \in B$ with the elements as follows

$$y_{nad,j} = \max\{f_j(\mathbf{x}) : \mathbf{x} \in X_{ND} \subset X, \}, \quad j \in J. \tag{11b}$$

The Nadir point can, but does not have to belong to the set of evaluations Y (Fig. II). The *satisfactory evaluation* $\mathbf{y}_S$ is the evaluation belonging to a part of the objective domain $Y_S \subset Y$ and ensuring for each objective function an achievement of the value not greater than earlier prescribed $\hat{\mathbf{y}}_{S,j}$ (Fig. II). The satisfactory evaluation does not have to belong to the set of nondominated evaluations Y_{ND}. The satisfactory evaluation corresponds to the *satisfactory solution* $\mathbf{x}_S \in X_S \subset X$ in the feasible domain.

3. Determination of nondominated solutions

In discrete optimization problems, values of design variables usually create a nonregular net in the feasible domain. This feature can be shown by the example of a catalogue of tubular sections; the variable x_1 corresponds to the external diameter D, and the variable x_2 corresponds to wall thickness g of a tubular section. The *regular net* is easier to use in computer realization, because the optimization methods are usually based on the notions of improvement direction of objective function and step size in iteration process. The mutual mapping of a design variable space A into a *regular solution space* A_r is introduced, which assigns a number m_n to particular values of design variables x_{n,m_n} (Fig. 3b)

$$\begin{gathered} f_r : A \to A_r, \quad \mathbf{x}_r = f_r(\mathbf{x}), \quad m_n = f_r(\mathbf{x}_{n,m_n}), \quad m_n \in M_n = \{1, 2, \ldots, m_n, \ldots, M_n\}, \\ n \in N = \{1, 2, \ldots, N\}, \quad \mathbf{x} \in X \subset A, \quad \mathbf{x}_r \in X_r \subset A_r. \end{gathered} \tag{12}$$

The mapping $f_r(\mathbf{x})$ assigns a vector of natural numbers $\mathbf{x}_r$ to each vector of design variables $\mathbf{x}$. Another decision variables defined in descriptive way, e.g., structural topology, system of stiffness zones, and type of material are subjected to similar mapping. The set of natural numbers M_n is

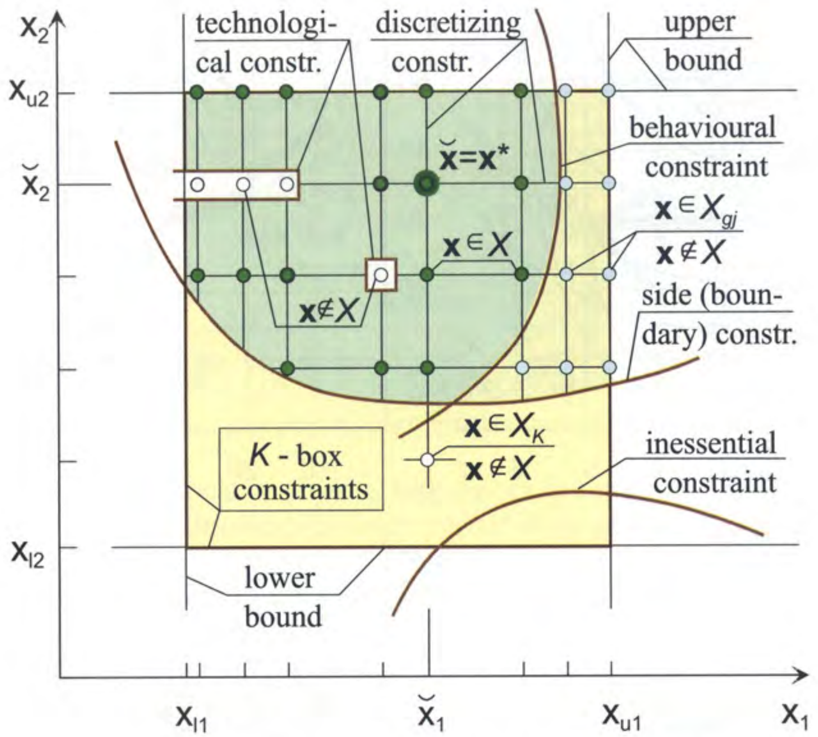

Fig. I. The constraints of the feasible domain X

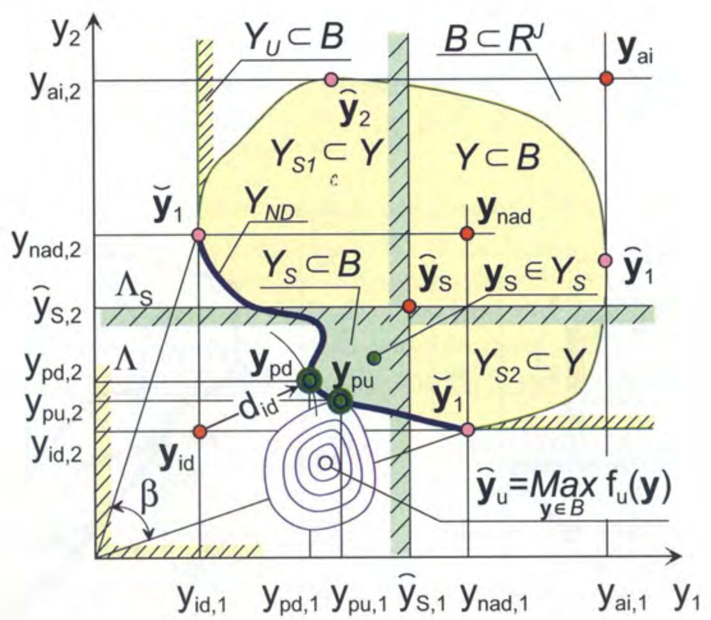

Fig. II. Space of evaluations of solutions B (objective space)

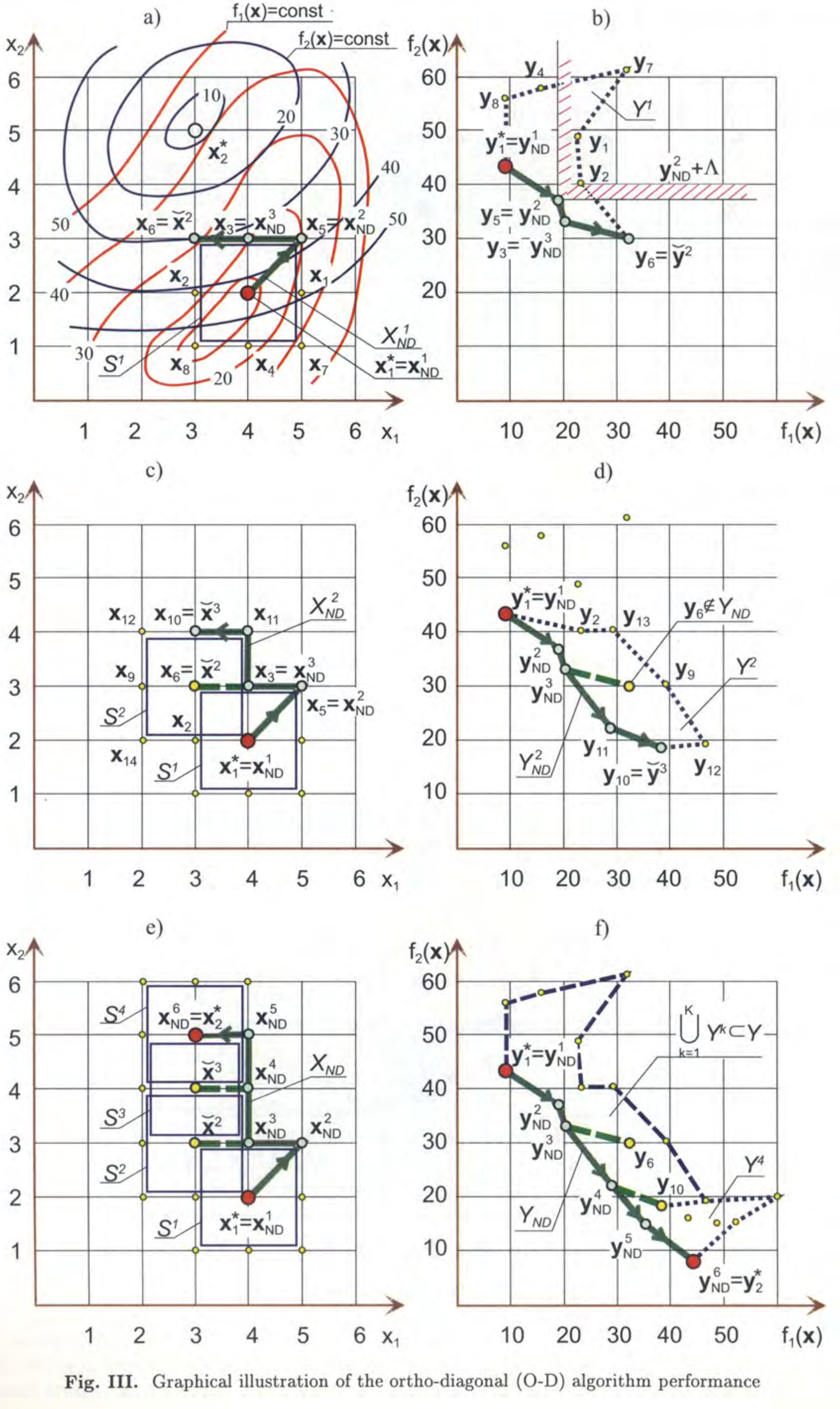

Fig. III. Graphical illustration of the ortho-diagonal (O-D) algorithm performance

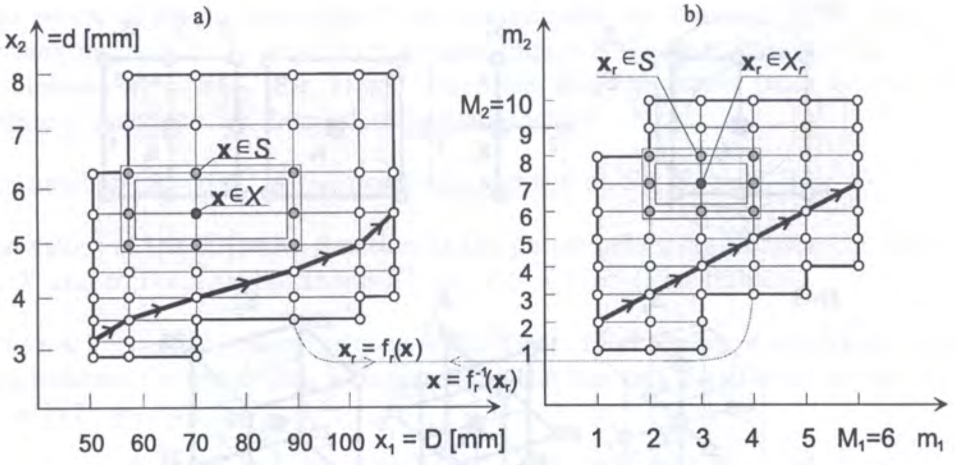

Fig. 3. Mutually explicit mapping of design variable space A into regular solution space A_r

assigned to particular geometrical or physical features. In the space A_r, the improvement direction lies on the straight line and the distance between two arbitrary points in a given direction is constant (Fig. 3b).

The analysis of the sets of feasible solutions X and their evaluations Y need some definitions concerning neighbourhoods of a given point, the inside, and boundaries of the set Y. The *surrondings* S of the point $\mathbf{x} \in R^N$ is any set such that there exists its open subset containing the point $\mathbf{x}$. The set obtained from the surroundings of the point $\mathbf{x}$ after removal of $\mathbf{x}$ is called the *neighbourhood* of the point $\mathbf{x} \in X$. For $Y \subset B$, if there exist the neighbourhood of the point $\mathbf{y}$ contained in the set Y, then the point $\mathbf{y} \in Y$ is called the *interior point* of the set Y. The point $\mathbf{y} \in Y$ is called the *boundary point* of the set Y, if each neighbourhood of the point $\mathbf{y}$ contain both some elements belonging and some not belonging to the set Y. The set of the interior points of the set Y is called the *interior of the set* Y and is denoted by $\mathrm{Int}\,(Y)$, while the set of boundary points of the set Y is called the *boundary of the set* Y and is denoted by $\mathrm{Fr}\,(Y)$.

The *discrete neighbourhood of the point* $\mathbf{x} \in R^N$ for $N > 1$ can be defined in various ways. All possibilities for $N = 2$ and $N = 3$ are shown in Fig. 4. They consist of combinations of increments with values of one δ_{n1} and zero δ_{n0}, described as follows

$$\delta_{n1} = m_{n\pm1} - m_n = \pm 1.0, \qquad \delta_{n0} = m_n - m_n = 0. \tag{13}$$

The consecutive points of a given neighbourhood S_i are obtained by adding the appropriate vectors $\delta = [\delta_1, \delta_2, \ldots, \delta_N]^T$ to the vector $\mathbf{x}$, where $\delta_n = \delta_{n1}$ or $\delta_n = \delta_{n0}$. It should be noticed that the number of points $l = 3N - 1$ belonging to the box neighbourhood $S = \sum_{i=1}^{N} S_i$, increases rapidly with the space dimension N. In the case $N \geq 3$, a reduced box neighbourhood $S_0 = S_1 \cup S_N$, or methods based on decomposition of the design variable space are mostly used. The number of points belonging to the box neighbourhoods S_1, S_N and S_0 is calculated as follows

$$\begin{aligned} l_1 &= 2N, && \text{for } S_1 \\ l_N &= 2^N, && \text{for } S_N \\ l_0 &= l_1 + l_N = 2N + 2^N && \text{for } S_0 . \end{aligned} \tag{14}$$

In this way, for $N > 2$, as many as $3^N - 2^N - 2N - 1$ points are eliminated from the full neighbourhood S. In this way, the process of finding the values of the objectives for the points belonging to the neighbourhood of a given point $\mathbf{x}$ is considerably accelerated. E.g., for $N = 4$ the acceleration exceeds 3 times, and for $N = 10$ it exceeds 56 times.

Finding the nondominated solutions in the discrete optimization problems depends on:

- problem size, i.e. the number of design variables and objectives,

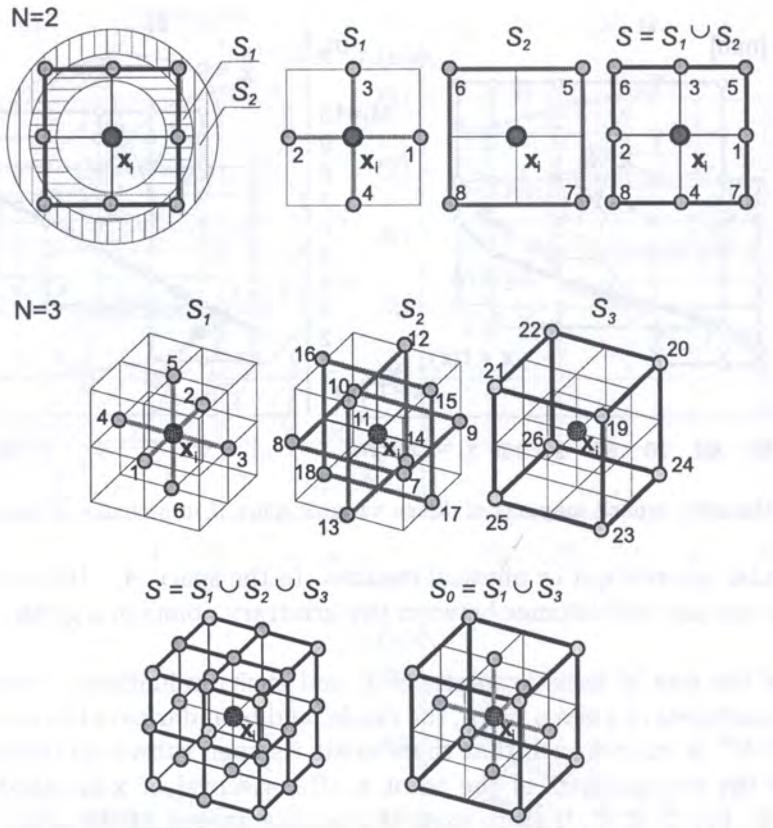

Fig. 4. Discrete neighbourhoods S of the point $\mathbf{x}_i$ for $N = 2$ and $N = 3$

- modality of the objectives,
- computational effort of finding the values of the objectives.

In structural optimization, the value of objective function is found using an algorithm and is usually connected with static and dynamic analysis. Therefore, the full enumeration of the solution space can only be applied for problems with small number of design variables, $N < 4$. The number of the objectives is essential for computation time in the cases when finding the values of the different objectives is concerned with change of the type of analysis, e.g., for finding a load capacity of elastic and elasto-plastic structures. For large number of objectives, the set of nondominated solutions grows considerably. This feature can raise a psychological barrier against choosing the preferred solution from among many nondominated solutions which differ only slightly from each other. Only the full enumeration is sure to find a global optimum and the complete set of nondominated solutions in the case of multimodal problems. But it can only be applied to small problems. For $N > 3$, the satisfaction problem can be formulated and solved, or else methods based on the decomposition of design variable vector can be used.

4. Algorithm for determination of nondominated solutions and evaluations

In the case of a unimodal function, methods for finding the set of nondominated solutions exist, e.g., the method of parametric scalarization [1, 13], the genetic algorithms method [6] or the method of weighting-correlation [13]. In the following, our own algorithm for finding the set of the nondominated discrete solutions will be shown. Figure III illustrates the algorithm in the case of two design variables and two objectives. In the case of minimization of all objectives the sequence of steps is as follows:

1. Find the vector of design variables $\mathbf{x}_1^*$ which minimizes the function $f_1(\mathbf{x})$ using any effective optimization method. If the solution is unique, then it is the first element of the set of nondominated solutions $\mathbf{x}_1^* = \mathbf{x}_{ND}^1$ (Fig. IIIa). Otherwise, find the corner point of the set Y_{ND} using the minimum condition for the next objective function.

2. Find the neighbourhood S^1 of the point $\mathbf{x}_{ND}^1$ defined as S_1, S_0 or S (Fig. 4).

3. Find the values of the objective function at the points belonging simultaneously to the feasible domain X and to the neighbourhood S^1, i.e. $\mathbf{x} \in X \cap S^1$ (Fig. IIIb).

4. Order the set $Y^1 = \mathbf{f}(S^1)$ using the cone relation $\prec_\Lambda$. Find the set of nondominated evaluations Y_{ND}^1 and indicate the evaluation $\breve{\mathbf{y}}^2$ minimizing the function $f_2(\mathbf{x})$ over the set Y_{ND}^1 (Fig. IIIb — $Y_{ND}^1 = \{\mathbf{y}_1^*, \mathbf{y}_5, \mathbf{y}_3, \mathbf{y}_6\}$).

5. If $Y_{ND}^1 = \{\breve{\mathbf{y}}^2 = \mathbf{y}_{ND}^1\}$, then the evaluation $\mathbf{y}_{ND}^1$ is dominating for the function $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$, i.e. $Y_{ND} = Y_{ND}^1 = Y_D = \{\mathbf{y}_{ND}^1\}$. If so, go to step 11, otherwise, go to step 6.

6. Find the inverse image $\breve{\mathbf{x}}^2 = \mathbf{f}^{-1}(\mathbf{y}_{ND}^2)$ and find the neighbourhood S^2 of the point $\breve{\mathbf{x}}^2$ (Fig. IIIc).

7. Repeat the steps 3–5 for the consecutive neighbourhood S^k of the points $\breve{\mathbf{x}}^k$ until the set of nondominated evaluations Y_{ND}^K becomes empty, i.e. $Y_{ND}^K = \emptyset$. This means that at the point $\breve{\mathbf{x}}^K = \mathbf{x}_{ND}^k$ the function $f_2(\mathbf{x})$ has reached minimum, and the point $\mathbf{y}^K = \mathbf{y}_2^*$ is the second corner point of the set Y_{ND}^{1-2} (Fig IIIe — $K = 4$, $\breve{\mathbf{x}}^4 = \mathbf{x}_{ND}^6 = \mathbf{x}_2^*$).

8. Create and order the set $Y_{ND}^{1-2} = \bigcup_{k=1}^{K} Y_{ND}^k$ using the cone relation $\prec_\Lambda$ (i.e., eliminate the dominated evaluations — Fig. IIIf — $\mathbf{y}_6$ and $\mathbf{y}_{10}$).

9. Find the inverse image $X_{ND}^{1-2} = \mathbf{f}^{-1}(Y_{ND}^{1-2})$ — Fig. IIIe.

10. If $Y_{ND}^{1-2} \subseteq Y$ then put $Y_{ND}^{1-2} = Y_{ND}$ and $X_{ND}^{1-2} = X_{ND}$ and go to step 17; if $Y_{ND}^{1-2} \subset Y$ then go to step 11.

11. In the case when the number of criteria $J > 2$ then go to step 12. If $J = 2$ then go to step 17.

12. Repeating the steps from 2 to 10, find the consecutive bicriteria sets of the nondominated evaluations $Y_{ND}^{2-3}, Y_{ND}^{3-4}, \ldots, Y_{ND}^{J-1}$ and the sets of nondominated solutions $X_{ND}^{2-3}, X_{ND}^{3-4}, \ldots, X_{ND}^{J-1}$.

13. Find the missing combinations of the nondominated bicriteria sets, e.g., for $J = 4$ Y_{ND}^{1-3} and Y_{ND}^{2-4} as well as X_{ND}^{1-3} and X_{ND}^{2-4}, using the corner points (of the set Y_{ND}) found earlier, and repeating the steps from 2 to 10 — Fig. 5a.

14. Find the nondominated three-criteria sets $Y_{ND}^{1-2-3}, Y_{ND}^{1-3-4}, Y_{ND}^{2-3-4}, Y_{ND}^{1-2-4}$, etc., as well as $X_{ND}^{1-2-3}, X_{ND}^{1-3-4}, X_{ND}^{2-3-4}, X_{ND}^{1-2-4}$, etc., using interior points of the sets $Y_{ND}^{j-(j+1)}$ and corner points $\mathbf{y}_j^*$, $j = 1, 2, \ldots, J$, repeating the steps from 2 to 10 — Fig. 5b.

15. Find the nondominated $4, 5, \ldots, J$-criteria sets, using interior points $3, 4, \ldots, (J-1)$-criteria sets and corner points $\mathbf{y}_j^*$, $j = 1, 2, \ldots, J$, repeating the steps from 2 to 10.

16. Create the set of nondominated evaluations Y_{ND} as the sum of the sets of the nondominated $2, 3, \ldots, J$-criteria evaluations. Create the set of nondominated solutions X_{ND} as the sum of the sets of the nondominated $2, 3, \ldots, J$-criteria solutions.

17. Interpret the results and find the preferred evaluation $\mathbf{y}_p$ and the preferred solution $\mathbf{x}_p$.

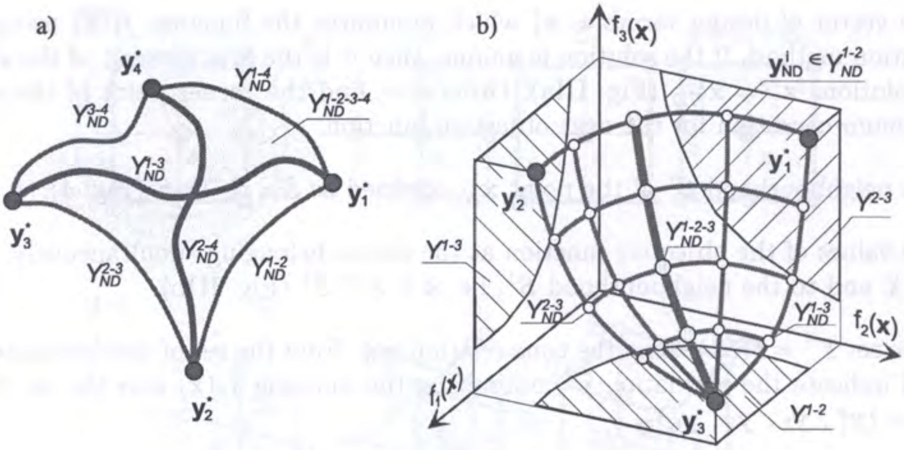

Fig. 5. Partial sets of the nondominated evaluations

The property of monotonicity of sets Y_{ND} is used for finding the sets of nondominated evaluations and solutions in the above algorithm [18]. According to this property, corner points corresponding to the extremal points of the particular objectives, sets of nondominated bicriteria, tree-criteria evaluations, and all other nondominated partial sets belong to the sets of nondominated evaluations. In the case of structure optimization, while more than two optimization criteria are considered, usually only the nondominated bicriteria sets are found, without determination of the nondominated $3, 4, \ldots, J$-criteria sets. In this way, computation time is reduced but not all nondominated solutions are found. In computer implementation of the algorithm, repeated analysis of the same vector $\mathbf{x}$ should be eliminated, as finding values of objectives for the particular structure $\mathbf{x} \in X$ needs a lot of computing time. This can be done by the use of a data base, updated after each analysis of a new structural variant. The values of the objectives for the points analyzed earlier should be taken from the data base before investigating the neighbourhood of a given $\mathbf{x}$.

The method of creation and search of neighbourhoods of a given point has a great significance for efficiency of the algorithm. The techniques of finding an extremum of one objective used in the ortho-diagonal method can be successfully applied here. An important role is played by the sequence of investigation of points belonging to the neighbourhood of a given point $\mathbf{x}$, and intelligent control of this process according to specific needs. The analysis of points chosen from second neighbourhood of a given point $\mathbf{x}$ and application of the superiority (predominance) coefficient allow to move along the bottom of the valley or jump over local minima. In this way, the fundamental drawbacks of often used combinatorial algorithms are eliminated. In regard to polyoptimization problems, it means that finding the concave (or created from several subsets) sets of nondominated evaluations Y_{ND} becomes also possible.

5. Tests of the ortho-diagonal algorithm

Test of the vector (O-D) algorithm is performed on the following bicriteria optimization problem. Find the set X_{ND} which minimizes $\mathbf{f}(\mathbf{x})$ on discrete feasible domain X

$$\begin{aligned}
&X_{ND} = f^{-1}(Y_{ND}), \qquad && Y_{ND} = \operatorname*{Min}_{\mathbf{x}\in X} \mathbf{f}(\mathbf{x}),\\
&\mathbf{x} = [x_1\,,\, x_2]^T, && \mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x})]^T,\\
&f_1(\mathbf{x}) = x_1^3 + x_2^2 - 3x_1 - 2x_2 + 10, &&\\
&f_2(\mathbf{x}) = \frac{(x_1+0.6)^2}{1.1} + \frac{(x_2-1.8)^2}{0.9} + 0.4\,x_1x_2 + 2, &&\\
&X = \{\mathbf{x} : \quad -2.5 \le x_1 \le 2.5, \quad 0.0 \le x_2 \le 3.0, \quad 4x_1 \in C, \quad 4x_2 \in C\}. &&
\end{aligned} \tag{15}$$

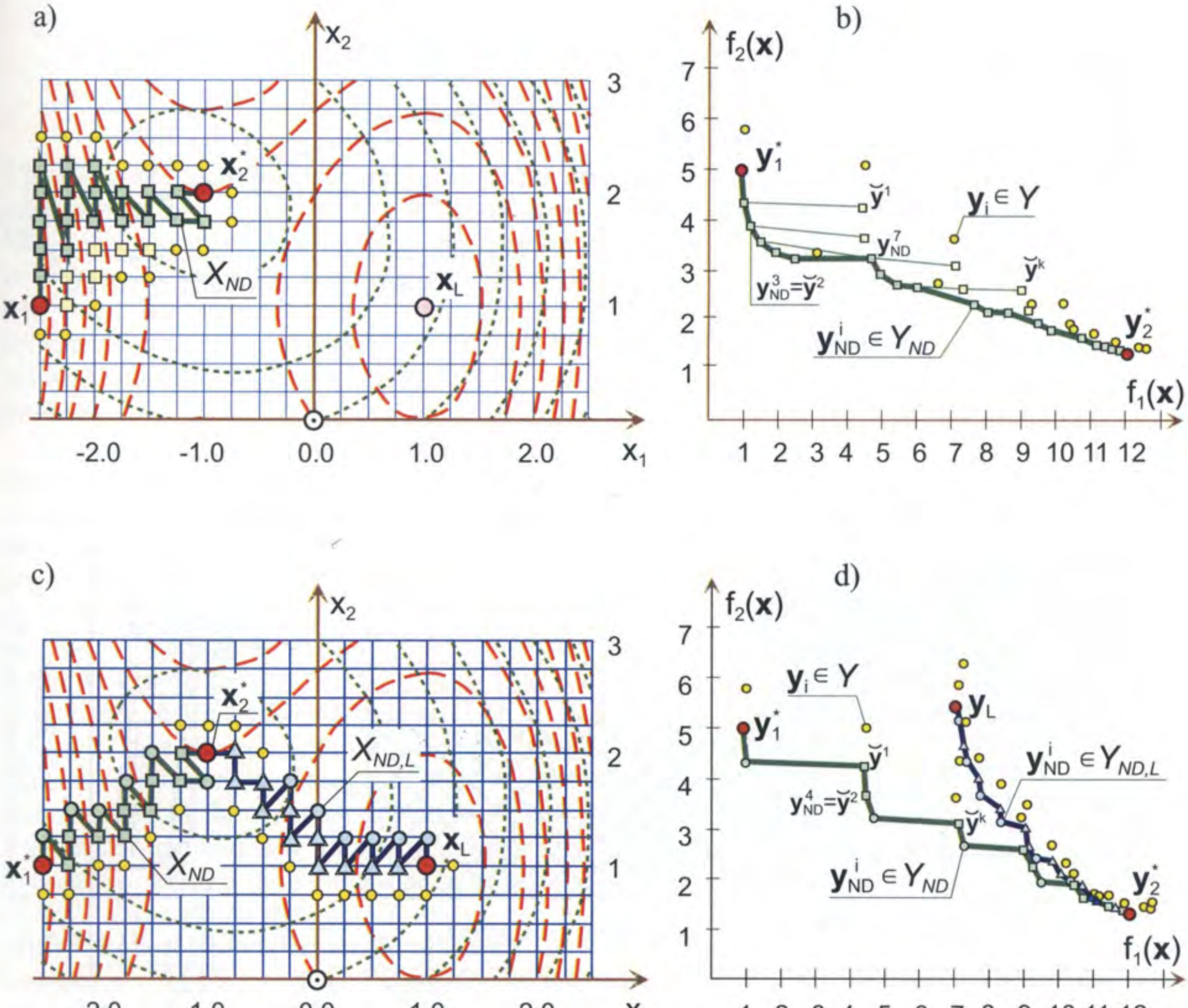

Fig. IV. Results of test problems: a), b) Test No. 5; c), d) Tests No. 1 and No. 6

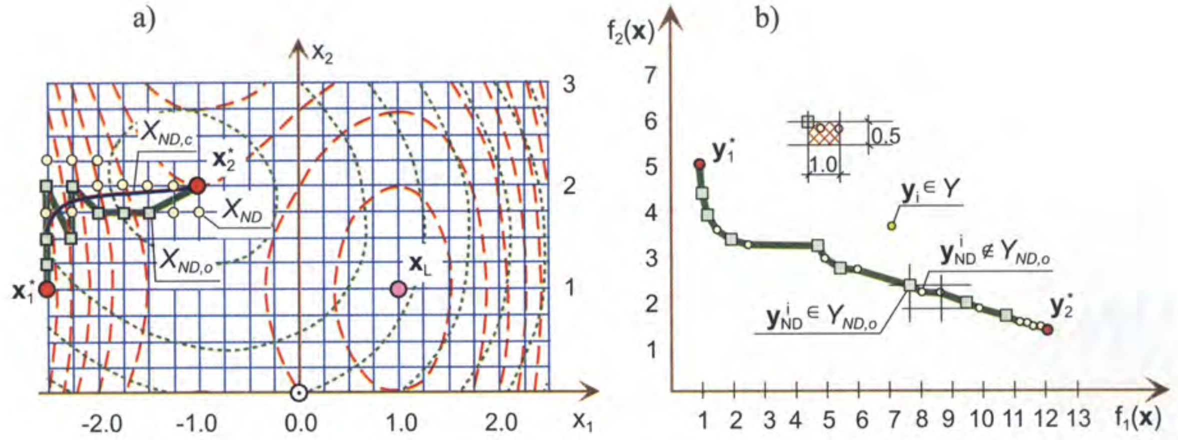

Fig. V. Limitating the number of nondominated sets

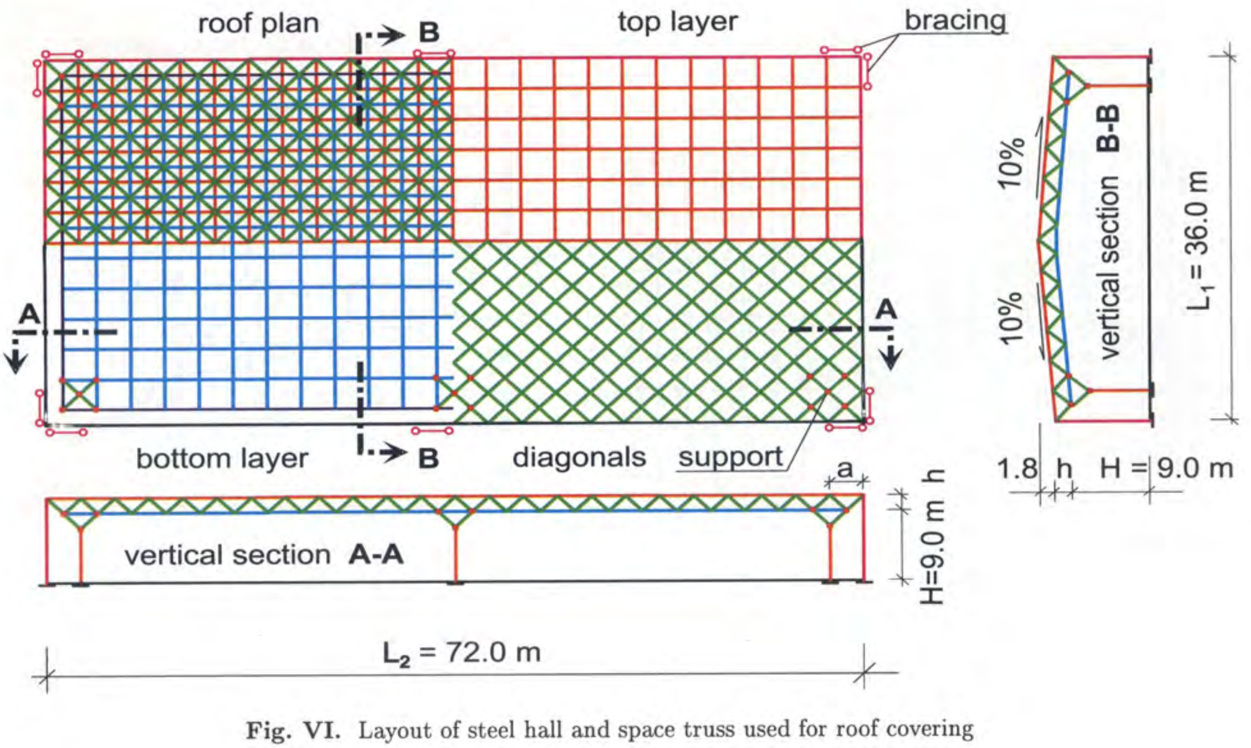

Fig. VI. Layout of steel hall and space truss used for roof covering

Table 1. Tests of a bicriteria optimization problem (15)

Test No.	Starting point	Analyzed neighbourhood	Number of solutions		
			checked	local *ND*	global *ND*
1	$\mathbf{x}_1^* = [-2.5,\ 1.0]^T$	S_1	26	17	10
2	$\mathbf{x}_1^* = [-2.5,\ 1.0]^T$	S_2	19	12	7
3	$\mathbf{x}_1^* = [-2.5,\ 1.0]^T$	S_0	37	20	13
4	$\mathbf{x}_2^* = [-1.0,\ 2.0]^T$	S_1	30	18	13
5	$\mathbf{x}_1^* = [-2.5,\ 1.0]^T$	S_1*	42	21	21
6	$\mathbf{x}_L = [\ 1.0,\ 1.0]^T$	S_1	32	19	1

*) The neighbourhood S_1 was found in the vicinity of all nondominated solutions $\mathbf{x}_{ND}$

The diagrams of objectives $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are shown in Fig. IV. Objective $f_1(\mathbf{x})$ has a local minimum at $\mathbf{x}_L = [1,1]^T$, $f_1(\mathbf{x}_L) = 7.0$ and a global one at $\mathbf{x}_G = \mathbf{x}_1^* = [-2.5, 1.0]^T$, $f_1(\mathbf{x}_G) = 0.875$. Ellipsoidal objective $f_2(\mathbf{x})$ attains its minimum at the point $\mathbf{x}_2^* = [-1,2]^T$, $f_2(\mathbf{x}_2^*) = 1.3899$ which is situated on the edge of the objective $f_1(\mathbf{x})$. The ortho-diagonal vector algorithm was applied a few times. The particular runs of the algorithm are described in Table 1. The starting points and definitions of neighbourhood of the point $\mathbf{x}_i$ are different. In each case, starting from a minimum point of the first function — $f_1(\mathbf{x})$ or $f_2(\mathbf{x})$ — a minimum point of the second function has been attained.

As it was expected, depending on the definition of the neighbourhood S_i, the sets of nondominated evaluations $Y_{ND}(S_i)$ were found which are more or less similar to the global set of nondominated evaluations Y_{ND} (Fig. IVb). Objective $f_1(\mathbf{x})$ is very sensitive with respect to x_1, and less sensitive with respect to x_2 in the vicinity of global minimum $\mathbf{x}_G$. Therefore, the set of discrete nondominated solutions X_{ND} occurs very rarely (see Fig. IVa). The neighbourhood points, e.g. $[-2.5, 1.5]^T$ and $[-2.25, 1.5]^T$ are separated by three other points situated on the constraint $x_1 = -2.5$. Finding of such a set X_{ND} is possible by the (O-D) method after modification of points 4 and 6 of the algorithm presented above. In typical cases, because of considerable saving of computing time, finding the neighbourhood S^k around points $\breve{\mathbf{x}}^k$ which minimize the objective $f_2(\mathbf{x})$ on the set Y^k has been proposed. Finding the full set X_{ND} will be more probable if the neighbourhood S_k around the nondominated solutions found earlier will be determined. The above modification has been applied in the test problem No. 5 and by using orthogonal neighbourhood S_1, the full set X_{ND} has been found (Fig. IVa,b). Figure IV and Table 1 show the results of the tests defined above. In tests No. 1 and 6 the same neighbourhood S_1 has been applied but the starting point in the case of No. 1 is the global minimum $\mathbf{x}_G = \mathbf{x}_1^*$, and in the case No. 6 is a local minimum $\mathbf{x}_L$ of objective $f_1(\mathbf{x})$. The local nondominated evaluations $Y_{ND,L}(S_1)$ are dominated by the global nondominated evaluations $Y_{ND}(S_1)$ — Fig. IVd. The sets of nondominated solutions occur between the corner points $\mathbf{x}_1^* - \mathbf{x}_2^*$, respectively, and between a local minimum $\mathbf{x}_L$, and corner point $\mathbf{x}_2^*$ (Fig. IVc.).

Limitating the number of nondominated sets Y_{ND} and X_{ND} might be essential in technical polyoptimization problems. Small number of solutions $\mathbf{x}_{ND}$ can make easier a choice of a preferred solution $\mathbf{x}_p$, e.g. by experts. The indistinguishability threshold of evaluations of solutions can be defined — Fig. Vb. The nondominated sets Y_{ND} and X_{ND} are appropriately filtered. The indistinguishable evaluations and corresponding solutions (in the sense of assumed thresholds for objectives) are eliminated.

Figure Vb shows the results of filtration of the set Y_{ND}, assuming that the evaluations which differ less than 1.0 for objective $f_1(\mathbf{x})$ and less than 0.5 for objective $f_2(\mathbf{x})$ are eliminated. The constrained set of nondominated solutions $X_{ND,o} \subset X_{ND}$ is shown in Fig. Va. For comparison, the set of nondominated solutions for the continuous problem $X_{ND,c}$ is also shown. Limitating the number of nondominated sets Y_{ND} and X_{ND}, both continuous and discrete, can be done by other methods e.g. by method of representative solutions [11].

6. Optimization of a hall with spatial grid structure

6.1. Formulation of a polyoptimization problem

A steel hall with span dimensions $L_1 \times L_2 = 36 \times 72$ m and a space truss with bars in the form of an orthogonal net used for roof covering is considered as the object of optimization (Fig. VI) [3, 5, 10]. The following optimization parameters are assumed: type of truss net — orthogonal, type of bar cross-sectional area — tubular, type of steel — R35 (calculated strength $f_d = 210$ MPa), metallurgical catalogue of steel tubes T_M [4], distribution of cross-sections of calculated catalogue T in the metallurgical catalogue T_M, $T \subset T_M$, type of loads acting on the structure, type of a technological system (bar-joint connections). The following design variables are considered: $p = L_1/a$ — mesh density of the truss (a is a distance between truss nodes in lower and upper layers), h is the space truss depth, t is a number of bar cross-sections in catalogue T, s is a kind of space truss supports, A_i are the cross-sectional areas of truss bars. This type of design variables leads to a discrete optimization problem. The design variable vector can be written in the following form:

$$\mathbf{x} = [p, h, t, s, A_i]^T. \tag{16}$$

The domain of feasible solutions is determined by design, technological and computational constraints listed in Table 2. The design constraints are concerned with limit load capacity state

Table 2. Optimization constraints

Design constraints		Technological constraints	
$N_i^t \le A_i f_d, \quad i = 1, \ldots, I$	(17.1)	$2.0 \le a \le 4.5$ m	(19.1)
$N_i^c \le \varphi_i A_i f_d, \quad i = 1, \ldots, I$	(17.2)	$8 \le (p = L_1/a) \le 16$	(19.2)
$\lambda_i^c = \dfrac{\mu l_i}{i_i} \le 250$	(17.3)	$L_1/30 = 1.2 \le h \le L_1/10 = 3.6$ m	(19.3)
$[K]_v \{\delta\}_{v\,cl} = \{P\}_{v\,cl}$	(17.4)	$h/0.3 \in \langle 4, 11 \rangle \subset C$	(19.4)
$\delta_z \le \dfrac{L_1}{250} = 14.4$ cm	(17.5)	$30 \le \left[\alpha = \arctan\left(\sqrt{2}\, h/a\right)\right] \le 60°$	(19.5)
		$A_i \in T \subset T_M$	(19.6)
Computational constraints		$2.9 \le g \le 11.0$ mm	(19.7)
$z_{\%} \in \langle 90, 100\% \rangle$	(18.1)	$31.8 \le D \le 273.0$ mm	(19.8)
$z_m \in \langle 0.1, 1.0\% \rangle$	(18.2)	$1 \le t \le 25$	(19.9)
$c_l \in C_l = \{1, 2, \ldots, C_l\}$	(18.3)	$t \in \{25, 13, 9, 7, 5\}$	(19.10)
$C_l \in \{1, 5, 17\}$	(18.4)	support pattern $s \in \{s_1, s_2\}$	(19.11)
		$\lambda_{\max} = \dfrac{l_{\max}}{i_{\min}} = \dfrac{4.58\text{ m}}{0.0103\text{ m}} = 445 < 500$	(19.12)

where: N_i^t, N_i^c – tension ($\cdot^t$) and compression ($\cdot^c$) of the i-th element,
ϕ_i – instability parameter of the i-th element,
λ_i^c, i_i – slenderness ratio and radius of inertia of the i-th element,
$[K]_v$ – stiffness matrix in the v-th iteration,
$\{P\}_{v\,cl}$ – load vector in the v-th iteration and c_l-th load combination,
δ_z – vertical displacement of a truss node,
C_l – a number of the load combinations,
$z_{\%}$ – rate of the number of bars which do not change cross-sections in consecutive iterations,
z_m – rate of variation of the truss mass in consecutive iterations,
C – the set of integers.

(17.1–17.4) and limit serviceability state of the truss (17.5). The buckling of compressed bars is taken into account. It is assumed that there is no weakening of bar cross-sections under tension due to their connections at joints, i.e., $A_{\text{net}} = A_{\text{gross}}$. The vertical displacement of truss nodes is limited to 1/250 of the span L_1.

The technological constraints limit variation of truss geometry (Fig. 6) through the acceptable mesh density of the truss p (19.1–19.2), its depth h (19.3–19.4) and the angle of inclination between diagonals and upper layer surface α (19.5).

The technological constraints also take into account the metallurgical catalogue T_M shown in Fig. 7. The minimal cross-sectional area is chosen in such a way that the slenderness of the longest bar $\lambda_{\max} \leq 500$. The maximal cross-sectional area is chosen on the basis of maximal compression and tension forces in the truss. The above conditions allow to choose the diameters and wall thicknesses of tubular bars, i.e., $D/g = 31.8/2.9$ mm to $273.0/11.0$ mm (19.6–19.8). For small wall thicknesses of tubular bars it is assumed that they are sufficiently protected against corrosion. The consecutive catalogues of cross-sections with numbers t (19.9–19.10) are created on the basis of the 25-elements catalogue through leaving every 2-nd, 3-rd, 4-th and 6-th cross-sectional area.

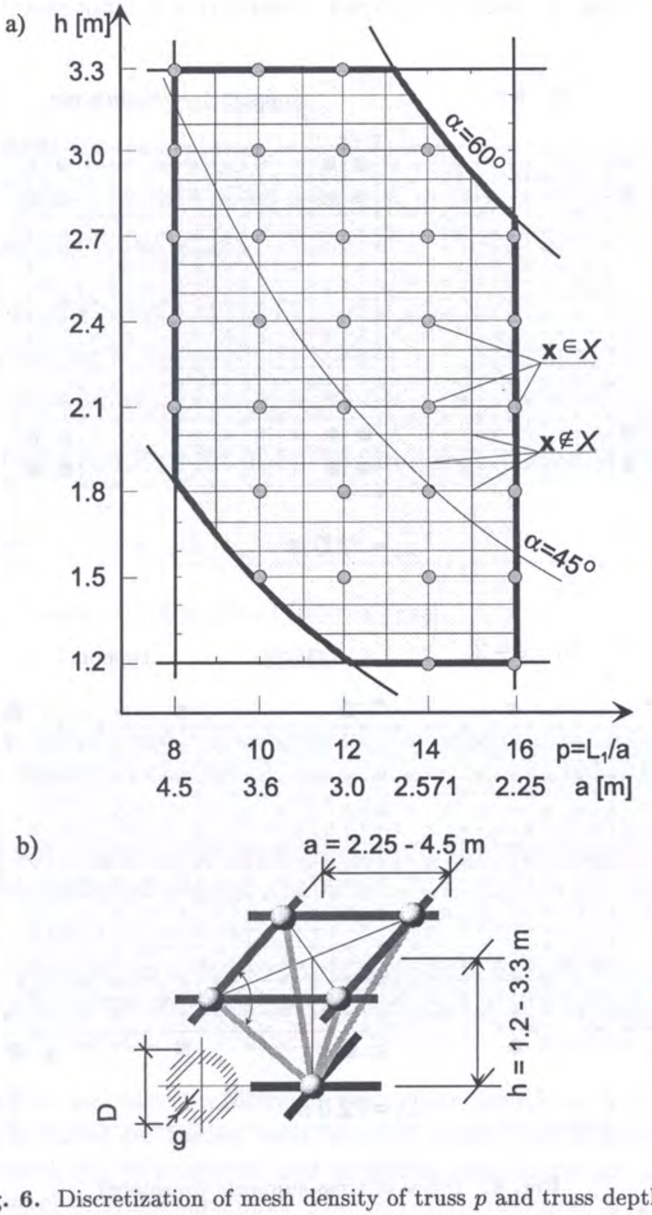

Fig. 6. Discretization of mesh density of truss p and truss depth h

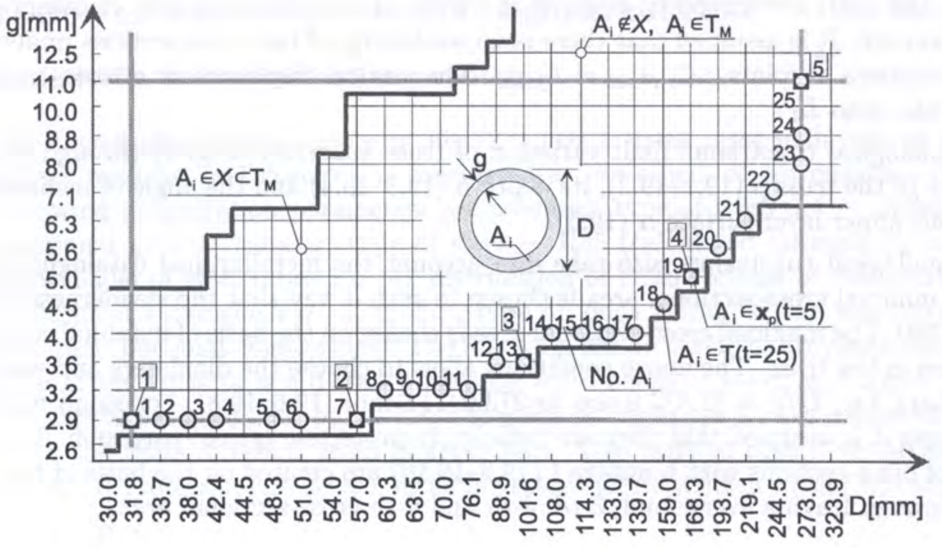

Fig. 7. Basic ($t = 25$) and preferred ($t = 5$) catalogues

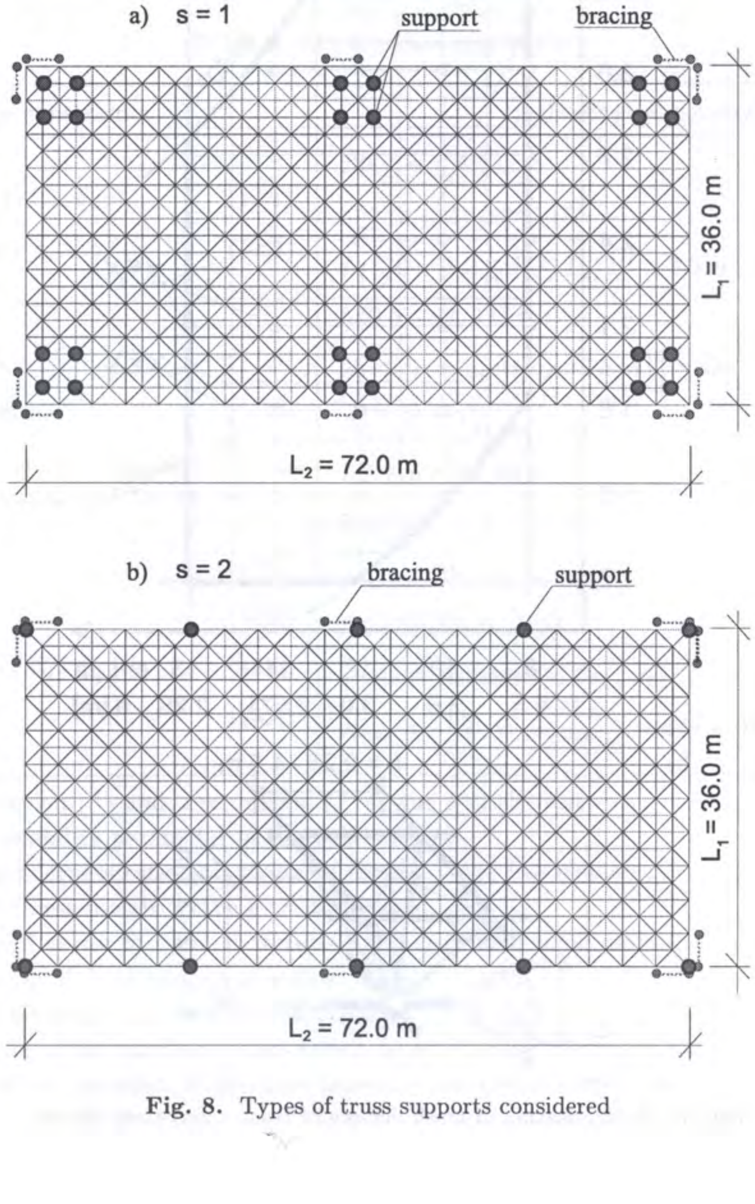

Fig. 8. Types of truss supports considered

The space truss is supported by 6 or 10 columns (19.11), distributed uniformly along longitudinal walls of the hall (Fig. 8). Hall diagonal braces placed in longitudinal and transverse walls play a role of elastic supports of the roof structure. In the s_1 case the trusses are simply supported at lower layer on six V-shape columns. In the s_2 case the upper layer of the space truss is supported by columns at individual points.

Computational constraints are concerned with the particular Truss program used for truss design. The program needs, on average, from 4–6 to 8–12 iterations, depending on the number of elements t in the catalogue, the number of load combinations C_l, and termination criteria $z_{\%}$ and z_m. It allows to find the distribution of cross-sectional areas of truss bars which minimizes the truss mass.

The structures are evaluated with respect to the following three objectives which are minimized.

- mass of the truss, columns and steel elements of walls needed for $1\,\mathrm{m}^2$ of the horizontal projection of the roof

$$f_1(\mathbf{x}) = \frac{a_d \gamma}{L_1 L_2} \sum_{i=1}^{I(p)} A_i\,(p, h, t, s, c_l)\; l_i + \frac{M_s + M_w + M_b}{L_1 L_2} \tag{20}$$

where:

$\gamma = 7850\,\mathrm{kg/m^3}$ – the bulk density of steel;

$a_d = 1.03$ – a supplement for truss joints;

$I(p)$ – the number of truss bars with mesh density p,

$I(p) \in \{576, 900, 1296, 1764, 2304\}$;

M_s – the mass of columns for truss depth h;

M_w – mass of steel elements of walls for truss depth h;

M_b – the mass of diagonal braces for truss depth h;

- the manufacturability function for $1\,\mathrm{m}^2$ of the horizontal projection of the roof [9, 17]

$$\begin{aligned} f_2(\mathbf{x}) = {} & \frac{p^2 + (p+1)^2}{L_1 L_2}\, a^2 \left[1 + \left| \sin\left(\frac{\pi}{4} - \arctan \frac{\sqrt{2}\,h}{a} \right) \right| \right] \\ & \times \left[1 + c_1 \left(1 + c_2 - 0.001 c_2 I(p) \right) t \right] \left[1 + c_3\,(s) \right] \end{aligned} \tag{21a}$$

where:

c_1 – the coefficient of catalogue size [17], designating how much the manufacturability is increased after doubling the number of elements in the catalogue, $c_1 \in \langle 0.005, 0.015 \rangle$, here it is assumed that $c_1 = 0.01$;

c_2 – the coefficient of series length [17], specifying how much the manufacturability is decreased after doubling the number of bars in the truss, $c_2 \in \langle 0.05, 0.15 \rangle$, here it is assumed that $c_2 = 0.1$;

c_3 – the coefficient depending on a type of roof supports; here is assumed that for the truss supported at lower layer on six V-shape columns $c_3(s_1) = 0.1$, and for the point support $c_3(s_2) = 0.0$;

Function $f_2(\mathbf{x})$ describes manufacturability of particular solutions belonging to a domain of feasible solutions. It is based on taking into account mainly an influence of assumed design variables on manufacturability of solutions and assigning smaller values of the function $f_2(\mathbf{x})$ to the solutions having better manufacturability. Therefore, the function $f_2(\mathbf{x})$ should be minimized.

Manufacturability of solutions, under given vector of design variables $\mathbf{x}$, depends mainly on the angle α of inclination diagonal bars to planes of upper and lower surface layers (Fig. 6), i.e.,

$$\alpha = \arctan \frac{h\sqrt{2}}{a}. \tag{21b}$$

The more the angle α tends to $\pi/4$, the smaller is the size of the joints and it is easier to manufacture them. The component $|\sin(\pi/4 - \alpha)|$ used in the function expression causes the contours of $f_2(\mathbf{x})$ to form a valley shape with bottom edge corresponding to $\alpha = \pi/4$ (Fig. 11).

The manufacturability of solutions depends strongly on the number of joints in a truss structure and on their shapes. The assumed vector of design variables $\mathbf{x}$ disregards such joint parameters as: type of joint, the way of connecting the bars, the size count of joint catalogue. Therefore, they are not represented in function $f_2(\mathbf{x})$. Only the total number of truss joints is taken into account. The factor $[p^2 + (p+1)^2]$ describes the number of truss joints with mesh density p on the surface $L_1 \times L_2$. The fewer truss joints, the faster is the truss manufacture. This tendency is countered by an equally important tendency, namely: the fewer elements (bars, joints), the larger, longer and heavier are they and thus harder to produce and assembly. This property is taken into account in function $f_2(\mathbf{x})$ by the component a^2 describing the lengths of bars in truss layers. Cumulative effects of number of joints and bar lengths is such that the greater is the mesh density p, the smaller value of function $f_2(\mathbf{x})$, i.e., the manufacturability of solution is improved. Assembly of light and short elements does not require an assistance of heavy equipment. It should be stressed that the labor cost is a separate issue here. Assuming that labor cost strongly depends on the number of joints, the greater mesh density of the truss, the more expensive is the structure.

The function $f_2(\mathbf{x})$ includes also a coefficient of catalogue size c_1 and a coefficient of series length of ordered elements c_2. They take into account the following properties:

1. The smaller is the number of different bar cross-sections in the catalogue, the easier is to mark (differentiate) them, the more identical elements are in the structure, and the smaller size count of joints catalogue is necessary.
2. The greater is the mesh density p, the bigger is the number of structural elements (bars and joints), and the longer are the series of identical elements.
3. The more identical elements is in the structure, the easier and cheaper is the realisation of orders (preparation of production, marking, packing, transportation).

- maximal vertical displacement of the truss node

$$f_3(\mathbf{x}) = \max \delta_z(\mathbf{x}). \tag{22}$$

The vector of the objective function for this polyoptimization problem has a form

$$\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x})]^T. \tag{23}$$

6.2. Designing the spatial grid structure

We analyze here a typical load combination for a hall without suspended transport, situated in the Polish climatic zone (the first zone for wind and snow loads according to the Polish Code). The vertical loads are as follows [16]: dead load of truss which is automatically calculated at each design iteration $\gamma_{f_1} = 1.1$ (γ_{f_i} — the coefficient for i-th loading, $\gamma_{f_i} \in \langle 0.9, 1.4 \rangle$), roofing $0.5\,\mathrm{kN/m^2}$, $\gamma_{f_2} = 1.2$, technological loading $0.5\,\mathrm{kN/m^2}$, $\gamma_{f_3} = 1.4$, snow $0.56\,\mathrm{kN/m^2}$, $\gamma_{f_4} = 1.4$ and wind

$-0.405\,\mathrm{kN/m^2}$ and $-0.225\,\mathrm{kN/m^2}$, $\gamma_{f_5} = 1.3$. The vertical loads are distributed to lower and upper truss layers according to their place of acting. The horizontal loads come from pressure and suction of wind and are transmitted by wall panels to the truss nodes of upper layer and then on to vertical braces of walls. As many as $C_l = 17$ combinations of loads have been created taking into account all possible loading conditions and coefficients of simultaneous occurrence of varying loads. The preferred solution is determined in the last stage of design process taking into account all possible combinations of loading conditions. The previous design experience [8, 9, 16, 17] allows to take into account only combinations of vertical loads in the first stages of design process which lead to determination of nondominated solutions.

The trusses are designed in linear elastic range using the method of limit states. The Truss program, using a displacement method, selects iteratively the cross-sectional areas of truss bars from a given catalogue containing $t \geq 1$ elements.

The global stiffness matrix is build in the first iteration with the minimal cross-sectional areas of truss bars taken from the catalogue T. The bar forces are calculated on the basis of node displacements under characteristic loading conditions and their combinations. The program chooses the cross-sectional area of each bar according to the criterion of full stress design under the worst combination of loads for that bar. The bars under tension are designed assuming that there is no weakening of bar cross-sections due to connections at joints, i.e. $A_{\text{net}} = A_{\text{gross}}$. The compressed bars are designed taking into account stability conditions, assuming a coefficient of buckling length $\mu = 1.0$. The slenderness for compressed bars is limited to $\lambda = 250$ (17.3), and for bars under tension to $\lambda = 500$ (19.12). The stiffness matrix in the v-th iteration, $[\mathbf{K}]_v$, is build on the basis of cross-sectional areas of bars determined in the previous iteration. The termination criteria for the first limit state are as follows: no changes in cross-sectional areas of bars in two consecutive iterations or simultaneous fulfilment of the following conditions: acceptable rate of bars without changes of cross-sections $z_{\%}$ and rate of variations of truss mass in consecutive iterations z_m.

Next, the combinations of computed loads are replaced by characteristic loads and the node displacements are determined. The program Truss does not design the trusses with respect to the second limit state, i.e., does not change the cross-sections of bars determined in the first limit state in the case of violation of the permissible displacement. The values $z_{\%}$ and z_m are given. To determine the nondominated solutions the following data is assumed: $z_{\%} = 90\%$ and $z_m = 1\%$ in the I-st and II-nd stage, and $z_{\%} = 99\%$ and $z_m = 0.1\%$ in the III-rd stage.

6.3. Decomposition of the optimization problem

The feasible domain of the above formulated optimization problem contains 450 solutions, i.e., variants of hall structure. Of that, 340 solutions satisfy explicit constraints of the feasible domain. Test of the (O-D) algorithm is done by comparing the results obtained by the algorithm and by a decomposed enumeration of variants. Time needed for realization of numerical calculations is initially estimated. Computations are made on a Pentium 100 MHz/48MB RAM computer.

Figure 9 shows the average times of roof design for the following data: one combination of four vertical loading conditions, 7-elements catalogue of cross-sections of bars, termination criteria — $z_{\%} = 90\%$ and $z_m = 1\%$. The average number of iterations is 4.6. It can be estimated that the time of numerical calculations (without data preparation) under above constraints for the full enumeration method is around 52 hours. In the case of $z_{\%} = 100\%$, the computing time is increased to about 100 hours. Taking into account the additional 16 schemes of wind and snow loading conditions would need 200 additional hours. The memory capacity needed to write the final results is 120 MB (21 MB after compression).

Testing of the (O-D) algorithm is done on decomposed polyoptimization truss problem. Decomposition concerns the following elements [9, 18]: vector of design variables $\mathbf{x}$, vector of objectives $\mathbf{f}(\mathbf{x})$ and optimization method. The vector of design variables is divided by a so called disjoint

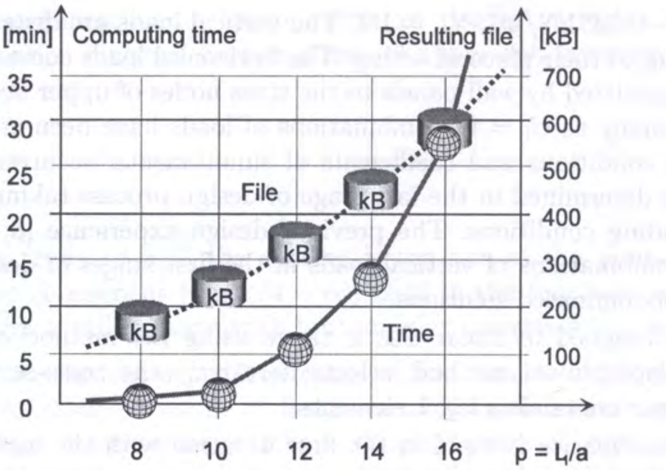

Fig. 9. Time of truss analysis and size of the resulting file

decomposition into two independent subvectors

$$\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]^T = \left[\mathbf{x}^I, \mathbf{x}^{II}\right]^T = \left[[x_1\ x_2\ x_5]^T, [x_3, x_4, x_5]^T\right]^T$$
$$\mathbf{x}^I = [x_1, x_2, x_5]^T = [p, h, A_i]^T, \qquad \mathbf{x}^{II} = [x_3, x_4\ x_5]^T = [t, s, A_i]^T \tag{24}$$

Subvector $\mathbf{x}^I$ contains design variables concerned with geometry of the space truss. Subvector $\mathbf{x}^{II}$ contains design variables concerned with catalogue of cross-sections of bars and a type of roof supports. The cross-sectional areas of bars A_i occur in both subvectors $\mathbf{x}^I$ and $\mathbf{x}^{II}$. But A_i does not become a joint variable for subvectors $\mathbf{x}^I$ and $\mathbf{x}^{II}$, because it is chosen individually by Truss program for each variant of the structure.

The vector of objectives $\mathbf{f}(\mathbf{x})$ is divided in such a way that particular criteria of optimization lead to the best technological and structural solutions for given subvectors $\mathbf{x}^I$ and $\mathbf{x}^{II}$. Variants containing the geometrical design variables p and h are evaluated with respect to the mass of truss, columns and hall walls — $f_1(\mathbf{x})$ and with respect to technological objectives — $f_2(\mathbf{x})$. Variants described by the subvector of design variables t and s are evaluated with respect to all criteria of the

Table 3. Stages of polyoptimization of spatial grid structures

Stage	$z\%$ z_m c_l	$\mathbf{x}$	$\mathbf{f}(\mathbf{x})$	S	$\mathbf{x}_0$	Number of solutions checked	Number of solutions local ND	$\mathbf{x}_p$
I	90 1.0 1	$[p, h, 7, 1]^T$	$[f_1(\mathbf{x}), f_2(\mathbf{x})]^T$	S_1 S_2 $S0$	$[14, 1.8, 7, 1]^T$	13 11 16	6 5 6	$[12, 2.1, 7, 1]^T$
II	90 1.0 1	$[12, 2.1, t, s]^T$	$[f_1(\mathbf{x}), f_2(\mathbf{x})]^T$	S_1 S_0	$[12, 2.1, 7, 1]^T$	10 10	9 9	$[12, 2.1, 5, 1]^T$
			$[f_1(\mathbf{x}), f_3(\mathbf{x})]^T$	S_1 S_0		10 10	5 5	
			$[f_2(\mathbf{x}), f_3(\mathbf{x})]^T$	S_1 S_0		5 6	2 2	
III	99 0.1 17	$[p, h, 5, 1]^T$	$[f_1(\mathbf{x}), f_2(\mathbf{x})]^T$	S_1	$[12, 2.1, 5, 1]^T$	14	6	$[12, 2.1, 5, 1]^T$

optimization problem. Three following bicriteria optimization problems are considered: $f_1(\mathbf{x})-f_2(\mathbf{x})$, $f_1(\mathbf{x}) - f_3(\mathbf{x})$ and $f_2(\mathbf{x}) - f_3(\mathbf{x})$.

Decomposition of the (O-D) method is based on three ways of defining point neighbourhood. Only the neighbourhoods S_1, S_2, and $S_0 = S$ are considered, because both subproblems are two-dimensional with respect to design variables (Fig. 4). The optimization problem is solved in three stages. Descriptions of design variables vector, objectives and analyzed point neighbourhoods in the particular stages are shown in Table 3.

6.4. Evaluation of the optimization process

The results of the optimization of truss mass with respect to the subvector $\mathbf{x}^I$ containing design variables concerned with geometry of the truss, obtained by full enumeration method and with the use of the Truss program, are presented in Fig. 10. Function $f_1(\mathbf{x})$ attains its minimum on the constraint $p = 8$ for $h = 2.1$ m, i.e. $(\mathbf{x}^I)^{*1} = [8,\ 2.1]^T$. In the above solution the lengths of bars in lower and upper layers are rather large, i.e. $a = 4.5$ m which is not appropriate from a technological point of view.

Analysis of the manufacturability of the structure is made using spread sheet. Contours of function $f_2(\mathbf{x})$ have a shape of a curved valley (Fig. 11), with bottom corresponding to the angle of inclination of diagonals of the truss, i.e., $\alpha = 45°$ $(\pi/4)$. Function $f_2(\mathbf{x})$ attains its discrete minimum at the point $(\mathbf{x}^I)^{*2} = [14,\ 1.8]^T$.

Determination of the sets of nondominated evaluations and solutions by the (O-D) algorithm started from the point $\mathbf{x}_0 = (\mathbf{x}^I)^{*2} = [14,\ 1.8]^T$. It uses the fact that the minimum of function $f_2(\mathbf{x})$ is easy to find because it does not require designing the new structure. The nondominated sets are found using the independent neighbourhoods S_1, S_2, and S_0. Figure 12 shows the results as seen in evaluation space and in the solution space, for the neighbourhoods S_1 and S_2. The

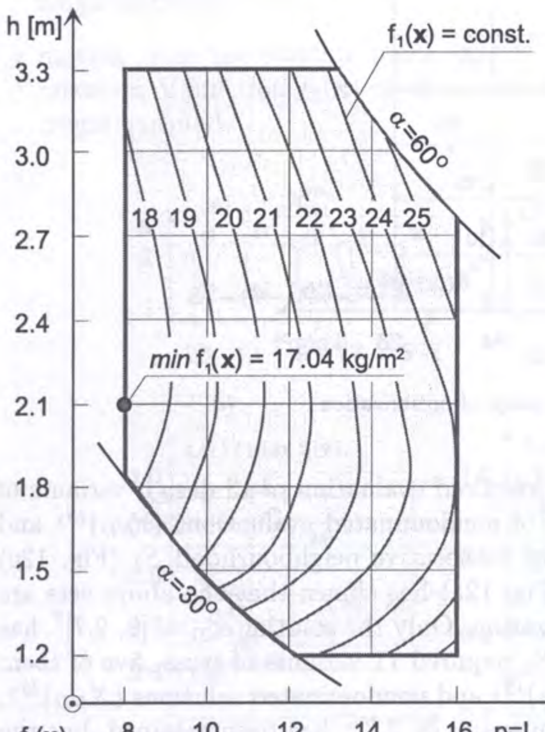

Fig. 10 Mass contours of truss, columns and walls

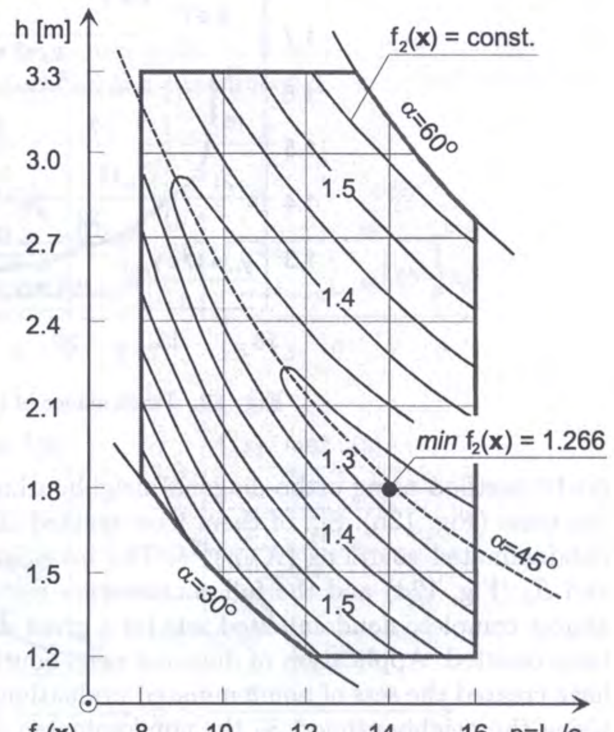

Fig. 11 Contours of space truss manufacturability function

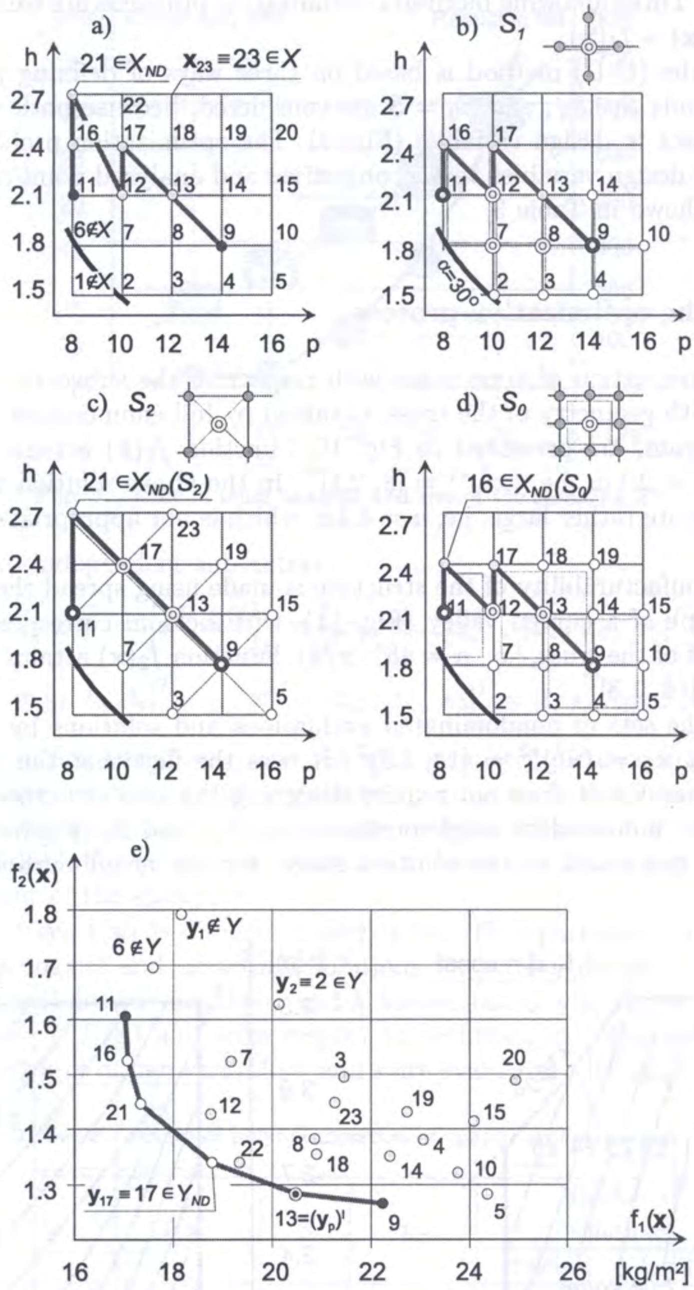

Fig. 12. Performance of the I-st stage of optimization

(O-D) method using ortho-diagonal neighbourhood S_1 required evaluation of 13 design variants of the truss (Fig. 12b). Six of them have created the sets of nondominated evaluations $(Y_{ND})^{IS_1}$ and nondominated solutions $(X_{ND})^{IS_1}$. The investigation of consecutive neighbourhood S_2 (Fig. 12c) and S_0 (Fig. 12d) and the full enumeration method (Fig. 12a) has shown that the above sets are almost complete nondominated sets for a given discretization. Only the solution $\mathbf{x}_{21} = [8,\ 2.7]^T$ has been omitted. Application of diagonal neighbourhood S_2 required 11 variants of truss, five of them have created the sets of nondominated evaluations $(Y_{ND})^{IS_2}$ and nondominated solutions $(X_{ND})^{IS_2}$. Using the neighbourhood S_2 the nondominated solution $\mathbf{x}_{21} = [8,\ 2.7]^T$ has been attained, but the solution $\mathbf{x}_{16} = [8,\ 2.4]^T$ has been omitted (Fig. 12c).

The preferred solution $(\mathbf{x}_p)^I$ has been chosen for the second stage on the basis of minimization of a scalar global function:

$$f_g(\mathbf{x}_{ND}) = w_1 \frac{f_1(\mathbf{x})}{\max f_1(\mathbf{x})} + w_2 \frac{f_2(\mathbf{x})}{\max f_2(\mathbf{x})} + w_3 \frac{f_3(\mathbf{x})}{\max f_3(\mathbf{x})} + w_4 \frac{h}{\max h}, \qquad \sum_{i=1}^{4} w_i = 1.0 \tag{25}$$

The following vector of weights is assumed: $\mathbf{w} = [0.25,\ 0.5,\ 0.1,\ 0.15]^T$. Only the nondominated solutions $\mathbf{x}_{ND} \in X_{ND}$ are evaluated. The preferred solution $(\mathbf{x}_p)^I = [12,\ 2.1,\ A_i]^T$ is chosen as a starting point $(\mathbf{x}_0)^{II}$ for the second stage.

The partial nondominated solutions are searched for at the second stage. The neighbourhoods S_1 and S_0, have been investigated for the three following bicriteria optimization problems: $f_1(\mathbf{x})-f_2(\mathbf{x})$, $f_1(\mathbf{x}) - f_3(\mathbf{x})$ and $f_2(\mathbf{x}) - f_3(\mathbf{x})$, for which 9, 5, and 2 nondominated solutions have been obtained, respectively (see Fig. 13 and Table 3). For the neighbourhoods S_1 and S_0, the same solutions have been investigated because of two-valued discretization of the variable s describing the type of truss supports. Only the order of their analysis changes. Finally, 9 solutions from 10 feasible solutions are considered for the three criteria nondominated sets $(X_{ND})^{II}$ and $(Y_{ND})^{II}$ at the second stage. Again, in the second stage, minimization of the global function $f_g(\mathbf{x})$ leads to the preferred solution $(\mathbf{x}_p)^{II} = [12,\ 2.1,\ 5,\ 1,\ A_i]$. The values $t = 5$ (a number of elements in the cross-sections catalogue) and $s = 1$ (six V-shape columns support) become the parameters at the third stage of the optimization. At this stage, all combinations of loading conditions are taken into account ($C_l = 17$), even those that increase by one the cross-sectional area index for only a few truss bars (for the structure designed with respect to combination of vertical loads). Accuracy of the solution, controlled by parameters $z_{\%}$ and z_m, is also increased (Table 3).

The preferred solution of the second stage $(\mathbf{x}_p)^{II}$ is taken as the starting point for the third stage, i.e., $(\mathbf{x}_0)^{III} = (\mathbf{x}_p)^{II}$. It requires modification of the (O-D) algorithm in such a way that determination of nondominated sets should take place in two steps:

- moving from the starting point $(\mathbf{x}_0)^{III}$ to the minimum point of function $f_1(\mathbf{x})$ — the minimization of function $f_1(\mathbf{x})$ on the actual neighbourhood is a criterion of choosing the consecutive neighbourhood,
- moving from the starting point $(\mathbf{x}_0)^{III}$ to the minimum point of function $f_2(\mathbf{x})$ — the minimization of function $f_2(\mathbf{x})$ on the actual neighbourhood is a criterion of choosing the consecutive neighbourhood.

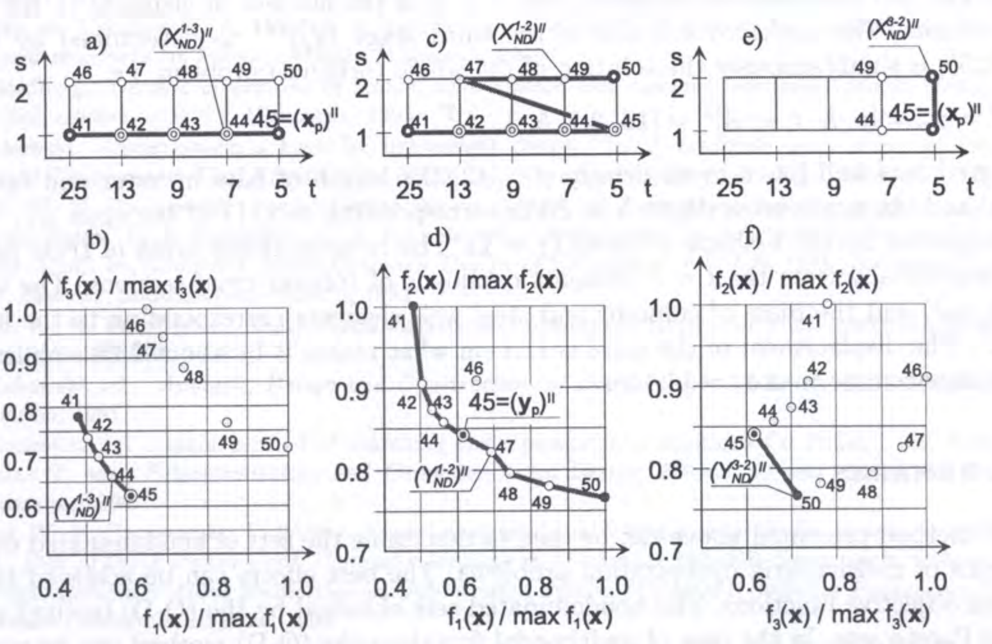

Fig. 13. Performance of the II-nd stage of optimization

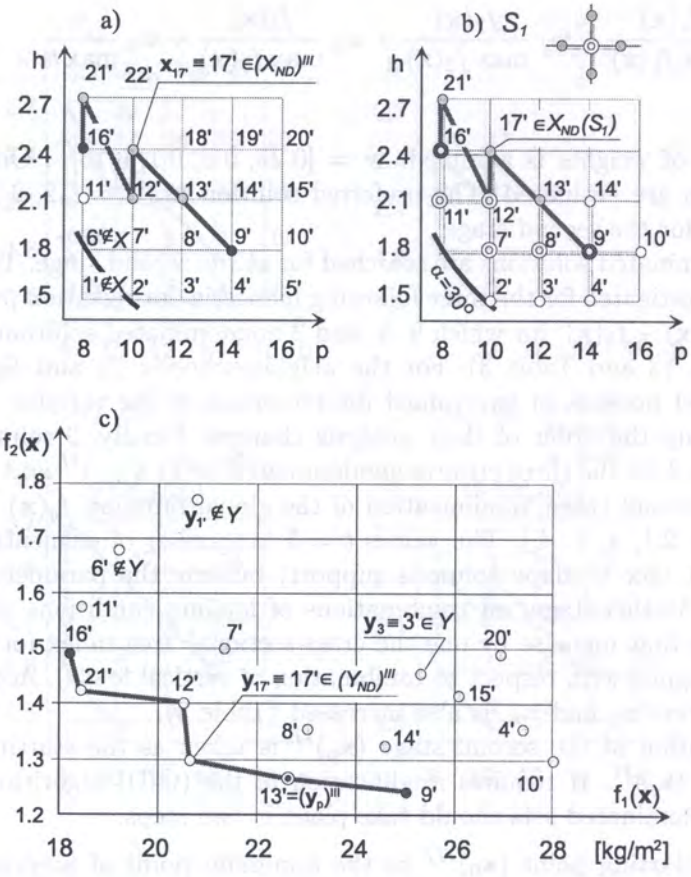

Fig. 14. Performance of the III-rd stage of optimization

The neighbourhood S_1 was only used in the third stage. The obtained set of nondominated solutions for design variables p and h is almost identical as in the first stage (Fig. 14). The truss mass is increased in average by about 10% due to the increase of the number of combinations of loading conditions and the simultaneous decrease from 7 to 5 of the number of elements in the catalogue of cross-sections. The preferred solution of the third stage $(\mathbf{x}_p)^{III}$ as determined by the global function (25), is simultaneously the solution of the whole (original) problem, i.e.,

$$(\mathbf{x}_p)^{III} = \mathbf{x}_p = [p,\ h,\ t,\ s,\ A]^T = [12,\ 2.1,\ 5, 1,\ A]^T. \tag{26}$$

The preferred roof hall has a mesh density $p = 12$, the length of bars in lower and upper layers $a = 3.0$ m, and the space truss depth $h = 2.1$ m corresponding to 1/17 of the span L_1. The space truss is supported by six V-shape columns ($s = 1$). The cross-sectional areas of truss bars A_i are matched individually from the $t = 5$ elements catalogue of tubular cross-sections. The truss mass is 18.31 kg/m^2, and the mass of columns and steel wall elements corresponding to the height h is 4.00 kg/m^2. The displacement of the truss is 11.1 cm what makes it by about 23% smaller than the allowable displacement $\max \delta_z = 14.4$ cm.

7. Final remarks

The (O-D) method presented above can be used to determine the sets of nondominated evaluations and solutions of multicriteria optimization problems. The best effects can be achieved in the case of unimodal objective functions. The nondominated sets obtained by the (O-D) method are in this case global Pareto sets. In the case of multimodal functions the (O-D) method can be used to find local nondominated sets.

Although the (O-D) method has been developed for discrete problems, it can be also used with success for continuous polyoptimization problems or mixed (continuous-discrete) problems. The (O-D) method can be additionally controlled in those cases by varying size and shape (spherical or ellipsoidal) of neighbourhood of the point.

The effectiveness of the discrete (O-D) method depends on the definition of neighbourhood of the given point. Unfortunately, increase of the number of design variables leads to exponential increase of the number of points in the neighbourhood S. The use of full neighbourhood S is recommended in the cases of 2 or 3 design variables. The use of ortho-diagonal neighbourhood S_0 is recommended in the cases of 4 to 6 design variables and the orthogonal neighbourhood S_1 is recommended in the case of larger numbers of design variables. The presented (O-D) algorithm is easy to modify in such a way that it can be started from neighbourhood S_1, and consecutive neighbourhoods can be added only when they are needed, i.e., when the nondominated solutions are not found in the neighbourhood S_n.

The (O-D) method can be used for polyoptimization of structures as well as for identification, management and control problems.

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