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Stability and accuracy of steady-state motions in loaded copying system: analytical approach

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An analytical study of stability is made for a hydromechanical servomechanism used in copying systems. In the framework of a nonlinear system of ODE mathematically modelling the servomechanism, we prove that for a ramp input the steady-state solution bifurcates into a stable limit cycle for a certain value of the underlap spool valve.

1. Introduction

A hydraulic copying system must reproduce a geometrical form known as the template in a workpiece as close as possible. So, from the manufacturing point of view, the stability and accuracy of the copying system are important.

The theoretical analyses of the hydraulic copying systems used until now the techniques specific to linear control systems. Nonlinear analysis was carried out in [3, 4, 5], using numerical simulation. no analytical approach to periodic solutions of the copying systems, neither to the behaviour of the system in the presence of a limit cycle being available in literature until recently [1]. In this paper such an analytical study of the behaviour of the hydraulic copying system having a ramp input signal and loaded with a constant cutting force, F_a, is performed.

A typical hydraulic copying mechanism is shown in Fig. 1. It consists of a hydraulic spool valve (a hydraulic amplifier) which commands a hydraulic linear motor. The spool of the valve is moved by a stylus which is in contact with the template. Because of the intrinsic reaction between the spool valve and the hydraulic motor, this one will receive the movement of the stylus $y_0(t)$ with an error $\varepsilon(t)$. So the displacement of the tool will finally be, [2, 3, 4, 5]:

$$y(t) = y_0(t) - \varepsilon(t). \tag{1}$$

The mathematical model of the hydraulic control system contains the equilibrium equation of the copying slide, the flow equation through the control valve and the equation of the feedback between the spool valve and copying slide. The proper nonlinearities of the hydraulic amplifier with underlap spool valve are the insensitive zone and the saturation zone.

The theoretical assumptions are: the hydraulic oil used in the servomechanism is a Newtonian compressible fluid; the flow equations are written for a turbulent flow at higher Reynolds numbers, [1]; the hydraulic coefficient c_d is constant and its value for a knife-type of flow port is: $c_d = 0.61$; the hydraulic servomechanism is supplied at a constant pressure, p_s; a hydraulic oil with

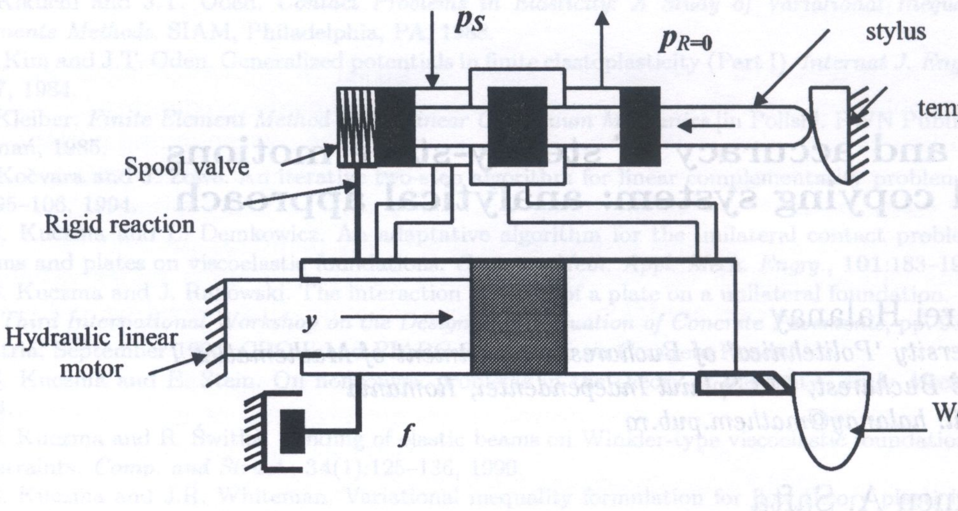

Fig. 1. The Hydraulic Copying System. Principle Schema

the kinematic viscosity $\nu = 80 \cdot 10^{-6}$ $[m^2/s]$ and the specific mass $\rho = 900[kg/m^3]$ at a constant working temperature is used; the bulk of oil module, E, has an usual value [2,3] of 10^9 [Pa] for the assumption of a less quantity of air in the oil column; the reservoir pressure p_R has a small value and can be considered to be zero.

2. MATHEMATICAL FORMULATION

The following model is essentially that proposed by Viersma [5, 6].

The flow characteristic equations through the valve ports, within the assumptions mentioned bellow, are given by:

$$Q_1 = \pi d c_d (h - x)\sqrt{\frac{2}{\rho}}\sqrt{p_s - p}; \quad Q_2 = \pi d c_d (h + x)\sqrt{\frac{2}{\rho}}\sqrt{p}, \tag{2}$$

$$Q = Q_2 - Q_1, \tag{3}$$

$$Q = A\frac{dy}{dt} - \frac{V_0}{E}\frac{dp}{dt}, \tag{4}$$

$$\frac{A}{2}p_s - A \cdot p = m\frac{d^2y}{dt^2} + k\frac{dy}{dt} + F_a, \tag{5}$$

$$x = y_0 - y, \tag{6}$$

with: d [m], the spool diameter, πd [m] is the area gradient of the valve; h [m], the underlap of the valve; A [m^2], effective cross-sectional area of the piston; V_0 [m^3], total volume of oil compressed in servo; m [kg], mass of copying slide; k [Ns/m], a coefficient related to the friction forces; $\tilde{F}_u$ [Ns/m], magnitude of dry friction coefficient in copying slide, [6]; F_a [N], total dynamic cutting force; Q [m^3/s], the flow rate of the hydraulic motor.

Equations (2)–(6) imply:

$$\begin{aligned} &F_a + my'' + (k + \tilde{F}_u)y' = A\left(\frac{p_s}{2} - p\right), \\ &\frac{V_0}{E}p' = Ay' - C[(h + y_0 - y)\sqrt{p} - (h - y_0 + y)\sqrt{p_s - p}], \end{aligned} \tag{7}$$

where, $C = \pi d c_d \sqrt{\dfrac{2}{\rho}}$.

3. THE STEADY-STATE MOTIONS: EXISTENCE, STABILITY AND BIFURCATION

We prove that if the input y_0 in the system given in Eq. (7) is $y_0(t) = \hat{a}t + \hat{b}$ the system has a solution $y(t) = \alpha t + \beta$, $p(t) = \tilde{p}$ (the steady-state motion). We study its stability and we prove that under a certain value of the underlap spool valve, h, the steady-state motion bifurcates into a stable limit cycle. The period and the amplitude of this motion can be computed as in [1].

The system given by means of Eqs. (7) can be rewritten as:

$$\begin{aligned} y' &= z. \\ z' &= -\frac{k}{m}z + \frac{A}{m}\left(\frac{p_s}{2} - p\right) - \frac{F_a}{m}, \\ p' &= \frac{AE}{V_0}z + \frac{CE}{V_0}\left[(y - \hat{a}t - \hat{b} - h)\sqrt{p} + (y - \hat{a}t - \hat{b} + h)\sqrt{p_s - p}\right]. \end{aligned} \tag{8}$$

Substituting in the second equation of Eqs. (8) the solution we are looking for, we find that $\gamma = 0$ and

$$\tilde{p} = \frac{p_s}{2} - \frac{F_a}{A} - \frac{k}{A}\hat{a}. \tag{9}$$

According to the notation used in Sec. 2 of the paper, the pressure $\tilde{p}$ must be positive. Then substitution in the third equation of Eqs. (8) gives:

$$\begin{aligned} \alpha &= \hat{a}, \\ \beta &= \frac{(\hat{b} + h)\sqrt{\tilde{p}} + (\hat{b} - h)\sqrt{p_s - \tilde{p}}}{\sqrt{\tilde{p}} + \sqrt{p_s - \tilde{p}}} - \frac{A\hat{a}}{C(\sqrt{\tilde{p}} + \sqrt{p_s - \tilde{p}})}. \end{aligned} \tag{10}$$

The steady-state motion is:

$$\begin{aligned} \tilde{y}(t) &= \hat{a}t + \beta, \\ \tilde{p}(t) &= \tilde{p} \end{aligned} \tag{11}$$

In order to investigate its stability we find it convenient to perform a translation of the system into zero, so we define:

$\xi = \tilde{y} - y, \quad \zeta = z - \tilde{z}, \quad \eta = \tilde{p} - p$ that satisfy:

$$\begin{aligned} \xi' &= -\zeta, \\ \zeta' &= -\frac{k}{m}\zeta + \frac{A}{m}\eta, \\ \eta' &= \frac{AE}{V_0}\zeta + \frac{CE}{V_0}\left[(\xi + h - \beta + \hat{b})\sqrt{\tilde{p} - \eta} + (\xi - h - \beta + \hat{b})\sqrt{p_s - \tilde{p} + \eta}\right] - \frac{AE}{V_0}\hat{a}. \end{aligned} \tag{12}$$

If we define:

$$C_1 = \tilde{p} = \frac{p_s}{2} - \frac{F_a}{A} - \frac{k\hat{a}}{A}, \qquad C_2 = p_s - C_1 = \frac{p_s}{2} + \frac{F_a}{A} + \frac{k\hat{a}}{A}, \tag{13}$$

$$\begin{aligned} H_1 &= h - \beta + \hat{b} = h\frac{2\sqrt{C_2}}{\sqrt{C_1} + \sqrt{C_2}} + \frac{A\hat{a}}{C(\sqrt{C_1} + \sqrt{C_2})}, \\ H_2 &= h + \beta - \hat{b} = h\frac{2\sqrt{C_1}}{\sqrt{C_1} + \sqrt{C_2}} - \frac{A\hat{a}}{C(\sqrt{C_1} + \sqrt{C_2})} = 2h - H_1 \end{aligned} \tag{14}$$

the system written in Eqs. (12) becomes:

$$\begin{aligned}
\xi' &= -\zeta,\\
\zeta' &= -\frac{k}{m}\zeta + \frac{A}{m}\eta, \\
\eta' &= \frac{AE}{V_0}\zeta + \frac{CE}{V_0}\left[(\xi + H_1)\sqrt{C_1 - \eta} + (\xi - H_2)\sqrt{C_2 + \eta}\right] - \frac{AE}{V_0}\tilde{a}.
\end{aligned} \tag{15}$$

The matrix of the linearised part of Eqs. (15) around zero is:

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -\dfrac{k}{m} & \dfrac{A}{m} \\ \dfrac{CE}{V_0}(\sqrt{C_1} + \sqrt{C_2}) & -\dfrac{AE}{V_0} & -\dfrac{CE}{V_0}\left(\dfrac{H_1}{\sqrt{C_1}} + \dfrac{H_2}{\sqrt{C_2}}\right) \end{pmatrix}. \tag{16}$$

If we denote:

$$\tilde{h} = \frac{H_1}{\sqrt{C_1}} + \frac{H_2}{\sqrt{C_2}}, \tag{17}$$

the characteristic polynomial of the matrix given by Eq. (16) is:

$$P(\lambda) = \lambda^3 + \lambda^2\left(\frac{k}{m} + \frac{CE}{2V_0}\tilde{h}\right) + \lambda\left(\frac{k}{2m}\frac{CE}{V_0}\tilde{h} + \frac{A^2E}{mV_0}\right) + \frac{ACE}{mV_0}(\sqrt{C_1} + \sqrt{C_2}) \tag{18}$$

and the asymptotic stability of the trivial solution of Eq. (15) is lost, according to the Hurwitz criterion if:

$$\left(\frac{k}{m} + \frac{CE}{2mV_0}\tilde{h}\right)\left(\frac{kCE}{2mV_0}\tilde{h} + \frac{A^2E}{mV_0}\right) = \frac{ACE}{mV_0}(\sqrt{C_1} + \sqrt{C_2}) \tag{19}$$

or equivalently

$$\frac{C^2Ek}{4V_0}\tilde{h}^2 + \frac{1}{2}\left(\frac{A^2CE}{V_0} + \frac{Ck^2}{m}\right)\tilde{h} + \frac{A^2k}{m} - AC(\sqrt{C_1} + \sqrt{C_2}) = 0.$$

Equation (19) has one positive root $\tilde{h}_0$ when the parameters are assigned the values corresponding to the standard copying systems, like in the numerical example given in Sec. 4 of the paper. The corresponding value of h_0 deduced from Eqs. (17), (13) and (14) is:

$$h_0 = \frac{\sqrt{C_1C_2}(\sqrt{C_1} + \sqrt{C_2})}{2p_s}\tilde{h}_0 + \frac{A\tilde{a}(\sqrt{C_1} - \sqrt{C_2})}{2Cp_s}. \tag{20}$$

If $\tilde{h} = \tilde{h}_0$ the polynomial given by Eq. (18) has two imaginary roots $\lambda_1 = i\omega_0$, $\lambda_2 = -i\omega_0$ and the third one λ_3 is strictly negative as is easily seen when $P(\lambda) = \lambda^3 - \lambda_3\lambda^2 + \omega_0^2\lambda - \lambda_3\omega_0^2$ is compared with Eq. (18). We infer that

$$\begin{aligned}
\lambda_3 &= -\frac{k}{m} - \frac{CE}{2V_0}\tilde{h}_0, \\
\omega_0 &= \left(\frac{k}{m}\frac{CE}{2V_0}\tilde{h}_0 + \frac{A^2E}{mV_0}\right)^{\frac{1}{2}}.
\end{aligned} \tag{21}$$

Generally, if $\lambda_1(\tilde{h})$ is a root of P in Eq. (18), derivation with respect to $\tilde{h}$ gives:

$$\lambda_1'(\tilde{h}) = \alpha_1'(\tilde{h}) + i\omega_1'(\tilde{h}) = \frac{CE}{2V_0} \frac{\lambda(\tilde{h})^2 + \dfrac{k}{m}\lambda(\tilde{h})}{3\lambda^2(\tilde{h}) + 2\lambda(\tilde{h})\left(\dfrac{k}{m} + \dfrac{CE}{2V_0}\tilde{h}\right) + \dfrac{kCE}{2mV_0} + \dfrac{A^2E}{mV_0}}.$$

Relations (17), (14) and (21) imply for $\lambda = \lambda(h) = \alpha(h) + i\omega(h)$ that:

$$\alpha'(h_0) = \alpha_1'(\tilde{h}_0)\frac{2(C_1 + C_2)}{(\sqrt{C_1} + \sqrt{C_2})\sqrt{C_1C_2}} = -\frac{CE\left(\omega_0^2 - \lambda_3\dfrac{k}{m}\right)(C_1 + C_2)}{2V_0(\sqrt{C_1} + \sqrt{C_2})\sqrt{C_1C_2}(\omega_0^2 + \lambda_3^2)}, \tag{22}$$

$$\omega'(h_0) = \omega_1'(\tilde{h}_0)\frac{2(C_1 + C_2)}{(\sqrt{C_1} + \sqrt{C_2})\sqrt{C_1C_2}} = \frac{C^2E^2\omega_0\tilde{h}_0(C_1 + C_2)}{4V_0^2(\sqrt{C_1} + \sqrt{C_2})\sqrt{C_1C_2}(\omega_0^2 + \lambda_3^2)}. \tag{23}$$

Since $\lambda_3 < 0$ we have $\alpha'(h_0) < 0$ so from Hopf theorem [8, 9] we conclude that for $h < h_0$, h close to h_0, the system given by (15) has a family of periodic solutions analytically depending on h. The algorithm developed in [8] allows us to prove that a stable limit cycle arises and to determine its period and amplitude. The computations are identical with those involved in the study of bifurcation for the output a step input in a loaded system, [1].

For given values of the constants involved, we compute the value of h_0 and then γ, the Lyapunov coefficient is computed, too. Since this coefficient is strictly negative a stable limit cycle arises. Some numerical results are given in the next section.

4. Numerical results and simulation

Computations performed using Mathematica, with given values of the constants involved according to usual standards, yield that a stable limit cycle arises for h close to h_0 and:

$$h_t < h < h_0 \tag{24}$$

where $h_t = 2.78 \cdot 10^{-5} m$ is the value of the underlap in turbulent flow [6]. We start with the following parameters of the hydraulic copying system in Eq. (7): $A = 0{,}028$ m^2, $C = 9.03386 \cdot 10^{-4}$ m$^{5/2}$ kg$^{-1/2}$, $k = 4000$ Ns/m, $m = 93$ kg, $p_s = 2 \cdot 10^6$ Pa, $E = 1 \cdot 10^9$ Pa, $V_0 = 3.5 \cdot 10^{-3}$ m^3, $\hat{a} = 3 \cdot 10^{-4}$ ms^{-1}, $\hat{b} = 1 \cdot 10^{-5}$ m. We determine from Eq. (19) the value of h_0 where bifurcation occurs and then, as in [1], the amplitude Δ and the period T of the stable limit cycle.

The results are presented in Table 1, for different values of the perturbation force F_a (the cutting force) and $h = 8 \cdot 10^{-5}$ m.

The steady-state error ε_{ss} of the copying system is given by the difference between the steady-state solution $y(t) = \alpha t + \beta$ and the ramp input $y_0(t) = \hat{a}t + \hat{b}$.

Using α and β given by Eq. (10) we obtain:

$$\varepsilon_{ss} = \hat{b} - \beta = \frac{h(p_s - 2\tilde{p})}{\left(\sqrt{\tilde{p}} + \sqrt{p_s - \tilde{p}}\right)^2} + \frac{A\hat{a}}{C\left(\sqrt{\tilde{p}} + \sqrt{p_s - \tilde{p}}\right)} \tag{25}$$

where $\tilde{p}$ is given by Eq. (9).

The results presented in Table 1 show that the steady-state error increases with the increase of F_a. The evolution in time of $\xi = \tilde{y} - y$ was simulated for different values of the underlap h of the spool valve. The results of numerical simulations (using MatLab with Simulink) for system of Eqs. (15) are given in Figs. 2, Fig. 3. With the underlap h smaller then the value of bifurcation h_0, the error remains within an interval determined by Δ.

Table 1

F_a[N]	300	1300	2000	3000
h_0[m]	$8.32 \cdot 10^{-5}$	$8.297 \cdot 10^{-5}$	$8.267 \cdot 10^{-5}$	$8.206 \cdot 10^{-5}$
$\hat{b}-\beta$[m]	$5.09693 \cdot 10^{-6}$	$6.57955 \cdot 10^{-6}$	$7.61041 \cdot 10^{-6}$	$9.06648 \cdot 10^{-6}$
Δ[m]	$4 \cdot 10^{-5}$	$3.69 \cdot 10^{-5}$	$3.348 \cdot 10^{-5}$	$2.685 \cdot 10^{-5}$
T[s]	0.004	0.004	0.004	0.004
γ	$-2.60451 \cdot 10^{8}$	$-2.59828 \cdot 10^{8}$	$-3.11115 \cdot 10^{8}$	$-3.71203 \cdot 10^{8}$

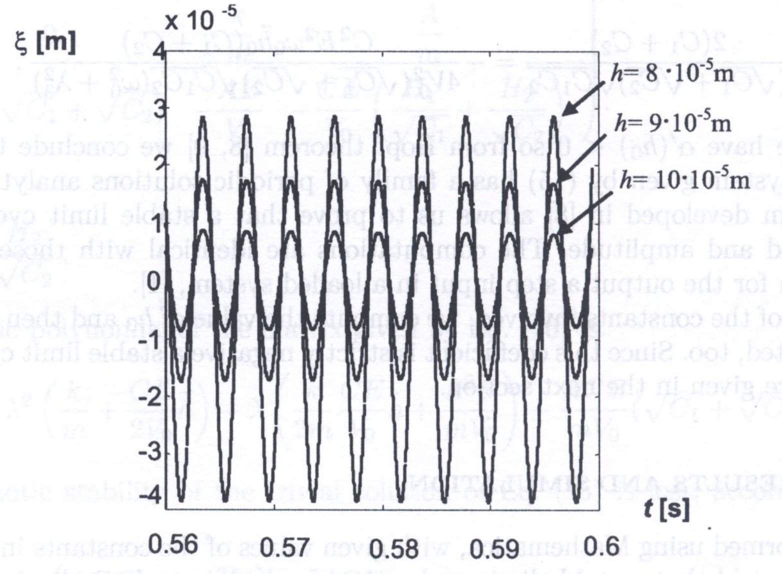

Fig. 2. Plot of ξ for different values of h.
The initial conditions are: $\xi(0) = 1 \cdot 10^{-3}$ m, $\eta(0) = 1 \cdot 10^5$ Pa, $\zeta(0) = 0.02$ m/s.
The tolerance of the integration method is $1 \cdot 10^{-8}$

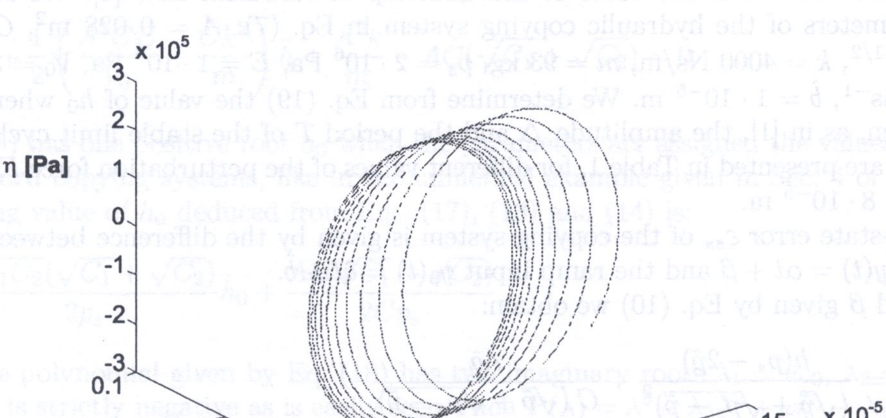

Fig. 3. Plot of ξ, ζ and η for $h = 8 \cdot 10^{-5}$ m.
The initial conditions are: $\xi(0) = 1 \cdot 10^{-3}$ m, $\eta(0) = 1 \cdot 10^5$ Pa, $\zeta(0) = 0.02$ m/s.
Time of simulation $t = 0.08$ sec

5. Conclusions

We prove that for a ramp input, the nonlinear system of ODE modelling the servomechanisms used in copying systems has a solution of the same form: *the steady-state solution.* Even when the steady-state solution is not asymptotically stable, the presence of a stable limit cycle ensures the steady-state error will not increase. The results of simulations indicate an increase of the steady-state error with perturbation force. Together with the analytical study, the simulations give some evidence on the initial conditions that will ensure a correct dynamic system response of the system. Practically when manufacturing with a given hydraulic copying system the choose of good initial conditions means a correct choose of the parameters in the cutting process.

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