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Axiomatic approach to the multicriteria structural optimization

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The paper presents the axiomatic approach for solving the multicriteria optimization of thin-walled structure such as vertical cylindrical reservoirs subject to pitting corrosion. The probabilities of derivation of the compromise optimal project based on MaxMin principle are investigated. The analytic dependencies for estimation of the partial criteria weighting coefficients are obtained. The project consists of the optimal thickness of reservoir shell along its height.

1. Introduction

The multicriterial problems appear in the majority of real design problems. It contains a compromise – the complex respect of uncountable factors and properties of optimal mechanical systems. The reason of compromise-optimal design use is closely connected with obtaining and utilisation of some additional information from the initial formulation point of view. This information particularly deals with the properties of many criteria and models of compromises, preferences and relative importance of these criteria [1, 3, 4]. For the approach to solution of multicriterial problems of structural design, proposed in this paper, such an additional information is uniqueness and axiomatic properties (optimality by Pareto and symmetry) of the solutions, obtained on a base of MaxMin [3, 4]. Using these properties it is possible to get the estimation of the weighting coefficients of particular criteria (with information about analogies and prototype of the structure [5]), and also to define the effective multistore optimization procedures. In the paper the possibilities of multicriterial optimization methods and the approach based on utility point of view with an example of optimal design of cylindrical reservoirs with respect to corrosion of external layer are compared.

2. Formulation of the problem

The vector of partial criteria of reservoir optimality consists of $F(X) = \{B(X) = f_1(X),\ H_I(X) = f_2(X),\ L(X) = f_3(X),\ E(X) = f_4(X)\}$, where $B(X)$ is the average income expected from the operation of its designed period of life T with regards to a possible failure at the moment of time $T_f < T$, $H_I(X)$ is the initial cost, $L(X)$ is the loss due to the structure failure, $E(X)$ is the coefficient of the cost of operation of structure for the period of life T.

To find a vector of the optimal structural parameters X maximising the function $\lambda(X)$

$$\{\lambda(X) = \min\left[\lambda_j(X)\right]\} \to \max, \quad j = 1 \ldots 4 \tag{1}$$

with the constraints in feasible domain:

$$D_X = \left\{ P_t(T) \geq P_*; P_i(\sigma_1(X) \leq [\sigma]) \geq P_*; h_i^1 \leq h_i \leq h_i^2 \right\}, \tag{2}$$

is the objective function, where

$$\lambda_1(X) = (f_1(X) - f_1^1)/(f_1^2 - f_1^1); \quad \lambda_j(X) = (f_j^2 - f_j(X))/(f_j^2 - f_j^1); \qquad j = 2...4;$$

$P_T(T)$, $P_i((\sigma_i)$ denote the reliability functions and P_* is the value of the assumed reliability [1, 2]; $P_T(T)$ denotes the reliability of the structure relatively to the corrosion process of the external layer of the reservoir and $P_i(\sigma_i)$ is the reliability of non-exceeding of the strength constraint, σ_i is the effective stress in the i-th sheet and h_i are the values of thickness along reservoir height, $i = 1,..,n$, T is the designed service life of the structure, $T^1 \leq T \leq T^2$, $[\sigma]$ is the allowable value of stress; f_j^2, f_j^1 are the highest and the lowest estimations of $f_j(X)$ in D_X (2).

The vector of variable parameters is assumed as follows

$$X = \{h_i, i = \overline{1,n};\ n;\ T\}^T = \{x_1, ..., x_n, x_{n+1}, x_{n+2}\}, \tag{3}$$

where n is the number of variable thickness belts into which the reservoir is divided.

The general reservoir dimension are the radius R and height H. It is subject to hydrostatic internal pressure of a liquid corrosive medium of density ρ [1].

The formulation of the problem (1)-(3), using $\lambda_j(X)$ instead of $f_j(X)$, it allows one not to define the values for some complicated coefficients of $f_j(X)$ [1].

3. Calculation aspect of the problem (calculation methods)

Now let one to consider the calculation aspect of the determination of the values included in (1)–(3). The functions of an average income $B(X)$ and the value of the losses reduced to the present time $L(X)$, the initial cost of the structure H_I are introduced in [1]. They are as follows

$$B(X) = \frac{bN}{\ln(1+r)} \left\{ [0.5 - \Phi(a - a_1)] - \exp\left(-b_1 + \frac{a_1^2}{2}\right) [0.5 - \Phi(a - a_1)] \right\}, \tag{4}$$

$$L(X) = L_T N \exp\left(-b_1 + \frac{a_1^2}{2}\right) [0.5 - \Phi(a - a_1)], \tag{5}$$

$$H_1(X) = \left\{ C_{om} = 1.07 \left(\sum_{i=1}^{n} c_{npi} k_{npi} G_i + 1.5G \right) \right\} + C_i + C_p + C_m. \tag{6}$$

In (4) and (5) b denotes the annual income in the absence of failure, N denotes the total number of pitting formations on the surface of the whole reservoir, r is the interest from the capital;

$$\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_0^a e^{-\frac{y^2}{2}} dy$$ is the Laplace integral and

$$a = \frac{h_i - c_i h_i - b_i t}{c_2 h_i}, \qquad b_i = \alpha + \beta \sigma_i \tag{7}$$

$$b_i = \frac{\ln(1+r) h_i (1 - c_1)}{b_i}, \quad a_i = \frac{\ln(1+r) c_2 h_i}{b_i}, \quad \sigma_i = \rho H \left(\frac{i}{n}\right) \left(\frac{R}{h_i}\right), \tag{8}$$

$$h_i = k + \sqrt{k^2 + l}, \quad i = 1...n, \tag{9}$$

where

$$k = \frac{\alpha t}{2(1 - c_1 - c_2 a)}, \quad l = \frac{\beta i H \rho R t}{n(1 - c_1 - c_2 a)},$$

and t presents the period of reservoir operation $T_1 \leq t \leq T_2$; c_1, c_2, α, β ; are the constant coefficients, characterising the corrosion process [1].

In (6) $C_i = 0.62\sqrt{G_n}$ denotes the cost of the structure's manufacture, $C_m = 0.641\sqrt{G_n}$ is the cost of the reservoir assembling, $C_m = 1.1406[1.005(C_{om} + C_i + 2.66)]$ is the structure's cost, C_{npi} is the whole sale price of sheets for i-cost of main structure's part, $G = \sum_i G_i$; $G_i = 2\pi R h_i H\gamma/n$ is the weight of the i-th part.

It is necessary to note that the thickness distribution along the height of the shell $\{h_i\}$ (9) is determined by the expression $P_T = (P_i^*)^N$ in such a way that the reliability of each sector is constant (equally reliable).

The exploitation expense parameter of the structure $E(X)$ is determined as follows:

$$E(X) = E_1 S(t - T_1) + E_2 S^2(t - E_3), \tag{10}$$

where $Ej = \text{const}$, $(j = 1, 2, 3)$ and

$$S(t-T) = \begin{cases} 0, & t < T \\ t - T & t \geq T \end{cases}$$

Let one determine the function of the reliability of the boundary stress (2) as: $Pi(\sigma_i(X) \leq [\sigma]) \geq P_*$. One denotes the initial depression of the values of $[\sigma]$ as a random variable with normal density $N(M_{[\sigma]}, \sigma_{[\sigma]})$

$$U([\sigma]) = \frac{1}{\sqrt{2\pi}\sigma_{[\sigma]}} \exp\left[-\frac{[\sigma] - [\sigma]}{2[\sigma]^2_{[\sigma]}}\right] \tag{11}$$

In this case one can found a fixed value of $[\sigma]_*$ from the equation $P_* = 0.5 + \Phi(Z)$ where $Z = \dfrac{[\sigma]_* - M_{[\sigma]}}{\sigma_{[\sigma]}}$. Thus the condition (2) is satisfied if the deterministic strength constraints $\sigma_i(X) \leq [\sigma]_*$ [2] are satisfied. As a calculated value of the external layer thickness one accepts the larger value of h_i (9) and $\sigma_i(X)/[\sigma]_*$, $i = 1, ..., n$.

Having current calculated values $B(X), L(X), H_I(X), E(X), \{h_i\}$ we draw our attention directly to the multicriteria optimization (MCO) of the reservoir.

4. AXIOMATIC ASPECTS OF THE MULTICRITERIA OPTIMIZATION

Problems of mutually conflicting criteria usually occur in practice but one does not know some formulation of this MCO property. This is the problem investigated in this paper. Let us restrict ourselves to the consideration of the axiomatic approach to the MCO for the explanation of the model of generalized objective function described in (1). The axiomatic approach peculiarity is the following: it is formulated *a priori* as specific properties of compromise – the optimal structures on a base of which the models of compromise – the partial criteria $\{f_j(X)\}$ and methods of the optimal solutions calculation X^* [3, 4] are developed. Giving required properties for compromise criteria means the introduction of the definite classification of MCO problems. Let one further consider the possibility of the axiomatic method for the model MinMax (1), (2)-(3). This function is axiomatically introduced to the MCO of the structure considered in [3, 4]. In paper [4] the class of the MCO problems has been defined for which the compromise-optimal solutions satisfy the following conditions:

- A1) $X^* \in P_x$ – Pareto's optimality [2, 3];
- A2) $\lambda_p(X^*)$, $\forall p, q \in [1...n]$ – the symmetry.

The symmetry condition A2 is basic one when the MCCP problems are being formulated. As the condition A2 is concern it can be interpreted as follows: the relations between all $\{f_j(X)\}$ are such that when the estimation of one (anyone) of the criteria in the compromise is reduced the estimations of the remaining criteria $f_j(X)$ might be improved:

$$(\overline{f}'_p) < f_p, \quad j \neq p, \quad j \in n$$

Here in the basis as a basis of classification the mutual conflict of the partial criteria $\{f_j(X)\}$ is understood. One calls such problems MCO as problems with mutual conflict criteria (MCCP).

Formally, the properties of conflict of some criteria $\{f_p(X), f_q(X)\}$ can be presented as follows. If the coefficients $C_q > 0$ are such that $\overline{D}_x(C_q) \neq 0$, (2)-(3) then

$$\left\{ \begin{array}{c} f_p(X^*) = \max f_p(X) \\ X \in D_X \\ f_q(X) \geq C_q \end{array} \right\} \Rightarrow f_q(X^*) = C_q \tag{12}$$

In (12) $\overline{D_x}(C_q) = D_x \cap D_q$, where $D_q = \{X / f_q(x) \geq C_q\}$.

MCCP means that the property (12) is satisfied for all $p, q \in$ [1..m]. In this case the use of (1) guarantees the obtaining of unique compromise-optimal project, satisfying axiom A1) and A2) [4].

Establishing a new separate class of problems with A2 which have the unique compromise solution of (1) gives one a new additional information about $\{f_j(X)\}_n$ compromise which was not previously used when MCO problem has been formulated.

As it is well known [2], the formulation of MCO problems must satisfy many requirements, among which there are the completeness and the unredundantly set of the optimization criteria $\overline{f_m}(X)$. The property of MCCP can be used for the normalisation of unredandentness of $\overline{f_m}(X)$ in such a way: in MCCP vector of $\overline{f_m}(X)$ is unredundant. Some examples of MCCP are given in [3]. One should note that -1) model of MCCP has a local character: the relations (12) are satisfied in some domain of estimations $\{f_j(X)\}$; -2) problem (1)-(3) does not satisfy the properties (12). A simple variant of step-by-step procedure allowing one to solve more general MCO problems is further considered.

In many cases when objectives $\{f_j(X)\}_m$ like (1) [2–4] are being formed as an additional source of information the relative importance of the partial criteria is used. For the model (1) it means the introduction of the coefficients $\hat{\gamma}_i = \lambda_j(X)\gamma_j \sum_{j=1}^m \gamma_j = 1$; $\gamma_j > 0$ (the partial criteria rate coefficients). The analytic dependencies for such coefficients estimation are obtained from the problem [5], which use an information about analogies and prototypes of the reservoirs constructions types being considered, for the values $\{\gamma_j\}_m$. Taking into account the fact that reservoir prototypes in a compromise domain $f_j : \left[f_j^1, f_j^2\right]$, $j = 1, ..., m$ are MCCP or these projects are obtained using model (1) one can get the analytical dependencies for the estimation of coefficients $\{\gamma_j\}_m$ as follows:

Let l be a number of structural prototypes, I be an information independent from the prototype parameters x_i

$$I = \left\{ Y_j^{(r)},\ j = 1, ..., m,\ \hat{f}_j^{(1)} = \min_{r\, inl} Y_j^{(r)} - \delta_j;\ \hat{f}_j^{(2)} = \min_{r \in l} Y_j^{(r)} + \delta_j \right\},$$

and $\hat{\lambda}_j^{(r)} = \left(Y_j^{(r)} - f_j^{(1)}\right) / \left(f_j^{(2)} - f_j^{(1)}\right)$, δ_i is a constant. Then the coefficients of importance γ_j are defined by relations given in [5].

$$\gamma'_j = \prod_{r=1}^{l} \gamma_j^{(r)} \Big/ \left[\prod_{k=1}^{m} \left(\prod_{r=1}^{l} \lambda_k^{(r)} \right) \right];\ \gamma_j = \gamma'_j \Big/ \sum_{k=1}^{m} \gamma'_j. \tag{13}$$

Presented formula (13) describes the expert estimation of the weight, expressed through the prototypes of the structure.

5. STEP BY STEP OPTIMIZATION PROCEDURE

Now let us consider the procedure of the solution (1)–(3), keeping in mind that in the domain of compromise criteria $\{f_j(X^*)\}$ the values $\overline{\Delta} = \left\{\Delta_j = (f_j^2 - f_j^1)^{-1}\right\}_m$ are defined by the value tradeoffs [2] – the compensation measure unit $f_p(X^*)$ on unit $f_q(X^*), p, q \in [1..n]$. In this case if symmetry conditions A2 are not satisfied it means that the assumption about values of coefficients $\overline{\Delta}$ are also not satisfied. Let one to define assumption about values of coefficients $\overline{\Delta}$ nonsatisfaction degree obtaining new values $\tilde{f}_q^1$:

$$\tilde{f}_q^1 = (1 - \lambda^0)^{-1}\Big[f_q^*(X^*) - \lambda^0 f_q^2\Big], \tag{14}$$

when for X^* project the equality A2: $\tilde{f}_q(X^*) = \lambda_j^0(X^*)$, λ_0 is the solution of (1). If $\tilde{\Delta}_q$ solved using (14) satisfy the relations $|\tilde{\Delta}_q - \Delta_q| \leq \varepsilon_q$ then the assumption about value tradeoffs $\overline{\Delta}$ are satisfied and X^* is a solution of (1)–(3).

Let us define the following step by step optimization procedure for MCO problem (1)–(3). Let S be a current procedure level. Then the procedure itself can be defined as follows:

P0. S$= 0, D_{(s)} = D_{(X)}$.

P1. Evaluate parameters $D_{(S)} : \left\{f_j^1, f_j^2, \Delta_j\right\}_m$.

P2. Evaluate (1) $\lambda_{(S)}^0, X_{(S)}^0; \left\{\lambda_{j(S)}^0, f_{j(S)}\right\}_m$.

P3. If $|\lambda_{q(S)}^0 - \lambda_{j(S)}^0| \leq \varepsilon_x, \quad p, q \in [1, ..., m]$, then P6.

P4. Evaluate (14): $\tilde{f}_{q(S)}^1 = C_{q(S)}; \tilde{\Delta}_{q(S)}; q \in [1, ..., m]$.

P5. Set constraints $f_{q(S+1)}^1 > C_{q(S)}, S \leftarrow (S+1)$; to P1.

P6. $X^0 = X_{(S)}^*$; $\overline{f}^0 = \overline{f}(X^0)$. End of procedure.

Let us consider the calculation procedures P1, P2. In general the task (1)–(3) is discrete problem of nonlinear programming. Because of assumption of equal reliability condition for reservoir external belts [1] $P_T(t) = P_*^*$ and technique of respect of strength constraint presented in Sec. 3, the task order is reduced to two: (n, T), where n is the number of reservoir belts. For double parametric problem solution the method of full enumeration has been used.

6. NUMERICAL EXAMPLE

Let us consider the solution of the optimization problem (1)–(3) with following reservoir initial data: R = 2 m, H = 4 m, $\rho = 0.8 \times 10^3$ kg/m^3; $\alpha = 0.6$ cm/year, $\beta_1 = 0.085$ cm/Ton·year, $C_1 = 0.05$, $C_2 = 0.1$; $N = 48$, $P^* = 0.99$; $b = 10^4$, $L_T = 10^4$ USD, $r = 10\%$; $m_{[\sigma]} = 170$ MPa, $\sigma_{[\sigma]} = 20$ MPa; the ranges of changes in variable parameters are $1 \leq n \leq 10$, 10 years $\leq T \leq 30$ years.

The results of the multicriteria optimization of the reservoir shell are given in the Table 1.

In the table $f_{(S)}$, $\lambda_{(S)}$ are the values of the objectives on a procedure step S; $\tilde{f}_{q(S)}^1$ are e-valuated according to (14) boundary values of the partial criteria where in the nominator the values (14) and in the denominator the accepted at P5 values $f_{q(S)}^1$ are given. The end condition as A2 and equality $\lambda_{j(S)}^0(X)$ was already active on the second step of the procedure P0-P6. The calculations showed considerable dependence of the optimization results on H_I characteristics. For example, if one chooses instead of H_I its component C_m – the cost of the materials, then the optimal reservoir project has $n = 10$ sections with following thickness h_i distribution: $h = (1.63, 2.11, 2.48, 2.79, 3.07, 3.32, 3.55, 3.76, 3.96, 4.16)$ (cm) when the cost was reduced about 25%. It should be noted that the difference estimation C_m and the result presented in [1] which characterizes in this case utility function change (20-24%) have very close values. In the numerical investigations the influence of the external layer corrosion processes on the optimal reservoir projects appeared to be active (most considerable) because of low level of $\sigma_i(X)$ stresses.

Table 1

$f_{(S)}$	$B(X)$	$L(X)$	$H_1(X)$	$E(X)$	(T^*, n^*)	h_1^*	$\sigma_1(X)$
$f_{(1)}^{(2)}$	1977.0	78.2	12.9	0.5			
$f_{(1)}^{(1)}$	1277.0	11.6	38.9	41.5			
$f_{(1)}^{*}$	1783.0	30.6	17.0	11	(20;1)	4.16 cm	≈ 0.2 MPa
$\lambda_{(1)}^{0}$	0.723	0.723	0.842	0.744			
$\tilde{f}_{q(1)}^{1}$	1277.0	78.2	27.7/24.4	38.3/27.7			
$f_{(2)}^{(2)}$	1977.0	78.2	24.4	27.0			
$f_{(2)}^{*}$	1750.0	33.0	16.6	9.0	(19;1)	4.04 cm	≈ 0.2 MPa
$\lambda_{(2)}^{0}$	0.676	0.676	0.692	0.675			
$\tilde{f}_{q(2)}^{1}$	1277.0	78.2	24.4	26.8			

7. Conclusions

The results presented in the Table 1 gives one the opportunity to characterise considerable aspects of multicriteria approach to the problem of reservoir optimal design, including the technique (1). Normalisation procedure, switch from the parameters $f_j(X)$ to $\lambda_j(X)$ allows one to estimate a relative change of the partial criteria $f_j(X)$ without giving hardly defining coefficients pre-set values [1]. Here the potential features of the optimization show up, structural variable parameters domain becomes clearer. Table 1 shows that in this particular case the change of all $f_j(X)$ is considerable, the compromise-optimal solution is *a priori* not defined.

The axioms system A1-A2 put in a base of the model (1), the opportunity of any compromise-optimal project $X^* \in P_x$ solution with respect to the errors of initial $\overline{\Delta}$ and the results X^* as boundary correction (14), simple analytical form of introduction into (1) the information about the coefficients of the partial criteria importance (13) make MaxMin optimality principle be used in the practical MCO problems. In some cases, set with axiomatic properties (12), the model (1) is the only one to be used.

We understand that after all in practise one uses such models of compromise and MCO procedures for which the results of their application are most understood for designer and predictable.

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