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Filtration in cohesive soils: mathematical model

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The paper discusses the physical basis of the process of filtration of water in a case of very low velocities and presents the mathematical model of the process, based on a new constitutive formula. The existence and uniqueness of a weak solution to the resulting nonhomogeneous initial boundary-value problem is then proven.

1. Introduction

The paper concerns the physical foundations and the mathematical model of the process of filtration of water through the porous media for low velocities of water (the so–called prelinear filtration).

The problem of water filtration through cohesive and organic soils is basic in physics of soils and its various numerical models have been recently investigated, see e.g. [3, 13, 18, 22] and the references cited therein. It is also fundamental in civil engineering, since such soils make a significant part of building sites or agricultural terrains. The evaluation of the filtration yield is required in the design and the exploitation of the drainage systems or earthern dams or in the description of the consolidation phenomena appearing e.g. in massive earthern structure bodies or in the soil underlying structure foundations.

The paper is organized as follows. Sections 2 and 3 are devoted to the description of the phenomenon of filtration and discuss various approaches to constitutive formulae. The mathematical description of the filtration process is presented in Secs. 4 and 5, being the main part of the paper contains existence-uniqueness results for the mathematical model, proved in the Appendix. The final Section contains various comments. Note that earlier results in this direction, obtained by authors have been presented in papers [15, 16].

2. Physical foundations of the filtration process

There are several reasons for which water filtrate differently through cohesive soil than through sand or another, medium-size grained materials:

a) The typical water movement in cohesive materials like clays or silts is characterized by a very small velocity, since their water permeability compared with this of pure sand is about 3-10 orders of magnitude smaller.

b) All phenomena of ground water occurring at the interfaces are dominating in the filtration process. In particular, the loosely attached pore water has a comparable or even a greater volume than the free one. This is a consequence of the fact that the specific surface, i.e. the surface of particles or domains per unit weight, of the cohesive soil skeleton is much greater than in the typical medium-grained soil. Particles of silt and clay usually have the form of plates (see Fig. 1 and illustrations in [7]), hence their surfaces are much more expanded than the surfaces of grain whose shape is more regular.

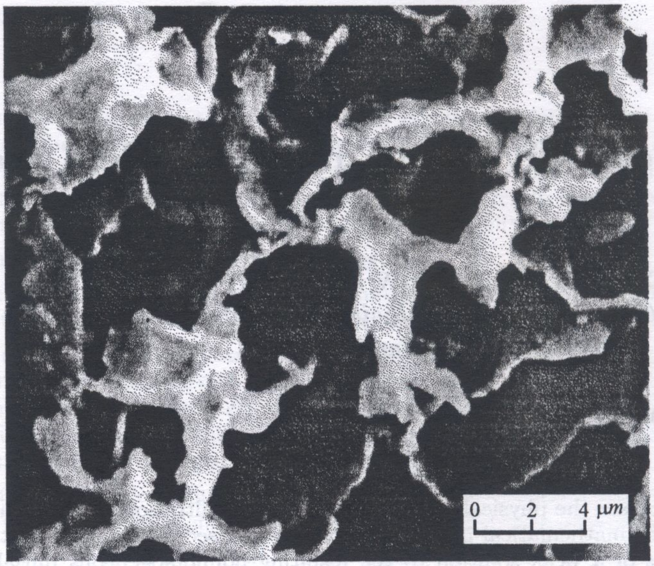

Fig. 1. Highly expanded surface of silt skeleton (5000 times enlarged view of sea bottom silt from Baltic sea)

c) The surfaces of skeleton particles are usually negative electrically charged. Semipolar water molecules and solution cations are attracted by the skeleton forming the so called double-layer (cf. [6, 19]), composed of the strongly attached layer – the *Stern layer* (cf. Stern [19]) in which water molecules and cations are attached with tension more than 20 – 50 MPa and the second one, much thicker – the *diffusive layer* in which the electric potential and the tension of attraction caused by it decrease exponentially to zero (see Fig. 2).

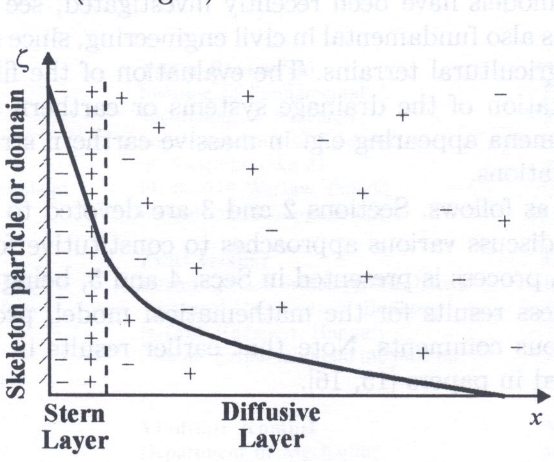

Fig. 2. Double-layer potential ζ versus the distance x from the skeleton border

Only the water molecules outside of the double-layer area (the free water) can move at the beginning of the filtration process. When the hydraulic forcing (the pressure gradient) intensifies, the increasing number of molecules break away from the diffusive layer and take part in a movement. The effective porosity increases and then the sudden drop of the hydraulic resistance appears accompanying the growth of the filtration velocity (see Swartzendruber [17]). The above phenomenon disappears in the range of large velocities, when the whole amount of losely attached water is involved in the filtration process.

The growth of conductivity in the initial range of velocities can be explained by the non-Newtonian behaviour of pore water in the cohesive soil (see Swartzendruber [17], von Engelhandt and Tunn [4]). The comparison of the average velocity of non-Newtonian liquid in a capilare with filtration curves provide a desired intuition. (see Fig. 3).

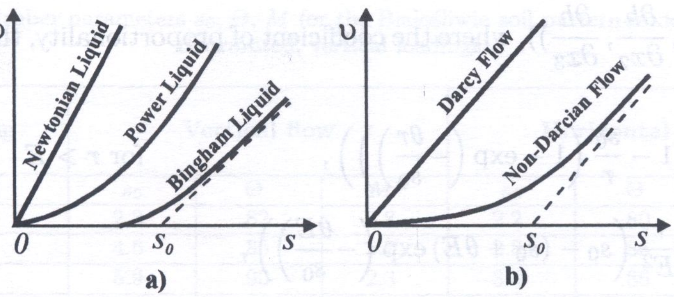

Fig. 3. The comparison of nonlinear dependence of the average velocity of the non-newtonian liquid flow in capillary with the non-darcian filtration in the initial range of slopes

d) Cohesive soils particles or flat domains usually are ordered in layers, which causes the dependence of hydraulic conductivity on the direction of the flow velocity, which can be observed as well in the dispersed sedimens (see Lambe [9]), turbulent domain stacks (see Aylmore and Quirk [1]) as in the naturally or artificially consolidated cohesive materials.

3. Local description of filtration process, parmeter identification

Numerous attempts have been undertaken to express phenomena discussed above in one formula relating the so called "filtration velocity" v, i.e. the unit flux of liquid through the porous medium (cf. Bear [2]) with the hydraulic slope s (measured in the direction parallel to y).

The simplest and most frequently used description of the one-dimensional filtration effects in a cohesive soil is given by the threshold formula

$$v = \begin{cases} 0 & 0 \le s \le s_0, \\ k(s - s_0) & s_0 < s. \end{cases} \tag{1}$$

Another well-known, more accurate formula is of a polynomial type (see Hansbo [8], Chowdury [3])

$$v = As + Bs^n, \qquad n > 1, \qquad s > 0. \tag{2}$$

The quantities k, A, B are non-negative material constants.

Swartzendruber in [18] has introduced the formula

$$v = M[s - s_0(1 - \exp((-\theta s)/s_0)], \qquad s > 0,$$

partially comprising (1) and (2). However his formula has not a direct generalization to the multidimensional case preserving the regularity properties of the former ones and, in addition, being invariant with respect to rotations of a coordinate system, which is the natural requirement for expressions describing the physical phenomena.

The new constitutive relationship, proposed by the first author in [13], overcomes the difficulties mentioned above.

Let us denote by $\mathbb{R}^3$ the three dimensional Euclidean space equipped with the scalar product $(\xi|\eta) = \sum_{i=1}^{3} \xi_i \eta_i$ ($\xi = (\xi_i), \eta = (\eta_i) \in \mathbb{R}^3$) and the norm $\| u \| = (u|u)^{1/2}$. Suppose the filtration process occurs in the domain (an open, simply connected and bounded set) Ω of $\mathbb{R}^3$,

Let $h(t, x)$ be the piesometric height (the pressure related to the specific water weight) measured at a point x of the filtration domain and at an instant t of time.

It is assumed that for isotropic flows the filtration velocity v at a point x in a moment t of time is parallel to the gradient of the pressure (cf. [13]) and

$$v(t, x, \nabla h) = \varphi(t, x, \| \nabla h \|) \nabla h, \tag{3}$$

($\nabla h = \operatorname{grad} h = (\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \frac{\partial h}{\partial x_3})$), where the coefficient of proportionality, the *permeability function* φ, is given by

$$\varphi(t,x,r) = \begin{cases} M\left(1 - \frac{s_0}{r}\left(1 - \exp\left(-\frac{\theta r}{s_0}\right)\right)\right), & \text{for } r > E \\ M\left[\frac{r}{E^2}\left(s_0 - (s_0 + \theta E)\exp\left(-\frac{\theta E}{s_0}\right)\right),\right. & \\ \left. + \left(1 - \frac{2s_0}{E} + \left(\frac{2s_0}{E} + \theta\right)\exp\left(-\frac{\theta E}{s_0}\right)\right)\right], & \text{for } E \geq r \geq 0. \end{cases} \tag{4}$$

In (4) E denotes a constant depending on the floating point arithmetic accuracy and on the features of the current soil pattern. The positive parameters M, s_0, θ may depend on (t, x) variables. In that case they are assumed to be strongly positive bounded functions, so regular that φ is of class C^1 in t, C^0 in x and C^1 in $r \in [0, \infty)$. For example, it suffices to assume that M, s_0 and θ are C^1 with respect to their arguments.

The form of φ is based on the variable permeability v/s which can be deduced for r in $[E, \infty)$ from the Swartzendruber's formula. The formula obtained is then extended linearly over the interval $[0, E]$ in order to avoid the non-physical singularities near zero. This imposes the following relationship between θ, M, E and s_0 : $0 < \theta < 1$, $\frac{M}{E^2}(s_0 - (s_0 + \theta E)\exp(-\frac{\theta E}{s_0})) > 0$.

Remark 1. For anisotropic flows, frequently occuring in silts, the constitutive formula (3) assumes the form

$$v(t, x, \nabla h) = \varphi(t, x, \|\nabla h\|_L) L(t, x) \nabla h, \tag{5}$$

where $\|\nabla h\|_L = (\nabla h | L \nabla h)^{1/2}$ and L denotes the 3×3 symmetric matrix representing the porous medium anisotropy with entries l_{ij}. The functions $l_{ij}(t, x)$ are assumed to be of class C^1 with respect to t and C^0 in x. It is also assumed that the matrix L is positive definite, uniformly with respect to $(t, x) \in [0, T] \times \Omega$ i.e.

$$l_0\,(\xi|\xi) \leq (\xi|L(t,x)\xi) \leq l^0\,(\xi|\xi), \quad \xi \in \mathbb{R}^3, \quad 0 < l_0 \leq l^0. \tag{6}$$

Without the loss of generality one can assume that $l^0 = 1$.

Note that in (5), similarly as in the isotropic case (3), all nonlinear effects are cumulated in the function φ being, as in the previous case, invariant with respect to rotations of a coordinate system.

Parameters appearing in the constitutive formulae have been obtained mainly from the laboratory tests. The methodology of experiments, equipement and results concerning the measurements have been described by Li Sung Ping [10] and Hansbo [8]. The first of them tested the big range of slopes (up to 60) for which the behaviour of the filtration process in cohesive soils (clay from Houston, Texas) is nonlinear.

Laboratory test performed by Wolski and others [21] for soil pattern from Białośliwie, Poland, made it certain that the effect of anisotropy together with the deviation of initial velocity is essential in consolidated river deposits (see Table 1.).

Laboratory tests for the soil pattern from the very principle give only the local information. More general information for the class of flows described by the constitutive relationships (3) or (5) can be obtained using the inverse solution technique introduced in [14, 20].

Table 1. Swartzendruber parameters s_0, Θ, M for the Białośliwie soil pattern under various, uniformly distributed, vertical loadings.

Loading	Vertical flow			Horizontal flow		
[kPas]	s_0	Θ	M^*	s_0	Θ	M^*
0	2.3	.82	5.8	2.2	.80	2.6
10	4.6	.85	2.7	2.8	.86	2.9
40	5.9	.90	2.3	3.4	.86	1.5
80	8.1	.92	2.1	3.9	.91	1.2

* (times 10^{-6}[m/s])

4. MODEL OF FILTRATION, MATHEMATICAL PRELIMINARIES

Suppose the process of the filtration through a body occupying the subset Ω of $\mathbb{R}^3$ occurs during the period T of time.

We will assume that the boundary $\partial\Omega$ of the domain Ω is piecewise C^1-regular, i.e. it is the union of a finite number of surfaces being the graphs of C^1-functions. The unit outer normal to the surface $\partial\Omega$ at a point x will be denoted by $n = n(x)$.

The piesometric height distribution $h : [0,T] \times \Omega \to \mathbb{R}$, describing the prelinear filtration process through Ω in time T is the solution of the following initial boundary-value problem (see [2]) consisting in the balance equation together with the boundary (Dirichlet, Neuman) and initial conditions

$$\beta p(t,x)\frac{\partial h}{\partial t} = \operatorname{div} v(t,x,\nabla h) + Q(t,x) \quad \text{for} \quad (t,x) \in (0,T] \times \Omega, \tag{7}$$

$$h(t,x) = h_b(t,x) \quad \text{for } (t,x) \in (0,T] \times \Omega, \tag{8a}$$

$$(n(x)|v(t,x,\nabla h)) = q(t,x) \quad \text{for } (t,x) \in (0,T] \times \partial\Omega_2, \tag{8b}$$

$$h(0,x) = h_0(x) \quad \text{for } x \in \Omega, \tag{9}$$

$\operatorname{div} v = \sum_{i=1}^{3} \dfrac{\partial v_i}{\partial x_i}$ is a divergence of the vector $v = (v_1, v_2, v_3)$ and $v(t,x,\nabla h)$ is given by (3). Q describes the yield of sources in the case of the undeformable skeleton or it comprehends both the yield of sources and the volume strain velocity of the skeleton, if the skeleton deformation occurs. Equation (7) can be considered as a part of the consolidation system of Biot's type (see e.g. [22]). β is a coefficient of the water compressibility, $p(t,x)$ is the distribution of the porosity of the medium, $q(t,x)$ is the boundary flux, $\partial\Omega_1$, $\partial\Omega_2$ are parts of the boundary $\partial\Omega$.

Remark 2. Observe that [7 – 9] has a classical (i.e. of class C^2) solution provided $\varphi(t,x,\|\nabla h\|)$ is differentiable everywhere which is not true, since the norm $\|.\|$ is nondifferentiable at zero. Thus one can expect only the existence of a weak solution (see definition below) to [7 – 9].

In what follows we will use notations (for more information consult for example [5, 11]). $C^k(A)$ is the space of k-times continuously differentiable real functions defined on A. $L^2(A)$ is the Hilbert space of square summable over A real functions with a scalar product $(u,v) = \int_A u(x)v(x)\,dx$ and a norm $\|u\| = \sqrt{(u,u)}$. $H^1(\Omega)$ denotes the Sobolev space on Ω of order 1, i.e. $H^1(\Omega) = \{u : u, D_i u \in L^2(\Omega), i = 1,2,3\}$, where $D_i u$ stands for the weak (distributional) derivative of u with respect to x_i. The norm in $H^1(\Omega)$ is denoted by $\|\cdot\|_{H^1}$. By γ (resp. γ_1) $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$, (resp. $\gamma_1 : H^1(\Omega) \to L^2(\partial\Omega_1)$) is denoted the operator of trace on $\partial\Omega$ (resp. $\partial\Omega_1$). Set $V = \{u \in H^1(\Omega) :$

$\gamma_1 u = 0\}$. Note that the condition $\int_{\partial\Omega_1} d\sigma > 0$ implies that $\|u\|_V = \|\nabla u\| = (\int_\Omega \|\nabla u\|^2\, dx)^{1/2}$ is the norm in V, equivalent to $\|u\|_{H^1}$.

Space of functions f defined on $(0,T] \times \Omega$, continuous with respect to t, such that for all fixed $t \in (0,T]$, $f(t,\cdot) \in Y$, where Y is a given function space, e.g. $Y = H^1(\Omega)$ or $C^1(\partial\Omega_1)$ etc., will be denoted by $C^0((0,T];Y)$. Similarly, $f \in L^2(0,T;Y)$ means that $f(\cdot,x) \in L^2(0,T)$ for almost all $x \in \Omega$ and $f(t,\cdot) \in Y$ for almost all $t \in (0,T]$.

For a Banach space Y, Y' denotes its dual, $\|\cdot\|_Y$ denotes the norm of Y, and $\langle\cdot,\cdot\rangle$ is the duality pairing between Y' and Y. Recall that $\langle\cdot,\cdot\rangle = (\cdot,\cdot)$ for $Y = L^2(\Omega)$.

By a *weak solution* of the problem [7 – 9] we mean the function $h \in L^2(0,T;H^1(\Omega))$ satisfying the Dirichlet condition (8a): $\gamma_1 h = h_b$ and for all $w \in V$ the variational equation

$$\int_\Omega \beta p(t,x)\frac{\partial h}{\partial t} w(x)\, dx = \int_\Omega \varphi(t,x,\|\nabla u\|)\,(\nabla v|\nabla w)\, dx + \int_\Omega Q(t,x)w(x)\, dx \qquad (10)$$

$$+ \int_{\partial\Omega_2} q(t,x)w(x)\, d\sigma.$$

5. Main results

The following theorem, being the main result of the paper, shows that under a very general, physically plausible conditions the initial boundary-value problem for the nonlinear filtration is well posed in the Hadamard sense, i.e. for the given initial and boundary data it has the unique solution, continuously depending on them. More precisely, we have the following theorem.

Theorem 1. *Assume $\partial\Omega$ is piecewise C^1-regular surface such that*

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2, \qquad \partial\Omega_1 \cap \partial\Omega_2 = \emptyset, \qquad \int_{\partial\Omega_1} d\sigma > 0, \qquad (11)$$

where the boundary of $\partial\Omega_1$ relative to $\partial\Omega$ is a union of a finite number of piecewise C^1 disjoint Jordan curves. Let φ be given by (10) and let $Q \in C^0([0,T];L^\infty(\Omega))$, $q \in C^0((0,T];C^1(\partial\Omega_2))$. Finally, suppose that

$$p \in C^1([0,T]\times\Omega), \qquad p(t,x) = \alpha(t)\, e(x), \qquad (12)$$

$$0 < \alpha_0 \le \alpha(t) \le \alpha^0, \qquad 0 < e_0 \le e(x) \le e^0 \quad \textit{for} \quad (t,x) \in [0,T]\times\Omega. \qquad (13)$$

Then for any $h_0 \in L^2(\Omega)$ and $h_b \in C^1([0,T]\times\partial\Omega_1)$ problem [7 – 9] has exactly one weak solution h satisfying

$$h \in L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega)) \qquad (14)$$

and depending continuously on the initial data.

Theorem 1 generalizes the result of [13] to the case of nonhomogeneous Dirichlet boundary conditions. Also we impose here slightly weaker regularity conditions on characteristics of the problem and on the function φ.

It can be verified that, apart from the regularity of φ, it follows from (9) that there exist positive constants $m, \widetilde{M}$ such that

$$\varphi(t,x,r)r - \varphi(t,x,s)s \ge m\,(r-s) \qquad \text{for} \qquad r \ge s, \qquad (15)$$

$$|\varphi(t,x,r)r - \varphi(t,x,s)s| \le \widetilde{M}\,|r-s| \qquad (16)$$

provided that $(t,x) \in [0,T] \times \Omega$. Note that in this paper besides the regularity of φ, we will only need properties (15), (16) in the proof of Theorem 1. Details are postponed to the Appendix.

The idea of the proof which we sketch here is standard: it consists in considering the problem of finding weak solutions to [7 – 9] as the problem of solving the nonlinear operator equation

$$Ch' + B(t,h) = f(t), \quad t \in (0,T]$$

with conditions (8a), (9), where $C : V' \to V'$, $B : [0,T] \times H^1(\Omega) \to V'$ are the operators associated with (7), (8b).

Applying the monotonicity and compactness methods to the sequence $\{h_n\}$ of Galerkin approximate solution to the operator problem above (compare [5, 11]) one gets the existence and uniqueness of the weak solution.

If V_n is the n-dimensional subspace of V (e.g. the finite element space) spanned by elements $\{w_1, \ldots, w_n\} \subset V$, then the Galerkin approximation h_n of the weak solution to [7 – 9] corresponding to V_n is the function $h_n : [0,T] \to V_n$ being the solution of the system

$$\langle Ch_n' + B(t,h_n) - f(t), w_i \rangle = 0, \quad i = 1, \ldots, n, \quad t \in (0,T]$$

with the appropriate initial conditions, specified later in the proof.

By Theorem 1, it is possible to obtain an information concerning the character of convergence of the sequence $\{h_n\}$ of approximate solutions in case where the sequence $\{V_n\}$ *approximates* V, i.e. when the set $\bigcup_{n=1}^{\infty} V_n$ is dense in V.

Theorem 2. *Suppose the conditions of Theorem 1 are satisfied. Let $\{V_n\}$ be the sequence of spaces approximating V and let h_n be the corresponding Galerkin approximation of a weak solution h to the problem [7 – 9]. Then*

$$h_n \to h \quad \text{in } C^0 \quad ([0,T]; L^2(\Omega)) \quad \text{as} \quad n \to \infty.$$

Theorems 1 and 2 admit a generalization to the case of anisotropic soils. Namely we have the following result:

Theorem 3. *Assume condtions of Theorem 1 are satisfied and suppose the flow is anisotropic i.e. constitutive formula is given by (5). If $l_{ij} \in C^1([0,T]; C^0(\Omega))$ for $i,j = 1,2,3$, matrix $L(t,x) = (l_{ij}(t,x))$ is symmetric and satisfies (6), then the assertions of Theorem 1 remain valid. For any sequence $\{V_n\}$ of spaces approximating V the corresponding sequence $\{h_n\}$ of Galerkin approximatie solutions satisfies the conclusion of Theorem 2.*

6. Conclusions

1) The constitutive formula proposed in the paper provides an accurate model of the three dimensional filtration process in cohesive soils taking into account various bounary conditions and anisotropy of the hydraulic conductivity.
2) Consitutive formulae (3), (6) are sufficiently flexible and general to describe the fitration phenomena in a broad range of hydraulic slopes (in both the initial or asymptotic ranges of velocities for a given process). Moreover their regularity enables to apply the advanced mathematical tools to the study the filtration equations.
3) The weak (variational) formulation of the initial boundary-value problems in the theory of filtration is natural since it does not impose strong, nonphysical assumptions on the process, required in the classical formulation.
4) The investigations of filtration of water through cohesive soils has been carried since about forty years. The model presented here is the first one which besides good points mentioned above has the reliable mathematical justification showing its well posedness (in the Hadamard sense).

5) The Faedo-Galerkin method proved to be very fruitful in the case of filtration equations since it provides a good starting point for various fast numerical methods of determining the filtration flow. The full description of the fast FE/FD linearized schemes will be published in the next paper.

Appendix

Notations, auxiliary lemmas. To formulate [7– 9] in terms of operator equation, certain notations will be required. Denote: $X = L^2(0,T;V)$, $W = \{u : u \in X, u' \in X'\}$, where $u' = \dfrac{du}{dt}$ denotes the time derivative of u in the sense of scalar distributions on $[0,T]$.

Observe that $X' = L^2(0,T;V')$ and the expression $\langle\langle f,u\rangle\rangle = \int_0^T \langle f(s),u(s)\rangle\, ds$ represents the duality between X' and X. For $u,v \in W$ the formula of integration by parts holds ([5, Ch. IV, Th. 1.17]).

For $u,v,z \in H^1(\Omega)$ and $t \in [0,T]$ set

$$b(t,u,v,z) = \frac{1}{\beta\,\alpha(t)} \int_\Omega \varphi(t,x,\|\nabla u\|)\,(\nabla v|\nabla z)\,dx, \tag{A-1}$$

$$f(t,z) = \frac{1}{\beta\,\alpha(t)}\left(\int_\Omega Q(t,x)z(x)\,dx + \int_{\partial\Omega_2} q(t,x)z(x)\,d\sigma\right). \tag{A-2}$$

Note that by (15) and (16), φ is nonnegative and bounded. Since $\varphi(t,x,\cdot)$ is C^1, it satisfies the Lipschitz condition with a constant, say K. In consequence for $u \in H^1(\Omega)$ $\varphi(t,x,\|\nabla u\|) \in L^\infty(\Omega)$. Using this fact and the observation that $D_i v, D_i w \in L^2(\Omega)$ for $v,w \in H^1(\Omega)$, by the Hölder inequality, we conclude that for fixed $t \in [0,T]$ $\varphi(t,\cdot,\|\nabla u\|)D_i v D_i w$ is in $L^1(\Omega)$, hence $|\int_\Omega \varphi(t,x,\|\nabla u\|)D_i v D_i w\,dx| \le c(t,\|u\|_{L^\infty(\Omega)})\,\|D_i v\|\,\|D_i w\|$, which shows that $b(t,u,v,w)$ is well defined. From the equivalence of norms $\|\cdot\|_{H^1}$, $\|\cdot\|_V$ and the above inequality it follows that

$$|b(t,u,v,w)| \le (\beta\alpha(t))^{-1} c(t,u)\,\|v\|_V\,\|w\|_V \qquad \text{for} \qquad t \in [0,T], \tag{A-3}$$

where $c(t,u)$ is bounded and $\|c(t,u) - c(t,z)\|_{L^\infty(\Omega)} \le K\|u - z\|$.

Let operators $C : V' \to V'$, $B : [0,T] \times (H^1(\Omega))^2 \to V'$ be defined by

$$V' \ni g \mapsto Cg = eg \in V' \quad (\langle e\,g,u\rangle = \langle g,e\,u\rangle \quad \text{for} \quad u \in V) \tag{A-4}$$

$$\langle B(t,u,v),w\rangle = b(t,u,v,w) \qquad \text{for} \qquad w \in H^1(\Omega).$$

From (10) and the definitions above it follows immediately that h is a weak solution of the problem [7 – 9] if it satisfies (8a), (9) and the operator equation

$$Ch' + B(t,h) = f(t), \qquad t \in (0,T] \tag{A-5}$$

where $B(t,h) = B(t,h,h)$ and $f \in L^2(0,T;V')$ is defined by $\langle f(t),z\rangle = f(t,z)$.

Fix $a \in C^1([0,T] \times \overline{\Omega})$ satisfying $\gamma_1 a = h_b$ (the existence of such a function follows from (11) and the regularity of $\partial\Omega_1$, cf. e.g. [12, Ch.III, Par.4, Th.2]).

The change of variables $h = u + a$ transforms (A-5), (8), (9) into the problem with homogeneous Dirichlet conditions:

$$C\frac{du}{dt} + B(t,u + a(t)) = f_1(t), \tag{A-6}$$

$$u(0) = h_1, \qquad u \in W, \tag{A-7}$$

where $a(t) = a(t,\cdot)$, $f_1(t) = f(t) - Ca'(t)$ and $h_1 = h_0 - a(0,\cdot)$. Recall that the space W is continuously embedded in $C^0([0,T];L^2(\Omega))$, hence (A-7) is meaningful.

The following lemmas are basic for the proof.

Lemma 1. *Suppose φ satisfies (15) and (16). Then for a fixed $a \in C^1([0,T]\times\overline{\Omega})$ the map $B(t,\cdot+a(t))$, $V \ni u \mapsto B(t,u+a(t)) \in V'$ is Lipschitz continuous, strongly monotone and coercive, uniformly in $t\in[0,T]$, i.e.*

$$\|B(t,u+a(t)) - B(t,v+a(t))\|_{V'} \le M_1\|u-v\|_V, \tag{i}$$

$$\langle B(t,u+a(t)) - B(t,v+a(t)), u-v\rangle \ge m_1\|u-v\|_V^2, \tag{ii}$$

$$\langle B(t,u+a(t)),u\rangle \ge \xi(\|u\|_V)\,\|u\|_V, \quad \xi(s)\to\infty \quad as \quad s\to\infty \tag{iii}$$

with M_1, m_1, ξ independent of t.

Proof of Lemma 1. To simplify notations, set $\nabla(u+a(t)) = y, \nabla(v+a(t)) = z, \nabla w = h, r = \|y\|, s = \|z\|, q = \|h\|$ and write φ_k for $\varphi(t,x,k)$.

Using the inequality $\|\alpha a - \beta b\| \le 2\alpha\|a-b\| + |\alpha\|a\| - \beta\|b\||$ valid for positive α,β and all $a,b\in\mathbb{R}^3$, for any $h\in\mathbb{R}^3$ from (15) and (16) we get

$$= |(\varphi_r\, y - \varphi_s\, z|h)| \le \|\varphi_r\, y - \varphi_s\, z\|\,\|h\| \le \|h\|(2\varphi_r\|y-z\| + |\varphi_r\, r - \varphi_s\, s|)$$

$$\le \|h\|(2\widetilde{M}\|y-z\| + \widetilde{M}|r-s|) \le 3\|h\|\widetilde{M}\|u-v\|.$$

Let $w\in V$. By (A-1) and (A-4), $\beta\alpha(t)|\langle B(t,u+a(t)) - B(t,v+a(t)),w\rangle| = |\int_\Omega(\varphi_r\, y - \varphi_s\, z|h)\,dx| \le \beta\alpha(t)\int_\Omega 3\|h\|\widetilde{M}\|u-v\|\,ds \le \beta\alpha(t)\|w\|_V 3\widetilde{M}\|u-v\|_V$, which proves (i).

Similarly, applying inequalities $(y|y-z) \ge r^2 - rs$, $(z|y-z) \le rs - s^2$ and (15) one gets for all $y,z\in\mathbb{R}^3$

$$(\varphi_r\, y - \varphi_s\, z|y-z) - m\|y-z\|^2 = (\varphi_r - m)(y|y-z) - (\varphi_s - m)(z|y-z) \ge$$

$$(\varphi_r - m)(r^2 - rs) - (\varphi_s - m)(rs - s^2)(\varphi_r\, r - \varphi_s\, s)(r-s) - m(r-s)^2 \ge 0.$$

Hence by the formula $\beta\alpha(t)\langle B(t,u+a(t)) - B(t,v+a(t)), u-v\rangle - m\|u-v\|_V^2 = \int_\Omega((\varphi_r\, y - \varphi_s\, z|y-z) - m\|y-z\|^2)\,dx$ and the above inequality we get (ii).

From (ii) and continuity of B it follows that

$$\langle B(t,u+a(t)),u\rangle \ge \langle B(t,a(t)),u\rangle + m_1\|u\|_V^2 \ge -c\|u\|_V + m\|u\|_V^2,$$

where c is a positive constant independent on t. The above inequality proves (iii).

Lemma 2. *For $e\in C^1(\Omega)$ satisfying (13), the mapping C is continuous and*

$$e_0\|u-v\|^2 \le \langle C(u-v),u-v\rangle \le e^0\|u-v\|^2 \quad \text{for} \quad u,v\in L^2(\Omega). \tag{iv}$$

$$\frac{d}{dt}\langle Cu,u\rangle = 2\langle Cu',u\rangle \quad \text{for} \quad u\in W. \tag{v}$$

Proof of Lemma 2. Continuity of C and property (iv) follow from the definition of C. To prove (v), observe that if $v\in W$, then $Cv\in W$ and $(Cv)' = Cv'$. Setting $u = Cv$ in the formula of the integration by parts one gets $\langle Cv(t),v(t)\rangle - \langle Cv(0),v(0)\rangle = 2\int_0^t\langle Cv'(s),v(s)\rangle\,ds$ from which (v) follows.

Proof of Theorem 1.

Passing to the proof of Theorem 1, note that by Lemma 1 and 2, the operators B and C are both monotone and coercive in V and $L^2(\Omega)$ respectively, which permits to apply to (A-6), (A-7), with necessary modifications, the classical Faedo–Galerkin approach. The proof is very close to the argument applied in [5] dealing with the case of C being the identity. For the sake of completeness we insert here an outline of the existence proof.

Let $\{w_n\}$ be a basis of V, i.e. for any k elements $w_1, \dots, w_k$ are linearly independent and all finite linear combinations of w_i form a dense subspace of V (such a basis exists, since V is separable). Denote by V_n the subspace of V spanned by $w_1, w_2, \dots, w_n$.

Choose $\{u_{0n}\} \subset V_n$, $u_{0n} = \sum_{i=1}^{n} \zeta_{jn} w_j$, such that $u_{0n} \to h_1$ in V as $n \to \infty$. Set $u_n(t) = \sum_{j=1}^{n} g_{jn}(t)\, w_j$, where $g_{jn}(t)$ are so defined that $u_n(t)$ is a solution of the initial value problem for a system of ordinary differential equations

$$\langle C\frac{du_n}{dt} + B(t, u_n + a(t)), w_i\rangle = \langle f_1(t), w_i\rangle, \quad i = 1, \dots, n, \tag{A-8}$$

$$u_n(0) = u_{0n} = \sum_{i=1}^{n} \zeta_{jn} w_j. \tag{A-9}$$

By the definition of operators C and B, from (A-8) and (A-9) it follows that the function $g_n(t) = (g_{1n}(t), \dots, g_{nn}(t))$ is the solution of the initial value problem

$$C_n g_n'(t) = A_n(t, u_n(t)) g_n(t) + k(t, u_n(t)), \tag{A-10}$$

$$g_n(0) = \zeta_n = (\zeta_{1n}, \dots, \zeta_{nn}), \tag{A-11}$$

where $A_n = (a_{ij})$ and $C_n = (c_{ij})$ are $n \times n$ matrices with entries $a_{ij}(t, u_n) = -b(t, u_n + a(t), w_j, w_i)$, $c_{ij} = (e\, w_i, w_j)$ and $k(t, u_n(t))$ has components $k_i(t, u_n) = \langle f_1(t), w_i\rangle - b(t, u_n + a(t), a(t), w_i)$.

The continuity of a_{ij} and k_i result from the continuity of b and f_1. It is easily seen that they are also bounded. Note that since $b(t, u_n + a, u_n + a, w_i) = \langle B(t, u_n + a), w_i\rangle$, from Lemma 1 it follows that $A_n(t, u_n)$ is Lipschitz continuous in u_n so the right hand sides of (A-10) have the same property with respect g_n and the problem (A-10), (A-11) is uniquely solvable and its solution g_n is defined and of class C^1 on $[0, T]$ and, in consequence, u_n is uniquely defined and exists on $[0, T]$.

Multiplying (A-8) by $g_{in}(t)$, summing over i and using (v) one arrives at the formula

$$\frac{1}{2}\frac{d}{dt}\langle Cu_n(t), u_n(t)\rangle + \langle B(t, u_n(t) + a(t)), u_n(t)\rangle - \langle B(t, a(t)), u_n(t)\rangle \tag{A-12}$$
$$= \langle f_1(t) - B(t, a(t)), u_n(t)\rangle.$$

By (A-1) – (A-3),

$$|\langle f_1(t) - B(t, a(t)), u_n\rangle| \le |\langle f(t), u_n(t)\rangle| + |(e\frac{\partial a}{\partial t}, u_n(t))|$$
$$+ |\langle B(t, a(t)), u_n(t)\rangle| \le \frac{1}{\beta\alpha(t)}\left(\|Q\|\,\|u_n(t)\| + \|q\|_{L^2(\partial\Omega_2)}\|\gamma u_n(t)\|_{L^2(\partial\Omega_2)}\right)$$
$$+ \|eq\frac{\partial a}{\partial t}\|\,\|u_n(t)\| + \frac{1}{\beta\alpha(t)} c(t, u_n)\|a(t)\|_V\|u_n(t)\|_V \equiv I(t).$$

Since V is continuously imbedded in $L^2(\Omega)$ and trace operator γ is continuous, $\|u_n(t)\| \le c_1\|u_n(t)\|_V$, $\|\gamma u_n(t)\|_{L^2(\partial\Omega_2)} \le c_2\|u_n(t)\|_V$, where c_1, c_2 are certain constants. Inequality above and (A-12) imply that for any $t \in [0, T]$

$$\frac{1}{2}(\langle Cu_n(t), u_n(t)\rangle - \langle Cu_{0n}, u_{0n}\rangle) + \int_0^t (\langle B(s, u_n(s) + a(s)) - B(s, a(s)), u_n(s)\rangle)\, ds \tag{A-13}$$
$$\le \int_0^t I(s)\, ds \le c\|u_n\|_X,$$

where c is a constant.

Finally, using (A-13) we obtain from (ii) and (iv) the inequality

$$\frac{1}{2}e_0\|u_n(t)\|^2 + \int_0^t m\|u_n(s)\|_V^2\, ds \le c\|u_n\|_X + \frac{1}{2}e^0\|u_{0n}\|^2$$

valid for any $t \in [0,T]$. In particular for $t = T$ we have

$$\frac{1}{2}e_0\|u_n(T)\|^2 + m\|u_n\|_X^2 \le c\|u_n\|_X + \frac{1}{2}e^0\|u_{0n}\|^2.$$

From the inequalities above it is clear that the sequence $\{u_n\}$ is bounded in X and in $C^0([0,T];L^2(\Omega))$. Setting $t = T$ in (13) we get

$$\langle\langle B(\cdot, u_n + a), u_n\rangle\rangle \le c\|u_n\|_X + \langle\langle B(\cdot, a), u_n\rangle\rangle - \frac{1}{2}\left(\langle Cu_n(t), u_n(t)\rangle - \langle Cu_{0n}, u_{0n}\rangle\right).$$

Since the sequence $\{u_n\}$ is bounded, the sequence $\{\langle\langle B(\cdot, u_n + a), u_n\rangle\rangle\}$ is also bounded, which by monotonicity of B, implies that $\{B(\cdot, u_n + a)\}$ is bounded in X' (see [5, Ch.III, Cor.1.2 and Ch.VI, Lemma 1.2]).

From the above considerations it follows that the sequence $\{u_n\}$ has a subsequence $\{u_\mu\}$ such that

$$u_\mu \to u \quad \text{weakly in } X, \tag{j}$$

$$u_\mu(T) \to z \quad \text{weakly in } L^2(\Omega) \tag{jj}$$

$$B(\cdot, u_\mu + a(\cdot)) \to v \quad \text{weakly in } X' \tag{jjj}$$

and the limits defined above satisfy

$$u \in W, \qquad u(0) = h_1, \qquad u(T) = z, \qquad Cu' + v = f_1. \tag{jv}$$

Properties (j), (jj), (jjj) are clear. To prove (jv), note that by (12), we have for $u, w \in X$

$$\langle\langle Cu, w\rangle\rangle = \int_0^T \langle Cu(s), w(s)\rangle\, ds = \int_0^T (u(s), ew(s))\, ds = \langle\langle u, e\,w\rangle\rangle,$$

hence condition (j) implies that $Cu_\mu \to Cu$ weakly in X. The reference to the argument used in [5, Ch.VI, Lemma 1.4] finishes the proof of property (jv).

To complete the existence proof, note that by (jj), $Cu(T) \to Cz$ weakly in $L^2(\Omega)$ and in consequence $\liminf\langle Cu_\mu(T), u_\mu(T)\rangle \le \langle Cz, z\rangle$. Applying formulae

$$Cu' + v = f_1, \quad \langle\langle Cu'_\mu, u_\mu\rangle\rangle + \langle\langle B(\cdot, a(\cdot) + u_\mu), u_\mu\rangle\rangle = \langle\langle f_1, u_\mu\rangle\rangle$$

we obtain the inequality

$$\limsup\left(\langle\langle f_1, u_\mu\rangle\rangle + \frac{1}{2}\left(\langle Cu_{0\mu}, u_{0\mu}\rangle - \langle Cu_\mu(T), u_\mu(T)\rangle\right)\right) \le \langle\langle f_1, u\rangle\rangle$$

$$+\frac{1}{2}\left(\langle Ch_1, h_1\rangle - \langle Cz, z\rangle\right) = \langle\langle f_1, u\rangle\rangle - \langle\langle Cu', u\rangle\rangle = \langle\langle v, u\rangle\rangle,$$

which by the monotonicity of B, (j) and (jjj), implies that $v = B(\cdot, u + a(\cdot))$ (cf. [5, Ch.III, Lemma 1.3]), so u satisfies (A-6). From conditions $u \in W$ and (jv) it follows that u satisfies also (A-7). Obviously $h = a + u$ satisfies (14), (A-5) and the initial condition (A-8).

Now suppose h_1, h_2 are solutions to (A-6), (8). Then by (ii),

$$\frac{1}{2}(\langle C(h_1(t)-h_2(t)), h_1(t)-h_2(t)\rangle - \langle C(h_1(0)-h_2(0)), h_1(0)-h_2(0)\rangle)$$

$$= \int_0^t \langle C(h_1(s)-h_2(s))', h_1(s)-h_2(s)\rangle\, ds$$

$$= -\int_0^t \langle B(s,h_1)-B(s,h_2), h_1-h_2\rangle\, ds \le 0,$$

which, by (iv), implies the inequality $e_0\|h_1(t)-h_2(t)\|^2 \le e^0\|h_1(0)-h_2(0)\|^2$ proving uniqueness and continuous dependence of solutions of (A-6), (A-7) on the initial data.

Proof of Theorem 2. From the formula $h_n = u_n + a$, where u_n and a are as in the previous proof, it follows that it suffices to prove that

$$u_n \to u \quad \text{in} \quad C^0([0,T]; L^2(\Omega)). \tag{k}$$

For this end, observe first that

$$u_n \to u \qquad \text{weakly in } X, \tag{kk}$$

$$B(\cdot, u_n + a(\cdot)) \to B(\cdot, u + a(\cdot)) \qquad \text{weakly in } X', \tag{kkk}$$

(kk) and (kkk) result from the boundedness of sequences $\{u_n\}$, $\{B(\cdot, u_n + a(\cdot))\}$ and the observation that by the uniqueness property of the problem (A-6), (A-7), all their converging subsequences must have the same limits u, $B(\cdot, u + a(\cdot))$ respectively.

For the proof of (k), choose a sequence $\{v_n\} \subset C^1([0,T]; V_n)$ such that $\|v_n - u\|_X \to 0$, $\|v_n' - u'\|_{X'} \to 0$ as $n \to \infty$.

Set (for simplicity arguments t are omitted)

$$I_1 = -\langle B(\cdot, a+u_n) - B(\cdot, a+u), u_n - u\rangle,$$

$$I_2 = -\langle B(\cdot, a+u_n) - B(\cdot, a+u), u - v_n\rangle,$$

$$I_3 = \langle C(u-v_n)', u_n - v_n\rangle.$$

From the continuity of C, boundedness of sequences $\{v_n\}$, $\{u_n\}$, $\{B(\cdot, a+u_n)\}$ resulting from (kk), (kkk) and conditions (i), (ii) it follows that

$$I_1 \le 0, \quad |I_2| \le M_1\|u_n - u\|_V \|u - v_n\|_V \le K_1\|u - v_n\|_V,$$

$$|I_3| \le \|C\|\, \|(u-v_n)'\|_{V'} \|u_n - v_n\|_V \le K_2\|u' - v_n'\|_{V'},$$

where K_i denotes suitable positive constants. We have

$$\langle C(u_n - v_n)', u_n - v_n\rangle = \langle -B(\cdot, a+u_n) + B(\cdot, a+u), u_n - v_n\rangle = I_1 + I_2 + I_3,$$

and a simple calculation gives

$$\frac{1}{2}(\langle C(u_n(t)-v_n(t)), u_n(t)-v_n(t)\rangle - \langle C(u_n(0)-v_n(0)), u_n(0)-v_n(0)\rangle)$$

$$= \int_0^t \langle C(u_n(s)-v_n(s))', u_n(s)-v_n(s)\rangle\, ds = \int_0^t (I_1+I_2+I_3)\, ds$$

$$\leq \int_0^t (I_2 + I_3)\,ds \leq K_1\|u - v_n\|_X + K_2\|u' - v_n'\|_{X'}.$$

The last formula and (iv) yield for $t \in [0, T]$ the inequality

$$e_0\|u_n(t) - v_n(t)\| \leq K_3\left(e^0\|u_n(0) - v_n(0)\| + \|u - v_n\|_X + \|u' - v_n'\|_{X'}\right),$$

from which (k) and the claim of Theorem 2 follows immediately.

Proof of Theorem 3. For v_i given by (5) the weak formulation of the problem can be again written in the form (A-6) with the scalar product $(a|b)$ $a, b \in \mathbb{R}^3$ appearing in (1) replaced by $(a|b)_L = (a|Lb)$. From symmetry of L and (6) it follows that $\|\cdot\|_L$ and $(\cdot|\cdot)_L$ represent respectively the norm and scalar product in $\mathbb{R}^3$ equivalent to the habitual ones, hence all argumentation used in previous proofs can be adapted to this more general case.

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