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Stress intensity factors computations using the singularity subtraction technique incorporated with the Tau Method

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In this paper we discuss the use of the singularity subtraction technique incorporated with the Tau Method for the numerical solution of singular partial differential equations which are relevant to the linear elastic fracture mechanics. To treat the singularity, we apply the singularity subtraction technique to the singular boundary value problems. The problems arising in this application are not in the standard form required by the Tau software. By introducing the pseudo-differential equations, $\lambda_k' = 0$, $k = 1(1)m$, to determine the stress intensity and higher order factors λ_k results in the standard boundary value problems. We consider two model crack problems including Motz' anti-plane crack problem and a plane strain problem defined by the biharmonic equation. We obtain results of considerable accuracy which compare favorably with those published in the recent literature.

1. INTRODUCTION

In 1981, Ortiz and Samara [13] proposed an operational technique for the numerical solution of ordinary differential equation with some supplementary (initial, boundary or mixed) conditions based on the Tau Method [5]. This technique has been extended by Ortiz and Samara [14] for the approximate solution of partial differential equations with bivariate polynomial or rational polynomial coefficients. The application of this technique to nonlinear partial differential equations and partial differential eigenvalue problems has been discussed by Ortiz and Pun [12], Liu, Ortiz and Pun [10] and Liu and Ortiz [9]. The basic idea of their method is the determination of the coefficients of the Tau approximants which represents the polynomial approximation to the solution of the problem.

It is well known that elliptic boundary value problems with boundary singularities are nortoriously difficult to treat numerically, as standard numerical methods lose accuracy in the vicinity of the singular point. In order to solve such problems precisely, numerical analysts naturally seek a combination of different methods. Some efficient methods for singular boundary value problems are the conformal transformation methods of Whiteman and Papamichael [21] and Rosser and Papamichael [16], the combined method of singularity subtraction and boundary integral equation methods of Symm [19] and Xanthis et al. [23], the conforming local mesh refinement finite element technique of Schiff et al. [17], the penalty-combined approaches to the Ritz-Galerkin and finite element methods of Li [6], and the combined method of singularity subtraction and finite difference methods of Liu et al. [7]. In this paper we discuss the use of the singularity subtraction technique incorporated with the operational approach to the Tau Method for the numerical solution of singular boundary value problems which are relevant to the linear elastic fracture mechanics. Our technique for determining the stress intensity and higher order factors is described and successfully applied to two model crack problems (mode I and III). In this method the stress intensity and higher order factors appear as some of the unknown parameters in the resulting system of linear algebraic equations, which is a most welcome feature since it avoids the uncertainties associated with the commonly employed J-integral [15] and displacement- or stress-extrapolation techniques [3]. We show that our numerical results compare favourably with the most accurate ones reported in the recent literature.

2. Lánczos' Tau Method

The Tau Method was first conceived by C. Lánczos in 1938 to construct polynomial approximations to the solution $y(x)$ of a given problem involving an ordinary differential equation of the form

$$Dy(x) = f(x), \quad a \le x \le b \tag{1}$$

with

$$(l_r, y) = \sigma_r, \quad r = 1(1)\nu, \tag{2}$$

where

$$D := \sum_{i=0}^{\nu} \sum_{j=0}^{\beta_i} p_{ij} x^j \frac{d^i}{dx^i} \in \mathcal{D},$$

the class of linear ordinary differential operators of order ν with polynomial coefficients. Let $l_r, r = 1(1)\nu$ be point evaluation functionals acting on ν-differentiable functions defined on an interval $[a, b]$, and let $(l_r, y) = \sigma_r, r = 1(1)\nu$, stand for the supplementary (initial, boundary or mixed) conditions of the given problem.

The basic idea of the Tau Method for problem (1)–(2) is essentially as follows. Assume that the approximate solution of problem (1)–(2) to be an n-th degree polynomial, $y_n(x) := \sum_{i=0}^{n} a_i x^i$, then substitute it into the differential equation (1) and the conditions (2) giving an overdetermined system of linear algebraic equations for the $n+1$ unknown coefficients a_i. A perturbation term $H_n(x) \in \mathcal{P}_{\beta(n)}$ is added to the right-hand-side of equation (1). The integer $\beta(n) \le n + h$, where h is the height of D and is defined by $h := \max\limits_{0 \le i \le \nu}(\beta_i - i)$.

With problem (1)–(2) we associate the perturbed problem (the Tau problem)

$$Dy_n(x) = f(x) + H_n(x), \quad a \le x \le b \tag{3}$$

with

$$(l_r, y_n) = \sigma_r, \quad r = 1(1)\nu, \tag{4}$$

where $H_n(x)$ is usually chosen to be a linear combination of the shifted Chebyshev or Legendre polynomials with $\nu + s$ free parameters τ_i which are to be determined in such a way that Tau approximant $y_n(x)$, is the exact polynomial solution of problem (3)–(4).

3. Ortiz and Samara's operational approach to the Tau Method

This approach is based on the systematic use of two simple and sparse matrices

$$\boldsymbol{\mu} := ((\mu_{ij})) \quad \text{and} \quad \boldsymbol{\eta} := ((\eta_{ij})) \quad \text{for} \quad i, j = 1, 2, 3, \ldots$$

where $\mu_{ij} = \delta_{i+1,j}$, $\eta_{ij} = j\delta_{i,j+1}$ with δ_{ij} as the Krönecker delta. The key point is that it makes it possible to transform problem (1)–(2) into an algebraic one. For simplicity, we let

$$a_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \tilde{\boldsymbol{a}}_n \cdot \boldsymbol{x},$$

with $\tilde{\boldsymbol{a}}_n = (a_0, a_1, a_2, \ldots, a_n, 0, 0, \ldots)$ and $\boldsymbol{x} = (1, x, x^2, \ldots, x^n, \ldots)^T$. Then

$$x a_n(x) = a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_n x^{n+1} = \tilde{\boldsymbol{a}}_n \boldsymbol{\mu} \boldsymbol{x}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}a_n(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} = \tilde{\boldsymbol{a}}_n\boldsymbol{\eta}\boldsymbol{x}.$$

The product $\boldsymbol{\eta}^i\boldsymbol{\mu}^j$ will transform the coefficients of $a_n(x)$ in the same way as operating $x^j\frac{\mathrm{d}^i}{\mathrm{d}x^i}$ on it. Thus, any linear ordinary differential operator $D \in \mathcal{D}$, as before defined by (1), can be expressed as a linear combination of $\boldsymbol{\eta}^i\boldsymbol{\mu}^j$ with coefficients p_{ij}. We recall the following result from Ortiz and Samara [13]:

Theorem 1. Let $a_n(x) = \tilde{\boldsymbol{a}}_n \cdot \boldsymbol{x} \in C^{(\nu)}[a,b]$ and D as before defined by (1). Then

$$Da_n(x) = \left(\sum_{i=0}^{\nu}\sum_{j=0}^{\beta_i} p_{ij}x^j\frac{\mathrm{d}^i}{\mathrm{d}x^i}\right)a_n(x) = \tilde{\boldsymbol{a}}_n\boldsymbol{\Pi}\boldsymbol{x}, \tag{5}$$

where

$$\boldsymbol{\Pi} := \sum_{i=0}^{\nu}\sum_{j=0}^{\beta_i} p_{ij}\boldsymbol{\eta}^i\boldsymbol{\mu}^j$$

is a matrix uniquely associated with D in the x-basis.

Liu and Ortiz [8] remarked that the matrix $\boldsymbol{\gamma}^{(ij)}$ corresponding to $p_{ij}x^j\frac{\mathrm{d}^i}{\mathrm{d}x^i}$ is one with a simple structure:

$$\boldsymbol{\gamma}^{(ij)} := \begin{pmatrix} \boldsymbol{O}^{(ij)} & \vdots & \boldsymbol{O} \\ \cdots & \cdots & \cdots \\ \boldsymbol{O} & \vdots & \boldsymbol{K}^{(ij)} \end{pmatrix}, \tag{6}$$

where $\boldsymbol{O}^{(ij)}$ is an $i \times j$ zero matrix and $\boldsymbol{K}^{(ij)}$ is a diagonal matrix with diagonal elements

$$K_{kk}^{(ij)} = \frac{(i+k-1)!}{(k-1)!}p_{ij}, \quad k \in \mathbb{N} = \{1,2,3,\ldots\}.$$

Therefore, $\boldsymbol{\Pi}$ is constructed in the computer as a result of the superposition of a suitable number of matrices of type (6). Suppose we choose $\boldsymbol{v} = (v_0(x), v_1(x), v_2(x), \ldots)^T = \boldsymbol{V}\boldsymbol{x}$ as the polynomial basis (Chebyshev or Legendre polynomial basis), where $\boldsymbol{V}$ is the matrix of coefficients of polynomial basis.

Let

$$y(x) = \alpha_0v_0(x) + \alpha_1v_1(x) + \alpha_2v_2(x) + \cdots = \tilde{\boldsymbol{\alpha}} \cdot \boldsymbol{v}$$

and let

$$f(x) = f_0v_0(x) + f_1v_1(x) + f_2v_2(x) + \cdots + f_mv_m(x) = \tilde{\boldsymbol{f}} \cdot \boldsymbol{v},$$

where $\tilde{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ and $\tilde{\boldsymbol{f}} = (f_0, f_1, f_2, \ldots, f_m, 0, 0, \ldots)$. Then from (1) and (5), we have

$$\tilde{\boldsymbol{\alpha}}\hat{\boldsymbol{\Pi}} = \tilde{\boldsymbol{f}}, \tag{7}$$

where $\hat{\boldsymbol{\Pi}} = \boldsymbol{V}\boldsymbol{\Pi}\boldsymbol{V}^{-1}$. Again from (2), we have

$$\tilde{\boldsymbol{\alpha}}\boldsymbol{B} = \tilde{\boldsymbol{\sigma}}, \tag{8}$$

where $\tilde{\boldsymbol{\sigma}} = (\sigma_1, \sigma_2, \ldots, \sigma_\nu)$, and the ij-element of matrix $\boldsymbol{B} = ((b_{ij}))$ is given by

$$b_{ij} = (l_j, v_{i-1}), \quad \text{for} \quad i = 1,2,3,\ldots; \quad j = 1(1)\nu.$$

Combining the two systems (7) and (8), we have

$$\tilde{\boldsymbol{\alpha}}\boldsymbol{G} = \tilde{\boldsymbol{s}}, \tag{9}$$

where $\boldsymbol{G} = (\boldsymbol{B} \mid \hat{\boldsymbol{\Pi}})$ and $\tilde{\boldsymbol{s}} = (\tilde{\boldsymbol{\sigma}} \mid \tilde{\boldsymbol{f}})$.

We assume that the Tau degree $n \geq \nu + h$, and write $\boldsymbol{G}_{nn}$ for the restriction of $\boldsymbol{G}$ to its first $n+1$ rows and $n+1$ columns, and $\tilde{\boldsymbol{\alpha}}_n$ for the restriction of $\tilde{\boldsymbol{\alpha}}$ to its first $n+1$ components. We call $y_n(x) = \tilde{\boldsymbol{\alpha}}_n \cdot \boldsymbol{v}$, the Tau approximant of degree n of problem (1)–(2) in the sense of the operational approach to the Tau Method if $\tilde{\boldsymbol{\alpha}}_n$ is the solution of the system of linear equations

$$\tilde{\boldsymbol{\alpha}}_n \boldsymbol{G}_{nn} = \tilde{\boldsymbol{s}}_n \,.$$

The same technique has been extended by Ortiz and Samara [14] to linear partial differential equations defined on a rectangular domain. We give now a brief account of the two dimensional formulation of the Tau Method, more details can be found in [14].

The effect of

$$x^m y^n \frac{\partial^{r+s}}{\partial x^r \partial y^s}$$

on a bivariate polynomial

$$a(x,y) = \sum_{i=1} \sum_{j=1} a_{ij} x^{i-1} y^{j-1} = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{y}, \tag{10}$$

where

$$\boldsymbol{A} = ((a_{ij})), \quad \boldsymbol{x} = (1, x, x^2, \ldots)^T, \quad \boldsymbol{y} = (1, y, y^2, \ldots)^T$$

is the same as that of applying to the coefficient matrix $\boldsymbol{A}$ the transformation of the form

$$(\boldsymbol{\eta}^r \boldsymbol{\mu}^m)^T \boldsymbol{A} (\boldsymbol{\eta}^s \boldsymbol{\mu}^n).$$

Let $\mathcal{L}$ be the class of linear partial differential operators $\boldsymbol{D}$ with bivariate polynomial coefficients

$$\boldsymbol{D} := \sum_r^{\nu_x} \sum_s^{\nu_y} \sum_i \sum_j p_{ijrs} x^i y^j \frac{\partial^{r+s}}{\partial x^r \partial y^s},$$

where $\sum_k$ stands for a finite sum. Then, the effect of an operator $\boldsymbol{D} \in \mathcal{L}$ on the bivariate polynomial $a(x,y)$ is given by

$$\boldsymbol{D}a(x,y) = \boldsymbol{x}^T \left(\sum_r^{\nu_x} \sum_s^{\nu_y} \sum_i \sum_j p_{ijrs} (\boldsymbol{\eta}^r \boldsymbol{\mu}^i)^T \boldsymbol{A} (\boldsymbol{\eta}^s \boldsymbol{\mu}^j) \right) \boldsymbol{y} \,. \tag{11}$$

Let $\boldsymbol{u} = \boldsymbol{U}\boldsymbol{x}$, $\boldsymbol{v} = \boldsymbol{V}\boldsymbol{y}$ be two polynomial basis defined by lower triangular matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ respectively. Then

$$a(x,y) = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{y} = \boldsymbol{u}^T \boldsymbol{B} \boldsymbol{v} = b(u,v),$$

where $\boldsymbol{B} = (\boldsymbol{U}^T)^{-1} \boldsymbol{A} \boldsymbol{V}^{-1}$ is the expansion of (10) in the basis $\{\boldsymbol{u}, \boldsymbol{v}\}$. From (11), we have

$$\boldsymbol{D}a(x,y) = \boldsymbol{D}b(u,v) = \boldsymbol{u}^T \boldsymbol{d}(\boldsymbol{B}) \boldsymbol{v},$$

where

$$\boldsymbol{d}(\boldsymbol{B}) = \sum_{ijrs} p_{ijrs} \boldsymbol{\alpha}_{ir}{}^T \boldsymbol{B} \boldsymbol{\beta}_{js} \,,$$

$$\boldsymbol{\alpha}_{ir} = \boldsymbol{U} \boldsymbol{\eta}^r \boldsymbol{\mu}^i \boldsymbol{U}^{-1} \quad \text{and} \quad \boldsymbol{\beta}_{js} = \boldsymbol{V} \boldsymbol{\eta}^s \boldsymbol{\mu}^j \boldsymbol{V}^{-1}.$$

The supplementary conditions can be treated in the same way and if we assume that the given differential equation is defined over a rectangular domain with sides parallel to the coordinate axes, then the conditions will depend on only one variable:

$$\boldsymbol{D}_{xj}a(x,y) = \left[\sum_{irs} q_{irs}x^i \frac{\partial^{r+s}}{\partial x^r \partial y^s}\right] a(x,y)\,|_{y=y_j} = f_j(x)$$

$$\boldsymbol{D}_{iy}a(x,y) = \left[\sum_{jsr} h_{jsr}y^j \frac{\partial^{s+r}}{\partial y^s \partial x^r}\right] a(x,y)\,|_{x=x_i} = g_i(x)$$

and the effect of these supplementary conditions on $a(x,y)$ is given by

$$\boldsymbol{D}_{xj}a(x,y) = \boldsymbol{u}^T\boldsymbol{R}_j(\boldsymbol{B})$$

and

$$\boldsymbol{D}_{iy}a(x,y) = \boldsymbol{L}_i(\boldsymbol{B})\boldsymbol{v}$$

where

$$\boldsymbol{R}_j(\boldsymbol{B}) = \sum_{irs} q_{irs}(\boldsymbol{U}\boldsymbol{\eta}^r\boldsymbol{\mu}^i\boldsymbol{U}^{-1})^T\boldsymbol{B}\boldsymbol{V}\boldsymbol{\eta}^s\boldsymbol{y}_j$$

$$\boldsymbol{L}_i(\boldsymbol{B}) = \sum_{jrs} h_{jsr}(\boldsymbol{U}\boldsymbol{\eta}^r\boldsymbol{x}_i)^T\boldsymbol{B}\boldsymbol{V}\boldsymbol{\eta}^s\boldsymbol{\mu}^j\boldsymbol{V}^{-1}.$$

4. THE SINGULARITY SUBTRACTION TECHNIQUE FOR THE MODEL PROBLEMS

4.1. The Motz problem

We first consider the Motz problem requires the solution of the Laplace equation

$$\nabla^2\phi = 0 \tag{12}$$

in a square region BCEF containing a slit (crack) OA as shown in Fig. 1, in which for convenience in subsequent comparisons of numerical results we take the dimensions of BC=CE=EF=14 units, OA=OG=AB=FG=7 units and the boundary conditions are $\phi = 1000$ on AB, $\phi = 0$ on FG, and

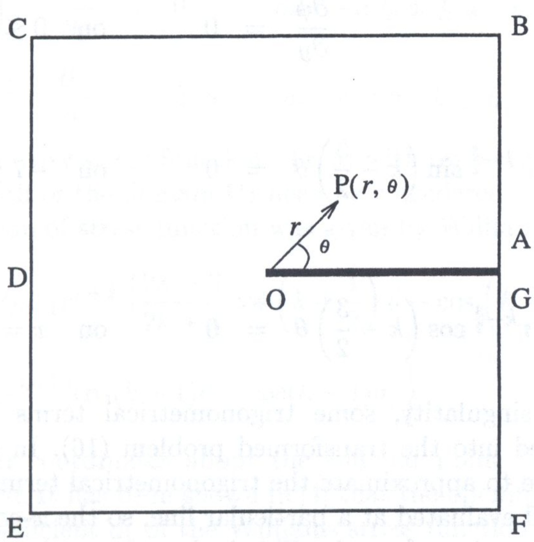

Fig. 1. Square region BCEF with a slit OA

$\frac{\partial \phi}{\partial n} = 0$ on the rest of the boundary including the slit, where $\frac{\partial}{\partial n}$ is the derivative in the direction of the outward normal to the boundary.

By taking account of the anti-symmetry of the function $\phi - 500$ about the line AOD, the Motz problem is reduced to the problem:

$$\begin{aligned}
\nabla^2 \phi &= 0 && \text{in} \quad \Omega_1 := (-7 \le x \le 7, \;\; 0 \le y \le 7), \\
\phi &= 1000 && \text{on} \quad x = 7, \;\; 0 \le y \le 7, \\
\phi &= 500 && \text{on} \quad -7 \le x \le 0, \;\; y = 0, \\
\frac{\partial \phi}{\partial n} &= 0 && \text{on other boundary segments of } \partial\Omega_1.
\end{aligned} \tag{13}$$

The analytic series solution is well known (see Motz [11]) to be

$$\phi(r,\theta) = 500 + \sum_{k=1}^{\infty} \lambda_k r^{k-\frac{1}{2}} \cos\left(k - \frac{1}{2}\right)\theta, \qquad (r,\theta) \in \Omega_1 + \partial\Omega_1 \tag{14}$$

where (r,θ) are local polar coordinates centred at O, and $\lambda_1, \lambda_2, \lambda_3, \ldots$ are unknown parameters to be determined. We put

$$\phi^* = \sum_{k=1}^{m} \lambda_k r^{k-\frac{1}{2}} \cos(k - \frac{1}{2})\theta,$$

and define

$$\overline{\phi} := \phi - \phi^* \tag{15}$$

as an m times continuously differentiable function on the domain Ω_1. The function $\overline{\phi}$ satisfies the Laplace equation since both the solution ϕ and the functions $r^{k-\frac{1}{2}} \cos(k - \frac{1}{2})\theta$ are harmonic. Then, the problem (13) becomes

$$\begin{aligned}
\nabla^2 \overline{\phi} &= 0 && \text{in} \quad \Omega_1, \\
\overline{\phi} + \sum_{k=1}^{m} \lambda_k r^{k-\frac{1}{2}} \cos\left(k - \frac{1}{2}\right)\theta &= 1000 && \text{on} \quad x = 7, \;\; 0 \le y \le 7, \\
\overline{\phi} &= 500 && \text{on} \quad -7 \le x \le 0, \;\; y = 0, \\
\frac{\partial \overline{\phi}}{\partial y} &= 0 && \text{on} \quad 0 \le x \le 7, \; y = 0, \\
\frac{\partial \overline{\phi}}{\partial y} + \frac{1}{2}\lambda_1 r^{-\frac{1}{2}} \sin\frac{\theta}{2} & \\
- \sum_{k=2}^{m} \lambda_k \left(k - \frac{1}{2}\right) r^{k-\frac{3}{2}} \sin\left(k - \frac{3}{2}\right)\theta &= 0 && \text{on} \quad -7 \le x \le 7, \;\; y = 7, \\
\frac{\partial \overline{\phi}}{\partial x} + \frac{1}{2}\lambda_1 r^{-\frac{1}{2}} \cos\frac{\theta}{2} & \\
- \sum_{k=2}^{m} \lambda_k \left(k - \frac{1}{2}\right) r^{k-\frac{3}{2}} \cos\left(k - \frac{3}{2}\right)\theta &= 0 && \text{on} \quad x = -7, \;\; 0 \le y \le 7.
\end{aligned} \tag{16}$$

After subtracting off the singularity, some trigonometrical terms with unknown coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$ are introduced into the transformed problem (16). In order to solve the problem by the Tau Method, we have to approximate the trigonometrical terms by polynomials. Since the trigonometrical terms are all evaluated at a particular line, so the x-coordinates are fixed and the terms can be written as a function of y only. This is done by setting up an initial value problem (IVP) with the trigonometrical term as the analytical solution, and then solve the IVP with the Tau

Method to obtain a polynomial approximation to the solution of the IVP. From the above transformation of the Motz problem, we have led to the following non-standard problem, in which the boundary conditions contain the unknown parameters $\lambda_k, k = 1(1)m$. By introducing the pseudo-differential equations,

$$\lambda_k' = 0, \quad k = 1(1)m, \tag{17}$$

to let the number of differential equations match with the number of unknown functions. Ascher and Russell [1] have discussed and successfully applied this technique to the numerical solution of ordinary differential eigenvalue problems. By examining the analytical solution

$$\overline{\phi}(r,\theta) = 500 + \sum_{k=m+1}^{\infty} \lambda_k r^{k-\frac{1}{2}} \cos\left(k - \frac{1}{2}\right)\theta$$

in the neighbourhood of the singular point O, we introduce m conditions to compensate the system for which the differential equations (17) being supplemented. The unknown parameters λ_k will be determined automatically by solving the well-posed problem with the Tau Method described in Section 3.

4.2. Biharmonic-type problem

The second example considered is a model problem of a rectangular plate in a plane strain situation. The dimensions of the plate are $2b$ units long and $2a$ units width. The plate contains a crack of length a units which is subjected to an uniform normal stress σ, perpendicular to the crack direction, acts over the two edges of the plate, as shown in Fig. 2.

In terms of the Airy stress function $\psi(x,y)$, the model problem is described as (see Williams [22] and Gross et al. [4])

$$\begin{aligned}
\nabla^4\psi &= 0 && \text{in } \Omega_2 := (-a \le x \le a,\ -b \le y \le b),\\
\psi = 0,\quad \frac{\partial\psi}{\partial y} &= 0 && \text{on } -a \le x \le 0,\ y = 0,\\
\psi = 0,\quad \frac{\partial\psi}{\partial x} &= 0 && \text{on } x = -a,\ 0 \le y \le b,\\
\psi = \sigma\left(\frac{x^2}{2} + ax + \frac{a^2}{2}\right),\quad \frac{\partial\psi}{\partial y} &= 0 && \text{on } -a \le x \le a,\ y = b,\\
\psi = 2\sigma a^2,\quad \frac{\partial\psi}{\partial x} &= 2\sigma a && \text{on } x = a,\ 0 \le y \le b.
\end{aligned} \tag{18}$$

Because of symmetry of the Airy stress function $\psi(x,y)$ with respect to the crack line OA as shown in Fig. 2, only the upper half of the domain Ω_2 need be considered.

The infinite series solution of stress function was given by Williams [22] as follows:

$$\begin{aligned}
\psi(r,\theta) = \sum_{k=1}^{\infty}\Bigg\{&(-1)^{k-1} d_{2k-1} r^{k+\frac{1}{2}}\left[\frac{2k-3}{2k+1}\cos\left(k+\frac{1}{2}\right)\theta - \cos\left(k-\frac{3}{2}\right)\theta\right]\\
&+ (-1)^k d_{2k} r^{k+1}\left[\cos(k+1)\theta - \cos(k-1)\theta\right]\Bigg\},
\end{aligned} \tag{19}$$

where (r,θ) are local polar coordinates about the singular point O and the $\{d_i\}$ are unknown coefficients to be determined. It has been shown in [4] that the opening mode stress intensity factor K_I is related to the first coefficient d_1 of the Williams stress function (19) by the formula

$$K_I = -(2\pi)^{\frac{1}{2}} d_1. \tag{20}$$

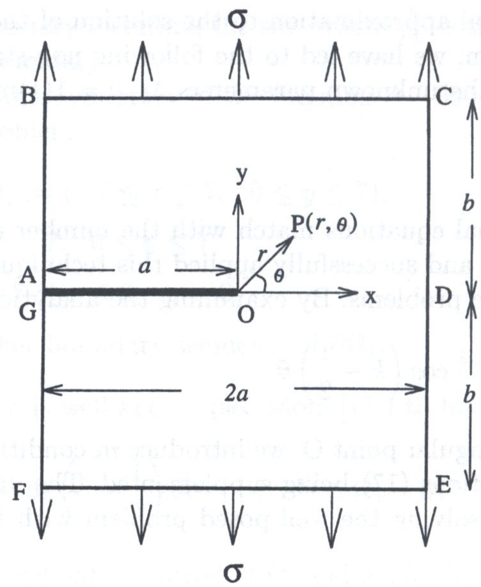

Fig. 2

Again, we put

$$\psi^* = \sum_{k=1}^{m}\left\{(-1)^{k-1}d_{2k-1}r^{k+\frac{1}{2}}\left[\frac{2k-3}{2k+1}\cos\left(k+\frac{1}{2}\right)\theta-\cos\left(k-\frac{3}{2}\right)\theta\right]\right.$$
$$\left.+(-1)^k d_{2k}r^{k+1}[\cos(k+1)\theta-\cos(k-1)\theta]\right\}, \tag{21}$$

and define $\overline{\psi} := \psi - \psi^*$ as an $m+1$ times continuously differentiable function on the domain OABCDO in Fig. 2. Then, the problem (18) becomes

$$\begin{aligned}
&& \nabla^4\overline{\psi} &= 0 && \text{in } \Omega_3 := (-a \le x \le a,\ 0 \le y \le b),\\
\overline{\psi}+\psi^*=0, && \frac{\partial\overline{\psi}}{\partial y}+\frac{\partial\psi^*}{\partial y} &= 0 && \text{on } -a \le x \le 0,\ \ y=0,\\
\overline{\psi}+\psi^*=0, && \frac{\partial\overline{\psi}}{\partial x}+\frac{\partial\psi^*}{\partial x} &= 0 && \text{on } \ x=-a,\ \ 0 \le y \le b,\\
\overline{\psi}+\psi^*=\sigma\left(\frac{x^2}{2}+ax+\frac{a^2}{2}\right), && \frac{\partial\overline{\psi}}{\partial y}+\frac{\partial\psi^*}{\partial y} &= 0 && \text{on } \ -a \le x \le a,\ \ y=b,\\
\overline{\psi}+\psi^*=2\sigma a^2, && \frac{\partial\overline{\psi}}{\partial x}+\frac{\partial\psi^*}{\partial x} &= 2\sigma a && \text{on } \ x=a,\ \ 0 \le y \le b,
\end{aligned} \tag{22}$$

where

$$\frac{\partial\psi^*}{\partial x} = \sum_{k=1}^{m}\left\{(-1)^{k-1}d_{2k-1}r^{k-\frac{1}{2}}\left[\left(k-\frac{3}{2}\right)\cos\left(k-\frac{1}{2}\right)\theta-\left(k+\frac{1}{2}\right)\cos\left(k-\frac{5}{2}\right)\theta\right.\right.$$
$$\left.\left.+2\sin\theta\sin\left(k-\frac{3}{2}\right)\theta\right]+(-1)^{k+1}2kd_{2k}r^k\sin(k-1)\theta\sin\theta\right\}$$

and

$$\frac{\partial\psi^*}{\partial y} = \sum_{k=1}^{m}\left\{(-1)^{k-1}d_{2k-1}r^{k-\frac{1}{2}}\left[-\left(k-\frac{3}{2}\right)\sin\left(k-\frac{1}{2}\right)\theta-\left(k+\frac{1}{2}\right)\sin\theta\cos\left(k-\frac{3}{2}\right)\theta\right.\right.$$
$$\left.\left.+\left(k-\frac{3}{2}\right)\cos\theta\sin\left(k-\frac{3}{2}\right)\theta\right]+(-1)^k d_{2k}r^k[-(k+1)\sin k\theta+k\sin(k-2)\theta-\sin k\theta]\right\}.$$

From the above formulation of singularity subtraction, unknown parameters $\{d_i\}$ are introduced in the boundary conditions. Therefore, in order that the transformed problem (22) is solvable, extra pseudo-differential equations, $d_i' = 0, i = 1(1)2m$ with $2m$ conditions have to be added into problem (22). The additional conditions may be obtained by examining the function ψ^* defined by (21) in the vicinity of the crack tip O. The resulting boundary value problem is solved by the Tau Method. The Tau approximations to ψ and $d_i, i = 1(1)2m$, are obtained directly from the computer output.

5. Numerical results for the model problems

The singularity subtraction technique incorporated with the Tau Method described in Section 3, were used to compute the stress intensity factors of the two model crack problems given in Section 4. All computations were performed on a SUN SPARC 10 machine at City University of Hong Kong using Tau software program written in MATLAB and double precision arithmetic with 16 significant digits. The Motz problem has been considered by Rosser and Papamichael [16], Xanthis et al. [23], Li [6] and Liu et al. [7], using the different combination of numerical methods. In Table 1 we report our numerical results of using $m = 1(2)5$ terms subtraction with Tau degree, $n = 14$ to approximate the stress intensity and higher order factors. We also report those results given by the authors of the reference papers.

Table 1. Estimation of the SIFs of Motz' problem

Techniques used	λ_1	λ_2	λ_3	λ_4	λ_5
Singularity Subtraction Technique plus the Tau Method					
$m = 1$	151.41				
$m = 3$	151.62523	4.73315	.13334		
$m = 5$	151.6251554	4.732974	.132964	−.0088943	.0002264
Rosser and Papamichel's result [16]	151.6251553	4.732975	.132966	−.0088940	—
Xanthis, Bernal and Atkinson's results [23]	151.63	4.73	.133	−.009	.0002
Li's result [6]	$\frac{400.989}{7^{0.5}}$ ≈ 151.560	$\frac{87.6563}{7^{1.5}}$ ≈ 4.733	$\frac{17.3445}{7^{2.5}}$ $\approx .134$	$\frac{-8.51435}{7^{3.5}}$ $\approx -.0094$	$\frac{1.68054}{7^{4.5}}$ $\approx .00026$
Liu, Lee and Pan's result [7]	151.634	4.729	.134	−.009	.0002

Table 2. SIF results for opening mode crack problem

Techniques used	d_1	d_2	K_I	K_I^*
Singularity Subtraction Technique plus the Tau Method with $n = 13$	−1.2654	−0.0980	3.1719	2.8295
Bernal and Whiteman [2], Motz' Technique with 10 near points	−1.279		$1.279\sqrt{2\pi} \approx 3.206$	$1.279\sqrt{5} \approx 2.860$
Whiteman [20], Collocation plus Linear Programming	−1.2651	—	$1.2651\sqrt{2\pi} \approx 3.1711$	$1.2651\sqrt{5} \approx 2.8288$
Sinclair and Mullan [18], Finite Element Method	$-\frac{2.86}{\sqrt{5}} \approx -1.28$	—	$2.86\sqrt{0.4\pi} \approx 3.21$	2.86

All the computations of problem (18) have been carried out with a=0.4, b=0.7 and $\sigma = 1$. This model problem has been treated using Motz' technique [11] by Bernal and Whiteman [2], and using the combined method of collocation and linear programming by Whiteman [20]. The approximate values of d_1, d_2, $K_I(= -(2\pi)^{\frac{1}{2}} d_1)$, $K_I^*(= \frac{K_I}{\sigma\sqrt{\pi a}})$ computed by the method described in Sections 3 and 4 with Tau degree, $n = 13$, are displayed in Table 2 in conjunction with the corresponding values reported in [2], [18] and [20].

REFERENCES

[1] U. Ascher, R.D. Russell. Reformulation of boundary value problems into standard form. *SIAM Review*, **23**: 238–254, 1981.

[2] M.J.M. Bernal, J.R. Whiteman. Numerical treatment of biharmonic boundary value problems with re-entrant boundaries. *Comp. J.*, **13**: 87–91, 1970.

[3] S.K. Chan, I.S. Tuba, W.K. Wilson. On the finite element method in linear fracture mechanics. *Engng. Fracture Mech.*, **2**: 1–17, 1970.

[4] B. Gross, J.E. Srawley, W.F. Brown Jr. Stress-intensity factors for a single-edge-notch tension specimen by boundary collocation of a stress function. Technical Note D-2395, N.A.S.A., Washington, D.C., 1964.

[5] C. Lánczos. Trigonometric interpolation of empirical and analytical functions. *J. Math. Phys.*, **17**: 123–199, 1938.

[6] Z.C. Li. Penalty-combined approaches to the Ritz-Galerkin and finite element methods for singularity problems of elliptic equations. *Numerical Methods for Partial Differential Equations*, **8**: 33–57, 1992.

[7] K.M. Liu, K.M. Lee, C.K. Pan. Numerical techniques for determining stress intensity and higher order factors using the finite difference methods. In: S.N. Atluri, G. Yagawa, T.A. Cruse, eds., *Computational Mechanics*, Springer, **2**: 2123–2128, 1995.

[8] K.M. Liu, E.L. Ortiz. Approximation of eigenvalues defined by ordinary differential equations with the Tau Method. In: B. Kågström, A. Ruhe, eds., *Matrix Pencils*, 90–102. Springer-Verlag, Berlin, 1983.

[9] K.M. Liu, E.L. Ortiz. Numerical solution of eigenvalue problems for partial differential equations with the Tau-Lines Method. *Comput. Math. Applic.*, **12B**: 1153–1168, 1986.

[10] K.M. Liu, E.L. Ortiz, K.S. Pun. Numerical solution of Steklov's partial differential equation eigenvalue problem with the Tau Method. In: J.J.H. Miller, ed., *Computational and Asymptotic Methods for Boundary and Interior Layers (III)*, 244–249. Boole Press, Dublin, 1984.

[11] H. Motz. The treatment of singularities of partial differential equations by relaxation methods. *Quart. Appl. Maths.*, **4**: 371–377, 1946.

[12] E.L. Ortiz, K.S. Pun. A bi-dimensional Tau-elements method for the numerical solution of nonlinear partial differential equations with an application to Burgers' equation. *Comput. Math. Applic.*, **12B**: 1225–1240, 1986.

[13] E.L. Ortiz, H. Samara. An operational approach to the Tau method for the numerical solution of nonlinear differential equations. *Computing*, **27**: 15–25, 1981.

[14] E.L. Ortiz, H. Samara. Numerical solution of partial differential equations with variable coefficients with an operational approach to the Tau method. *Comput. Math. Applic.*, **10**: 5–13, 1984.

[15] J.R. Rice. A path independent integral and the approximate analysis of strain concentration by notches and cracks. *J. Appl. Mech.*, **35**: 379–386, 1968.

[16] J.B. Rosser, N. Papamichael. A power series solution of a harmonic mixed boundary value problem. *MRC Technical Summary Report*, No. 1405, 1975.

[17] B. Schiff, D. Fishelov, J.R. Whiteman. Determination of a stress intensity factor using local mesh refinement. In: J.R. Whiteman, ed., *The Mathematics of Finite Elements and Applications III*, 55–64. Academic Press, London, 1979.

[18] G.B. Sinclair, D. Mullan. A simple yet accurate finite element procedure for computing stress intensity factors. *Int. J. Num. Meth. Engrg.*, **18**: 1587–1600, 1982.

[19] G.T. Symm. Treatment of singularities in the solution of Laplace's equation by an integral equation method. *National Physical Laboratory Report* NAC 31, 1973.

[20] J.R. Whiteman. Numerical treatment of a problem from linear fracture mechanics. In: D.R.J. Owen, A.R. Luxmoore, eds., *Numerical Methods in Fracture Mechanics*, 128–136, 1978.

[21] J.R. Whiteman, N. Papamichael. Treatment of harmonic mixed boundary problems by conformal transformation methods. *ZAMP*, **23**: 655–664, 1972.

[22] M.L. Williams. Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *J. Appl. Mech.*, **24**: 526–528, 1952.

[23] L.S. Xanthis, M.J.M. Bernal, C. Atkinson. The treatment of singularities in the calculation of stress intensity factors using the boundary integral equation method. *Comput. Meth. Appl. Mech. Engrg.*, **26**: 285–304, 1981.