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Filtration in cohesive soils: numerical approach

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Paper presents a numerical method for solving the initial boundary-value problem for a certain quasilinear parabolic equation describing the low velocity filtration problem. The convergence of the method is proved.

1. Introduction

The paper concerns the numerical solving of the nonlinear problem of filtration of water through cohesive soils.

Consideration of a nonlinear model of filtration instead of the classical linear one based on the Darcy law is motivated by significant discrepancies between theoretical results based on the linear model of flow and experimental data. These differences, visible in the range of low velocities (for the so called prelinear filtration flows) provided the starting point for the new constitutive formula for the flow (cf. [10, 11] and references therein) and, in consequence for the new mathematical model of the prelinear filtration flows which is described by the inital boundary-value problem for a certain quasilinear parabolic equation.

The accurate numerical modelling of ground water flows is of a crucial importance in many important design and exploitation enterprises. For example, the prediction of the behaviour of the ground water in peats is required for the drainage design and control of boggy terrains. The high accuracy modelling is important in the case of the earthern dam monitoring. Yet another application consists in using computed results in identification and updating the permeability parameters (see [12]).

The numerical solving of quasilinear parabolic equations has been considered by various authors, see e.g. works [2, 10, 13, 16] where further references can also be found.

The monograph [2] and the work [13] concern the case when nonlinear terms depend on the unknown function, not on its derivatives. Zlamal in [16] discusses equations modeling the heat transport. The paper contains important observations concerning discretization procedures. In [13] the estimates allowing to get some information concerning the speed of convergence of the proposed method has been obtained. However to get such results it was necessary to make rather strong assumptions concerning the regularity of solutions: the piecewise continuity of the second derivatives of h with respect to the spatial variables and C^2–continuity in t in the interval $[0, T]$, which in turn required strong regularity conditions on the considered equation.

In the present paper regularity assumptions imposed on parameters of the model have been considerably weakened. Their regularity seems to be more adequate to the physical characterization of parameters describing the process. As a result, one can only expect the weak solution of the equation of the prelinear filtration.

Section 2 of the paper presents the mathematical formulation of prelinear filtration based on a new constitutive formula in the local and the weak (variational) formulation. The latter provides the starting point for the numerical treatment of the problem, which is discussed in Sec. 3. Section 4, containing the main results of the paper, discusses the mixed scheme for solution of Galerkin equations. The MUBS package of programs and numerical experiments are described in Sec. 5. The last paragraph contains conclusions. Proofs are given in the Appendix.

2. Weak formulation

Denote by $\mathbb{R}^3$ the Euclidean 3-space with the scalar product $(\xi|\eta) = \sum_{i=1}^{3} \xi_i \eta_i$ ($\xi = (\xi_i)$, $\eta = (\eta_i)$) and the norm $\|\xi\| = \sqrt{(\xi|\xi)}$. $\partial\Omega$, $\overline{\Omega}$ denote respectively the boundary and closure of the set Ω. $n(x)$ is the unit vector of the outer normal to $\partial\Omega$ at x. ∇h is the gradient of $h : \Omega \to \mathbb{R}$, $\operatorname{div} v$ is the divergence of the vector function $v : \Omega \to \mathbb{R}^3$. The following formula (cf. [11])

$$v(t, x, \nabla h) = \varphi(t, x, \|L \nabla h\|) L \nabla h \tag{1}$$

relating the filtration velocity vector v with the gradient of the piesometric height distribution h in the porous medium will be used. In (1) $L = L(t, x) = (l_{ij}(t, x))$ denotes the 3×3 symmetric matrix (*an anisotropy matrix*) such that for all $\xi \in \mathbb{R}^3$, $\xi \neq 0$ $0 < l_0\|\xi\|^2 \leq (\xi|L\xi) \leq l^0\|\xi\|^2$. The function φ is given by

$$\varphi(t,x,r) = \begin{cases} M\left(1 - \dfrac{s_0}{r}\left(1 - \exp\left(-\dfrac{\theta r}{s_0}\right)\right)\right), & \text{for } r > E \\ M\left[\dfrac{r}{E^2}\left(s_0 - (s_0 + \theta E)\exp\left(-\dfrac{\theta E}{s_0}\right)\right) \right. \\ \left. + \left(1 - \dfrac{2s_0}{E} + \left(\dfrac{2s_0}{E} + \theta\right)\exp\left(-\dfrac{\theta E}{s_0}\right)\right)\right], & \text{for } E \geq r \geq 0. \end{cases}$$

Here M, S_0, θ are assumed to satisfy the following conditions: $0 < \theta < 1$ and $\dfrac{M}{E^2}(S_0 - (S_0 + \theta E)\exp(-\dfrac{\theta E}{S_0}) > 0$.

The quantities l_{ij}, M, S_0, θ, characterize the physical properties of the medium, in general, they may depend on the time t and the position x and in that case they are assumed to be functions strongly positive and bounded in their domain of definition and so regular (e.g. continuous differentiability with respect to t and x suffices) that φ is continuously differentiable in t and $r \in [0, \infty)$ and continuous in x. The constant E has no physical significance; it is a computer acceptable small real number, introduced to the formula to make φ sufficiently regular.

Assuming v is given by (1), the prelinear filtration in a body occupying the domain (an open, simply connected and bounded set) $\Omega \subset \mathbb{R}^3$ during the period T of time is described by the piesomeric height distribution $h : [0, T] \times \overline{\Omega} \to \mathbb{R}$ being the solution of the following initial boundary-value problem for a quasilinear parabolic equation:

$$\beta p(t,x)\frac{\partial h}{\partial t} = \operatorname{div} v + Q \quad \text{for} \quad (t,x) \in (0,T] \times \Omega$$

$$h(t,x) = h_b(t,x), \quad \text{for} \quad (t,x) \in (0,T] \times \partial\Omega_1, \quad h(0,x) = h_0(x) \quad \text{for} \quad x \in \Omega.$$

$$(n(x)|v(t,x,\nabla h)) = q(t,x) \quad \text{for} \quad (t,x) \in (0,T] \times \partial\Omega_2, \quad \partial\Omega_2 = \partial\Omega \setminus \partial\Omega_1.$$

The functions appearing in the formulae above describe various influences (external and internal) on the filtration process. Q describes the joint influence of the intensity of sources and the strain velocity of the skeleton. β is the coefficient of the water compressibility, q is the intensity of the boundary flux.

It is assumed that the domain and functions are sufficiently regular to make the formulas above meaningful (see [11] for details). In particular, it is assumed that $\partial\Omega$ is piecewise C^1 and the boundary of the set $\partial\Omega_1$ relative to $\partial\Omega$ is so regular that the function $h_b \in C^1([0,T]; C^1(\partial\Omega_1))$ can be extended as a C^1 function on the set $[0,T] \times \overline{\Omega}$.

Before stating the variational formulation of the problem above some extra notations will be needed (consult [3] for additional informations). $L^2(\Omega)$ denotes the Hilbert space of square summable over Ω real functions with a scalar product $(u,v) = \int_\Omega u(x)v(x)\,dx$ and a norm $\|u\|_{L^2(\Omega)} = \sqrt{(u,u)}$. $H^1(\Omega) = \{u : u, D_i u \in L^2(\Omega), i = 1,2,3\}$, where $D_i u$ stands for the weak (distributional) derivative of u with respect to x_i, is the Sobolev space on Ω of order 1. $\|u\|^2_{H^1} = (u,u_0) + \sum_{i=1}^3 (D_i u, D_i u)$. $\gamma_1 : H^1(\Omega) \to L^2(\partial\Omega_1)$ denote operator of trace on $\partial\Omega_1$. Set $V = \{u \in H^1(\Omega) : \gamma_1 u = 0\}$, $B = L^2(0,T;V)$. Let V' be the dual to V with the duality relation $\langle g, \xi\rangle$ $(g \in V', \xi \in V)$. Note that for $g, \xi \in L^2(\Omega)$ the duality $\langle g, \xi\rangle$ reduces to the usual scalar product (g,ξ) in $L^2(\Omega)$.

Finally, put $W = \{u : u \in B, u' \in B'\}$, where $B = L^2(0,T;V)$ and $B' = L^2(0,T;V')$.

For the numerical treatment of the prelinear filtration problem it will be convenient to apply the weak formulation of the problem which can be stated as follows: For a fixed, arbitrarily chosen $a \in C^1([0,T] \times \overline{\Omega})$ such that $\gamma_1 a = h_b$ find a function $u : [0,T] \to V$ satisfying

$$C\frac{d}{dt}u(t) + A(t, u(t) + a(t)) = f_1(t), \qquad t \in (0,T], \qquad u(0) = h_1, \qquad u \in W, \tag{2}$$

(the derivative d/dt is understood in the distributional sense), where $a(t) = a(t,\cdot)$, $f_1(t) = f(t) - Ca'(t)$ and $h_1 = h_0 - a(0,\cdot)$.

The function $f \in C^0(0,T;V')$ and operators $A : [0,T] \times V \to V'$, $C : V' \to V'$ are defined by the formulae:

$$\langle f(t), \xi\rangle = \frac{1}{\beta\,\alpha(t)}\left(\int_\Omega Q(t,x)\xi(x)\,dx + \int_{\partial\Omega_2} q(t,x)\xi(x)\,d\sigma\right),$$

$$\langle A(t, a(t) + u), \xi\rangle = \frac{1}{\beta\,\alpha(t)}\int_\Omega (v(t,x,\nabla(a(t)+u))|\nabla\xi(x))\,dx, \qquad \xi \in V,$$

$$\langle Cw, \xi\rangle = \langle w, e\xi\rangle, \qquad w \in V', \qquad \xi \in V.$$

Note that $W \subset C^0(0,T;L^2(\Omega))$, hence u is continuous in t and the initial condition in (2) is meaningful, moreover the function $h = u + a$ is independent on the choice of a.

It can be verified (cf. [10, 11]) that under the conditions stated, concerning v, β, p the operator $A(t,h)$ is strongly monotone and coercive with respect to h uniformly in $t \in [0,T]$, derivatives $(\partial/\partial t)A(t,h)$, $D_h A(t,h) = A'(t,h)$ (D_h denotes the Frechet derivative) exist and are contiuous, C is linear, invertible and continuous, hence (2) has the unique solution $u \in W$.

3. Galerkin approach

Let $\{X_n\}$, $\dim X_n = n$, $X_n \subset X_{n+1}$, $(n = 1, 2, \ldots)$ be a sequence of spaces approximating V i.e. $\bigcup_{n=1}^\infty X_n$ is dense in V. As in [11] approximate (2) by the system

$$C_n\frac{du_n(t)}{dt} + A_n(t, a(t) + u_n(t)) = f_{1n}(t), \qquad u_n(0) = (h_1)_n, \tag{3}$$

where $(h_1)_n$ is the projection of h_1 on X_n. The maps C_n, A_n and f_{1n} are defined respectively by $\langle C_n h, \xi\rangle = \langle Cu, \xi\rangle$, $\langle A_n(t, a(t) + u), \xi\rangle = \langle A(t, a(t) + u), \xi\rangle$, $\langle f_{1n}(t), \xi\rangle = \langle f_1(t), \xi\rangle$ $(u, \xi \in X_n)$.

Since X_n is closed and finite dimensional, from the differentiability of A and the continuity of f_1 it follows that A_n and f_{1n} are continuous in t and A_n is Lipschitz continuous in u_n, hence the Cauchy problem (3) has the unique solution $u_n \in C^1(0,T;X_n)$ (see e.g. [4, Chapt. III]). The function $h_n = a + u_n$, where u_n satisfies (3) and approximates the solution h of (2) in a sense that

$$\lim_{n\to\infty} \|h_n - h\|_{C(0,T;L^2(\Omega))} = 0. \tag{4}$$

(see [11, Thms 2 and 3]).

4. MIXED SCHEME

Now we are going to describe the numerical method for solving the initial value problem (3) which, for the sake of simplicity, for fixed $n \in \mathbb{N}$, will be written in the form

$$R(t)u(t) = r(t), \qquad u(0) = (h_1)_n, \tag{5}$$

where $R(t)u(t) = C_n \dfrac{du(t)}{dt} + A_n(t, a(t) + u(t))$, $u(t) \in X_n$, $r(t) = f_{1n}(t)$.

Let $\tau \in (0, t_0)$ be fixed and let S denote the time-grid over the interval $[-\tau, T]$: $S = \{\tau i : i = -1, 0, 1, \ldots, k; \quad k\tau = T\}$.

Denote by $g_\tau = g|_S$ the grid function corresponding to the function g defined on $[-\tau, T]$. Set $g_\tau^i = g_\tau(i\tau)$. Let $\widetilde{V}, \widetilde{W}$ denote the spaces of grid functions defined on S with values respectively in X_n and X'_n ($X'_n = C_n(X_n)$) equipped with norms

$$\|u_\tau\|_{k\tau} = \max\{\|u_\tau^i\| : i = -1, 0, \ldots, k\}, \; \|v_\tau\|_{k\tau}^* = \max\{\|v_\tau^i\|_{X'_n} : i = -1, 0, \ldots, k\},$$

where $\|\cdot\|$ is the norm in X_n defined by $\|u\| = \sqrt{\langle C_n u, u\rangle}$ and $\|\cdot\|_{X'_n}$ is the norm in X'_n, arbitrarily chosen.

Let $R_\tau : S \times \widetilde{V} \to \widetilde{W}$ be the three-level linearized difference approximation of the operator R defined by the formula:

$$(R_\tau u_\tau)^i = \frac{1}{2\tau} C_n (u_\tau^{i+1} - u_\tau^{i-1}) + A_n^i u_\tau^i + \frac{1}{2}(A'_n)^i (u_\tau^{i+1} - 2u_\tau^i + u_\tau^{i-1}), \tag{6}$$

with $A_n^i u_\tau^i = A(i\tau, a(i\tau) + u_\tau^i)$, $(A'_n)^i = A'_n(i\tau, u_\tau^i)$, $\langle A'_n(t, w)h, \xi\rangle = \langle A'(t, w)h, \xi\rangle$.

The difference scheme for (5), associated with R_τ has the form (cf. [13]):

$$R_\tau u_\tau = r_\tau, \qquad u_\tau^{-1} = \tilde{u}(-\tau), \qquad u_\tau^0 = (h_1)_n, \tag{7}$$

where $\tilde{u} \in C^1(-t_0, T; X_n)$ is given, and $u_\tau : S \to X_n$ is an unknown function, which leads to the system of algebraic equations for unknowns u_τ^{i+1}:

$$(R_\tau u_\tau)^i = r_\tau^i, \qquad i = 0, \ldots, k-1, \qquad u_\tau^{-1} = \tilde{u}(-\tau), \quad u_\tau^0 = (h_1)_n. \tag{8}$$

Note that one can approximate R by the two-level Crank-Nicolson difference operator L_τ defined by $(L_\tau u_\tau)^i = \frac{1}{\tau} C_n (u_\tau^{i+1} - u_\tau^i) + \frac{1}{2}(A_n^{i+1} u_\tau^{i+1} + A_n^i u_\tau^i)$. The corresponding difference scheme assumes then the form $(L_\tau u_\tau)^i = r_\tau^i$, $i = 0, 1, \ldots, k-1$, $u_\tau^0 = (h_1)_n$. Its convergence has been proved in [10].

The function $\tilde{u}$ is introduced to provide the additional initial data required in the case of the three-level schemes. The proper selection of $\tilde{u}$ improves the accuracy of computations. As a matter of fact, the influence of $\tilde{u}$ on u_τ, introduced for computation purposes, can be made as small as desired, which is a consequence of the following convergence result.

Theorem 1. *If A and f satisfy assumptions stated in [11], then the difference scheme (7) for the Galerkin initial problem (3) (or (5)) has for any $\tau \in (0, t_0)$ the unique solution u_τ such that*

$$\lim_{\tau \to 0} \|u_\tau - u\|_{k\tau} = 0,$$

where u denotes the solution of (5).

As an immediate corollary of Theorem 1 we get the result clarifying the relationship between the approximate solution of the Cauchy problem for the Galerkin equation and those of the initial value problem (2).

Theorem 2. *Assume the conditions of Theorem 1. Let $h \in W \subset C(0, T; L^2(\Omega))$ be the solution to (2) and let $h_{n\tau} \in \widetilde{V}$ be the solution to (7) corresponding to $\tilde{u}_n \in C^1(-t_0, 0; X_n)$. Then for an arbitrarily chosen, fixed $\tilde{u}_n$*

$$\lim_{\substack{n \to \infty \\ \tau \to 0}} \|h - h_{n\tau}\|_\tau = 0,$$

where $\|u\|_\tau = \max\{\|u(i\tau)\|_{L^2(\Omega)} : i = 0, 1, \ldots, k\}$.

Proof of Theorem 2. By the triangle inequality and definitions of $\|\cdot\|, \|\cdot\|_\tau, \|\cdot\|_{k\tau}$, it follows that $\|h-h_{n\tau}\|_\tau \le \|h-h_n\|_\tau + \|h_n - h_{n\tau}\|_\tau \le \|h-h_n\|_{C(0,T;L^2(\Omega))} + \|h_n - h_{n\tau}\|_{k\tau}$.

By (4), the first component tends to zero as $n \to \infty$. Theorem 1 implies that the second member also converges to zero.

5. MUBS PACKAGE

The mixed FE/FD scheme described in this paper has been implemented as a part of the package of programs MUBS (Multipurpose Underground Basin Simulator).

MUBS routines are focused mainly on solving various kinds of nonlinear filtration problems. Generally, they fall into three groups according to their purposes.

The first group is aimed to one- or two-dimensional free surface stationary or nonstationary filtration in medium grained layers described by Dupuit–Forcheiner and Boussinesq equations. Lagrange first degree triangles are used for spatial approximation and weighted Crank–Nicolson scheme in nonstationary cases.

The second group of routines may be applied to the prelinear as well as to the linear Darcy filtration in a saturated soil. Both the two-layer Crank–Nicolson and the three-layer linearized schemes are employed.

The third group provides tools for the permeability parameters identification giving methods of solving inverse problems (see eg. [13]) and processing the measured data (like grain size distribution, cracks widths, their directions and frequencies).

The basic version of MUBS codes have been implemented in a traditional way, using procedural programming technique in FORTRAN and C++ languages and may be run on a single-processor computers. Both PC and UNIX platforms are available as well as the special version for vector units CONVEX series 3200 and 3800 with the extensive use of VECLIB library for tiering linear algebra operations. The full documentation of the MUBS package can be provided by authors upon a serious request.

The MUBS applications are embedded in the graphic-network OCTOPUS environment (see [1]) which provides pre- and post-processing operations like automatic mesh generation and adaptation or visualizing the piesometric pressure or the filtration distributions.

The MUBS package has been extensively used in numerous geotechnical designs. Two of them will be briefly reported below.

Comparison of simulated pressures with the measured ones in consolidated organic soils. A one-dimensional vertical ground water flow under the center of a prismatic embankment (cf. Fig. 1) founded on two organic strata (peat and gytia) has been considered.

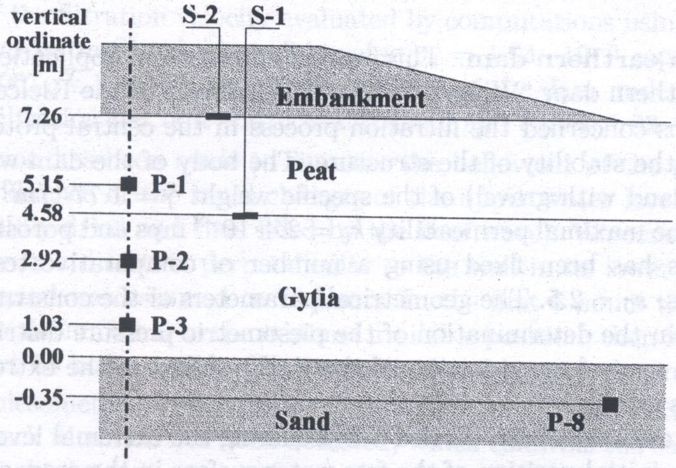

Fig. 1. Cross section through the consolidated organic layers

The organic soil is supported by a stable, preconsolidated, well permeable sand layer. The flow in the organic sediments results from the artesian pressure observed in underlying sand and from the deformation process caused by the pressure of the embankment.

The cross-section under consideration has been equipped by four piesometers (P1, P2, P3, P8) and two bench–marks S1, S2 (cf. Fig. 1). The volume strain velocity G has been calculated on the base of measured bench–mark and piesometer displacements, the functions $M, S_0, \theta, e, \alpha$ has been obtained empirically during the laboratory tests (see [15]). Note that pressures in P1,...,P8 have been measured during the whole consolidation process.

The seven days during which the embankment has been upgraded and the next 74 following days have been choosen as the simulation period. The observed free water surface and the mesurments in P8 constituted the basis for the evaluation of boundary conditions. The extrapolated P1, P2, P3 measurments at the beginning of the process served as the initial conditions. Computations have been performed using the three-level linearized scheme (6), the standard Crank–Nicholson sheme and performing the total linearization of the filtration equation (Darcy's flow). Comparison of the above results with the measurments are depicted in Fig. 2.

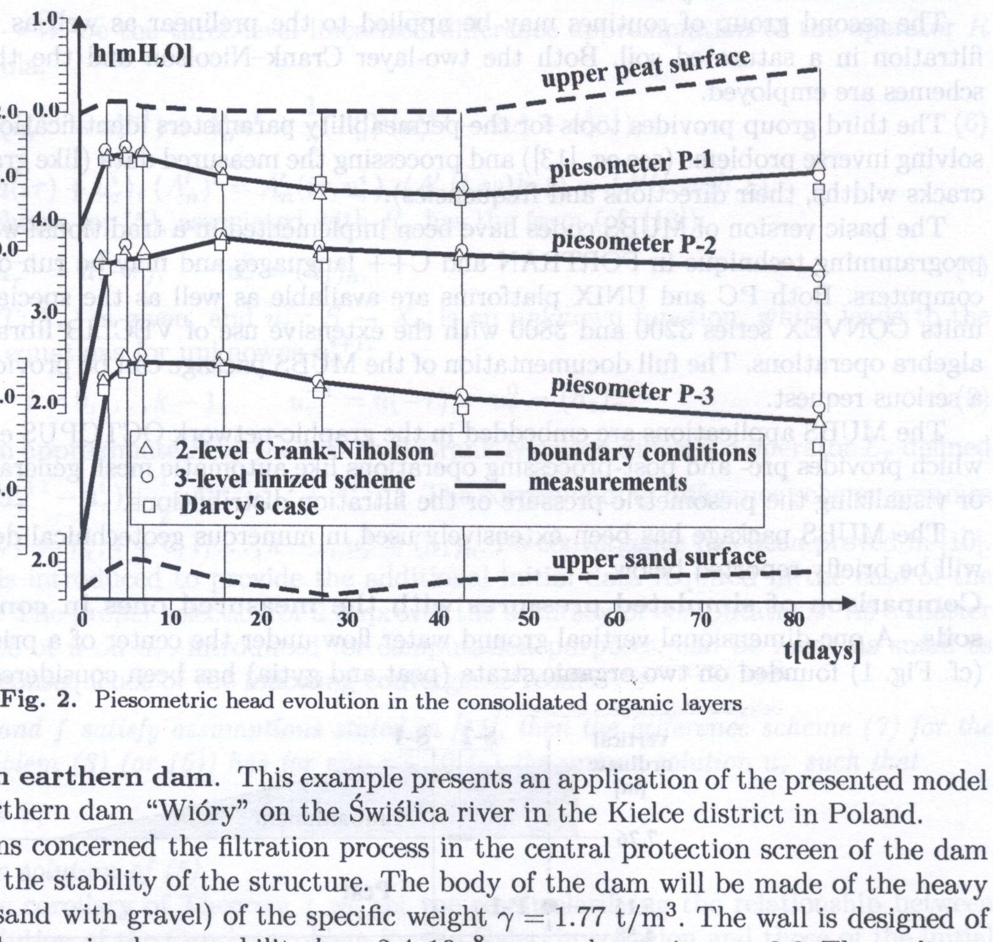

Fig. 2. Piesometric head evolution in the consolidated organic layers

Filtration in an earthern dam. This example presents an application of the presented model in designing the earthern dam "Wióry" on the Świślica river in the Kielce district in Poland.

Our computations concerned the filtration process in the central protection screen of the dam which is crucial for the stability of the structure. The body of the dam will be made of the heavy semigravel (coarse sand with gravel) of the specific weight $\gamma = 1.77$ t/m^3. The wall is designed of cohesive clay with the maximal permeability $k_e = 2.4 \cdot 10^{-8}$ mps and porosity $p = 0.8$. The prelinear fitration parameters has been fixed using a number of comparative tests. They are: $\theta = 2,5$, $M = 0.24 \cdot 10^{-9}$ mps, $s_0 = 2.5$. The geometrical parameters of the construction are given in Fig. 3. We were interested in the determination of the piesometric pressure distribution and the filtration velocity field in the central cross-section of the wall in cases of the extremal loading of the dam which may cause its erosion.

The first computation has been carried out assuming the extremal level of water (27.6 m above the wall's foot). The initial position of the free water surface in the cross section has been obtained by the Wieczysty method (see [15] for details) and then corrected in several iterations up to the assumed local accuracy (0.01 m). The final piesometric pressure distribution is shown in Fig. 3 as a colour map and a perspective view.

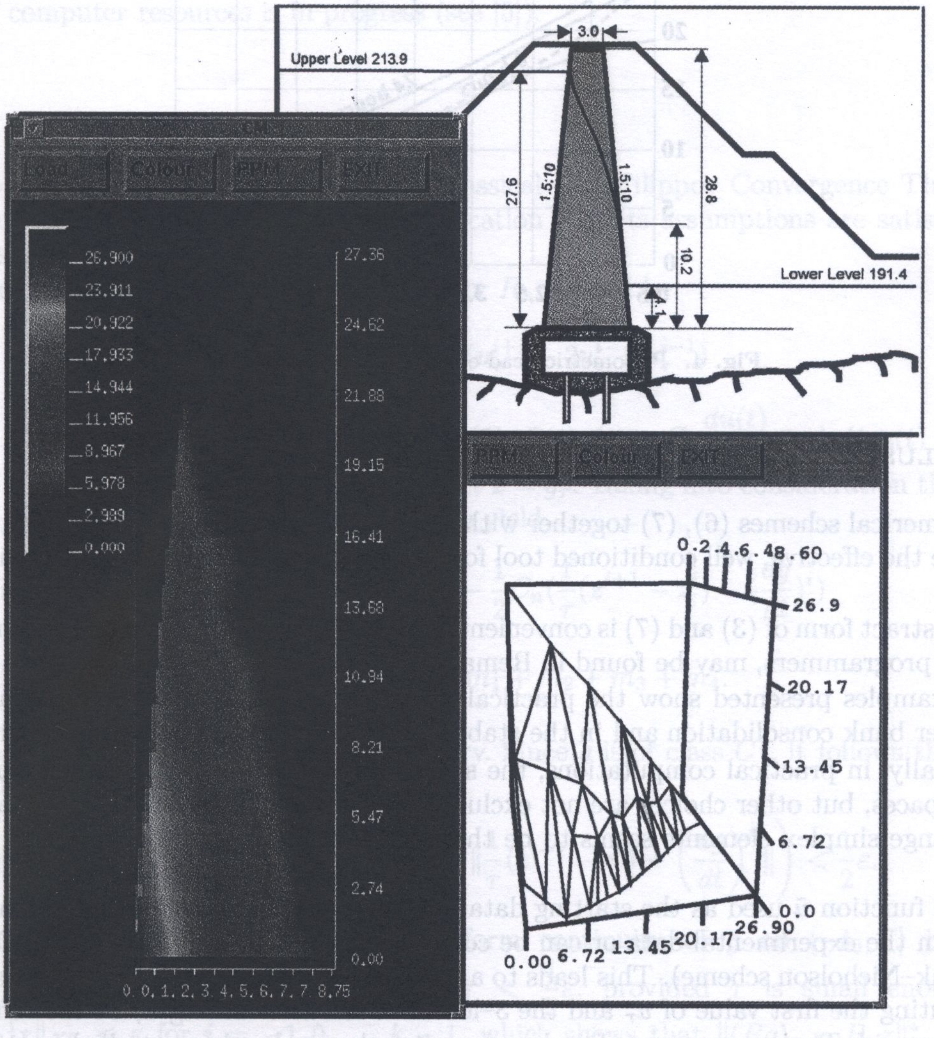

Fig. 3. Earthern dam study using MUBS and OCTOPUS packages

The maximal value of the filtration velocity evaluated by computations using (2.1) equals $0.571 \cdot 10^{-7}$ mps and is far from its maximal admissible value $v_e = 1.04\ 10^{-5}$ mps obtained from the Darcy criterion ($v_e = 0.067\sqrt{k_e}$), which proves the stability of the dam under this hydrodynamic conditions. The overall filtration per unit width is also small and equals $1.03 \cdot 10^{-7}$ m^2/sec.

The second computation has been made to simulate the behaviour of the piesometric pressure distribution under the influence of the rapid changes of the upper water level (e.g. in a situation when the reservoir is rapidly dried and then filled by the floof wave). We considered a one dimensional nonstationary flow in a direction perpendicular to the wall in its lower part, near the foot. The overal simulation period was assumed to 72 hours. In the first 6 hours the upper water level dropped down 5.4 m from the maximal to the minimal available position, then it grew up back to its maximum during the next 18 hours and finally, it remained stable until the end of the simulation.

The evolution of the piesometric pressure computed in the four characteristic instants of time is presented in Fig. 4. No back filtration has been detected, which confirms the safety of the analyzed structure. The shape of obtained curves confirms also the non-Darcian effects in the analyzed fitration field.

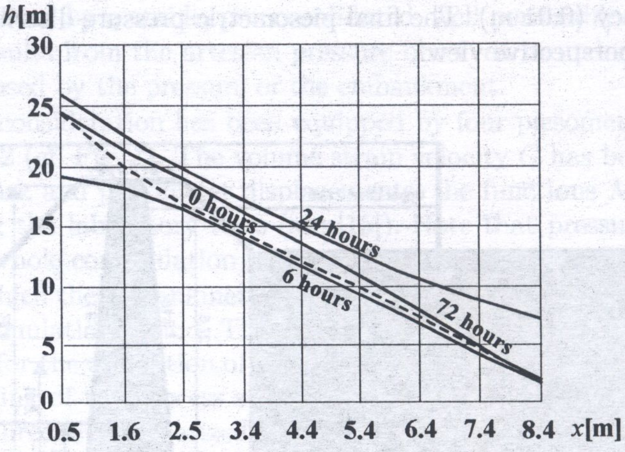

Fig. 4. Piesometric head evolution near the foot of the seal

6. Conclusions

1) Numerical schemes (6), (7) together with the mathematical convergence results (Theorem 1, 2) provide the effective, well conditioned tool for the numerical investigations of prelinear filtration processes.

The abstract form of (3) and (7) is convenient to express main results. An equivalent formulation, useful for programmers, may be found in Remark of Appendix.

The examples presented show the practical value of this formulation in the simulation of the boggy river bank consolidation and in the stability evaluation of an earthern dam silt.

2) Usually, in practical computations, the spaces X_n appearing in (7) are chosen as the finite element spaces, but other choices are not excluded. In the case of the prelinear filtration problems the Lagrange simplex elements seems to be the most suitable (see [13] or [16] and the references therein).

3) The function $\tilde{u}$ used as the starting data in the proposed difference scheme can be obtained either from the experimental data or can be computed by the aid of a two-level difference scheme (e.g. Crank–Nicholson scheme). This leads to a method using two grid operators: Crank–Nicholson for computing the first value of u_τ and the 3–level one for determining u_τ at the remaining points of the time grid. The convergence of such composite schemes has been studied in [13].

4) All computational results confirm the high accuracy of the proposed mixed scheme (6), (7) applied to the FE Galerkin equations. The comparison of the three-level scheme with the standard procedure using the two-level Crank–Nicolson formula (cf. Fig. 2) shows their similar quality. In addition, the linearized scheme applied to the problem of the small size, like the one presented in the first example, in comparison with the standard one performs computations 7 to 8 times faster. Moreover, the difference of the computational speeds grows faster than linearly when the number of degrees of freedom is increased. An extra acceleration of the computational speed may be obtained by the use of the variable time-step strategies (e.g. using the Gear method).

5) Results of the first example as well as the shape of the piesometric head (see Fig. 4) obtained in the next one confirm the non-Darcian effects in the analyzed filtration field, thus proving the considerable advantage of the proposed model over the linear one.

6) Much work have been recently done for preparing parallel versions of algorithms described above and implementing them in the distributed environment of the computer network. The first result in this directions is connected with the domain decomposition technique allowing to tacle with the large-scale two dimensional nonstationary problems (cf. [6, 7]). The second, recently added to the MUBS package, concern the ill posed identification computations using the parallel optimization algorithms (see [12]).

Both application packages have been implemented using the PVM programming system (the Parallel Virtual Machine) in the UNIX environment. The load balancing strategy is based on the Markovian schematic decision model of the computer environment.

The object oriented project which enable to carry out the cooperative design and the more flexible use of computer resources is in progress (see [5]).

APPENDIX

Proof of Theorem 1. The proof is based on the classical Lax–Filippov Convergence Theorem ([9, Ch. 1, Th.1] or [8, Ch. 3]) and it consists in verification that its assumptions are satisfied in the considered case.

We will prove at first that the difference operator R_τ defined by

$$(R_\tau u_\tau)^i = \frac{1}{2\tau}C_n(u_\tau^{i+1} - u_\tau^{i-1}) + A_n^i u_\tau^i + \frac{1}{2}(A_n')^i(u_\tau^{i+1} - 2u_\tau^i + u_\tau^{i-1}) \tag{A-1}$$

approximates the operator in the left hand side of (5): $R(t)u(t) = C_n\dfrac{du(t)}{dt} + A_n(t, a(t) + u(t))$.

Let $g \in C^1(-t_0, T; X_n)$ be arbitrary, fixed and let $z = g_\tau$. Taking into consideration the equality $A_n^i z^i = A_n(i\tau, a(i\tau) + g(i\tau))$, the above expressions yield

$$(Rg)_\tau^i - (R_\tau z)^i = -\frac{1}{2}C_n\left(\frac{1}{\tau}(z^i - z^{i-1}) - \left(\frac{dg}{dt}\right)^i\right) - \frac{1}{2}C_n\left(\frac{1}{\tau}(z^{i+1} - z^i) - \left(\frac{dg}{dt}\right)^i\right)$$
$$+\frac{1}{2}(A_n')^i(z^{i-1} - z^i) + \frac{1}{2}(A_n')^i(z^{i+1} - z^i) = m_1 + m_2 + m_3 + m_4.$$

By the choice of the norm in X_n, C_n is an isometry. Since g is of class C^1, it follows that for any $\varepsilon > 0$, and sufficiently small τ

$$\|m_1 + m_2\|_{X_n'} \le \frac{1}{2}\left(\left\|\frac{1}{\tau}(z^i - z^{i-1}) - \left(\frac{dg}{dt}\right)^i\right\| + \left\|\frac{1}{\tau}(z^{i+1} - z^i) - \left(\frac{dg}{dt}\right)^i\right\|\right) < \frac{1}{2}\varepsilon.$$

Similarly, the continuity of $(A_n')^i$ and the uniform continuity of g on $[-t_0, T]$ imply that $\|m_3 + m_4\|_{X_n'} \le \frac{1}{2}L\left(\|z^{i+1} - z^i\| + \|z^{i-1} - z^i\|\right) < \frac{1}{2}\varepsilon$, provided τ is small enough. Thus $\|(Rg)_\tau^i - R_\tau^{i+1}z\|_{X_n'} < \varepsilon$ for $i = -1, 0, \ldots, k-1$, which shows that $\|(Rg)_\tau - R_\tau z\|_{k\tau}^* < \varepsilon$, i. e. R_τ approximates R.

Next observe that for given u_τ^i, u_τ^{i-1} Eq. (8) has exactly one solution u_τ^{i+1}. In fact, rewrite (8) in the form

$$B^i u_\tau^{i+1} = -2\tau\left(A_n^i u_\tau^i - (A_n')^i u_\tau^i - r^i\right) + G^i u_\tau^{i-1}, \tag{A-2}$$

where $B^i = C_n + \tau(A_n')^i$, $G^i = C_n - \tau(A_n')^i$, $i = 1, \ldots, k$.

Since A_n' is positive definite and Lipschitz continuous it follows that operators B^i are coercive with constant 1 and Lipschitz continuous with a Lipschitz constant independent of i, hence B^i are invertible (see [3, Ch. III]), i.e. (A-2) is solvable.

For the proof of stability of (A-2) with respect to the right hand sides, we will apply a lemma, being an extension of the classical stability result ([9, Ch. 2]) concerning 3–level schemes to the case of nonlinear operators R_τ. The proof of the lemma will be postponed to the last part of the Appendix.

Lemma. *Suppose there are positive constants c_1, c_2, independent of τ, such that for any $0 \le m < k$ and $w, v \in \widetilde{V}$ the operator $(R_\tau u_\tau)^i$ satisfies:*

$$(R_\tau w)^j = (R_\tau v)^j, \quad j = m+1, \ldots, k \Rightarrow$$

$$\|w^{m+1} - v^{m+1}\| + \|w^m - v^m\| < (1 + c_1\tau)(\|w^m - v^m\| + \|w^{m-1} - v^{m-1}\|), \tag{A-3}$$

$w^i = v^i, i = -1, 0, \ldots, m-1$ and $\|(R_\tau w)^m - (R_\tau v)^m\|_{X'_n} < \delta \Rightarrow$

$$\|w^m - v^m\| < c_2\tau\delta, \tag{A-4}$$

then (A-2) is stable with respect to r_τ.

Using coercivity of B^i, Lipschitz continuity of A^i_n, $(A'_n)^i$ and G^i (the latter is Lipschitz continuous with the constant $1 + \tau L$) one gets from (A-2) the inequality

$$\|w^{i+1} - v^{i+1}\|^2 \le \langle B^i(w^{i+1} - v^{i+1}), w^{i+1} - v^{i+1}\rangle = -2\tau\langle A^i_n w^i - A^i_n v^i, w^{i+1} - v^{i+1}\rangle$$

$$+2\tau\langle (A'_n)^i(w^i - v^i), w^{i+1} - v^{i+1}\rangle + \langle G^i(w^{i-1} - v^{i-1}, w^{i+1} - v^{i+1}\rangle$$

$$\le (4L\tau\|w^i - v^i\| + (1 + L\tau)\|w^{i-1} - v^{i-1}\|)\|w^{i+1} - v^{i+1}\|,$$

from which (A-3) follows immediately.

Let $w, v \in \widetilde{V}$ satisfy the assumptions stated in condition (4) of Lemma. Then from (7) and (A-2) it follows that

$$\|w^m - v^m\|^2 \le \langle B^{m-1}(w^m - v^m), w^m - v^m\rangle = 2\tau\langle R^m_\tau w - R^m_\tau v, w^m - v^m\rangle$$

$$\le 2\tau\|R^m_\tau w - R^m_\tau v\|_{X'_n}\|w^m - v^m\|,$$

which proves (A-4).

Proof of Lemma. The proof will be carried out in two steps.

Step 1. From (3) it follows that for any $w, v \in \widetilde{V}$ and an arbitrary number $M \ge m$

$$\|w^{M+1} - v^{M+1}\| + \|w^M - v^M\| \le (1 + c_1\tau)^{M-m}(\|w^M - v^M\| + \|w^{M-1} - v^{M-1}\|).$$

Obviously $(M - m)\tau \le T$ and for $\tau > 0$ small enough $(1 + c_1\tau)^{M-m} \le \exp((c_1 T) = C$, hence for $M \ge m$

$$\|w^M - v^M\| \le C(\|w^m - v^m\| + \|w^{m-1} - v^{m-1}\|). \tag{A-5}$$

Step 2. Fix $\delta > 0$. Let $f, \tilde{f} \in \widetilde{W}$ satisfy $\|f - \tilde{f}\|^*_{k\tau} \le \delta$ and suppose $w, v \in \widetilde{V}$ satisfy conditions $R_\tau w = f$,$R_\tau v = \tilde{f}$, $w^i = v^i$ for $i = -1, 0, 1, \ldots, m$, i.e. w, v are solutions of (7) corresponding to different right hand sides. Let functions $h_m \in \widetilde{V}$, $m = 0, 1, \ldots, M$ $(0 \le M \le k, k\tau = T)$ be defined by

$$(R_\tau h_m)^i = \begin{cases} f_i, & \text{for} \quad i = 0, \ldots, m-1, \\ \tilde{f}^i & \text{for} \quad i = m, m+1, \ldots, k. \end{cases}$$

Since $\|f^{m-1} - \tilde{f}^{m-1}\|_{X'_n} \le \delta$, assumption (A-4) implies that

$$\|h^m_{m-1} - h^m_m\| \le c_2\tau\,\delta. \tag{A-6}$$

Functions h_{m-1}, h_m satisfy (A-3). By (A-5) and (A-6), $\|h^M_{m-1} - h^M_m\| \le c_2\tau\delta$ for $M \ge m$. Noting that $h_0 = w$, $h_M = v$, the inequalities above imply that

$$\|w^M - v^M\| \le \sum_{m=1}^{M} \|h^M_{m-1} - h^M_m\| \le c_2 M, \tau\delta \le c_2 T\delta,$$

hence $\|w - v\|_{k\tau} \le C\delta$, $C = c_2 T$, completing the proof of Lemma.

Remark. If the basis $\{\eta_i^n\}$, $i = 1, \ldots, n$ in X_n is selected, then finding the solution u_n to (3) amounts to determinig its components g_{in} $(i = 1, \ldots, n)$ relative to $\{\eta_i^n\}$: $u_n(t) = \sum_{j=1}^n \eta_j^n g_{in}(t)$. Inserting this expression into formulae

$$\langle C_n \frac{du_n(t)}{dt} + A_n(t, a(t) + u_n(t)) - f_{1n}(t), \eta_i^n \rangle = 0, (i = 1, \ldots, n)$$

$u_n(0) = (h_1)_n$ $((h_1)_n = \sum_{i=1}^n \alpha_i \eta_i^n)$, equivalent to (3) and using definitions of C_n, A_n, f_1, after a direct computations one arrives to the initial problem for an unknown function $g_n(t) = (g_{1n}(t), \ldots, g_{nn}(t))$:

$$Kg'(t) + F(t, g(t)) = 0, \quad g(0) = \alpha, \quad (\alpha = (\alpha_1, \ldots, \alpha_n)), \tag{A-7}$$

where $F(t, g(t)) = B(t, g(t))g(t) + H(t, g(t)) - F(t)$, $K = (k_{ij})$, $B(t, g) = (b_{ij}(t, g))$ are $n \times n$ matrices with entries $k_{ij} = \langle C_n \eta_i^n, \eta_j^n \rangle$,

$$b_{ij}(t, g) = (1/\beta\alpha(t)) \int_\Omega \varphi(t, x, \|\nabla(a(t) + \sum_{i=1}^n \eta_i^n g_{in}))\|)(\nabla \eta_i^n | \nabla \eta_j^n)\, dx$$

and vectors $F(t)$, $H(t, g)$ have components $f_i(t) = \langle f_{1n}(t), \eta_i^n \rangle$,

$$h_i(1/\beta\alpha(t)) \int_\Omega \varphi(t, x, \|\nabla(a(t) + \sum_{i=1}^n \eta_i^n g_{in}))\|)(\nabla a | \nabla \eta_i^n)\, dx.$$

The problem (A-7) is then solved by the 3–level formula for $i = 0, 1, \ldots, k - 1$

$$K \frac{g_\tau^{i+1} - g_\tau^{i-1}}{2\tau} + F(\tau i, g_\tau^i) + (1/2) D_g F(\tau i, g_\tau^i) \left(g_\tau^{i+1} - 2g_\tau^i + g_\tau^{i-1} \right) = 0,$$

$$g_\tau^0 = \alpha, \qquad g_\tau^{-1} = \beta,$$

where $\beta = (\beta_1, \ldots, \beta_n)$, $u_\tau^{-1} = \sum_{i=1}^n \beta_i \eta_i^n$ and $D_g F$ denotes the Jacobian matrix of the map $F(\tau i, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$.

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