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D-adaptive model for the elasticity problem

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The paper presents some aspects of the formulation and numerical implementation of combined mathematical model "elastic body — Timoshenko plate". The variational problem is formulated. The existence of solution of combined model is considered. The numerical investigation of the problem is performed by coupling Direct Boundary Element and Finite Element Methods. Numerical example is presented supporting the analysis.

In spite of impressive imagination of professionals and progress in computer technology effective solution of many problems, which is based on general 3-D models, remains still problematic. Therefore, a good deal of current research is dedicated to the problem of construction of mathematical models of different dimension for solution of mechanical problems.

The most common approaches for solution of such problems are: use of the asymptotic method [1, 2], construction of special boundary conditions [4, 10], design of special transition elements [13].

In this paper like in [9–11], the mathematical models of an elastic body is formulated by combining the equations of the theory of elasticity and Timoshenko's plates theory. Such models are called combined [7] or D-adaptive models [12]. They are convenient in numerical computer methods such as finite element method [10] or coupling boundary and finite element methods [11], which is used in paper.

1. Statement of the problem

Assume that the elastic continuum occupies the domain $\Omega = \Omega_1 \cup \Omega_2^*$, $\Omega_1 \cap \Omega_2^* = \emptyset$, (Fig. 1) where Ω_1, Ω_2^* arbitrary, connected sets of Euclidean space R^3.

We shall assume that the domain Ω_1 is bounded by a Lipschitz boundary Γ_1, and the domain

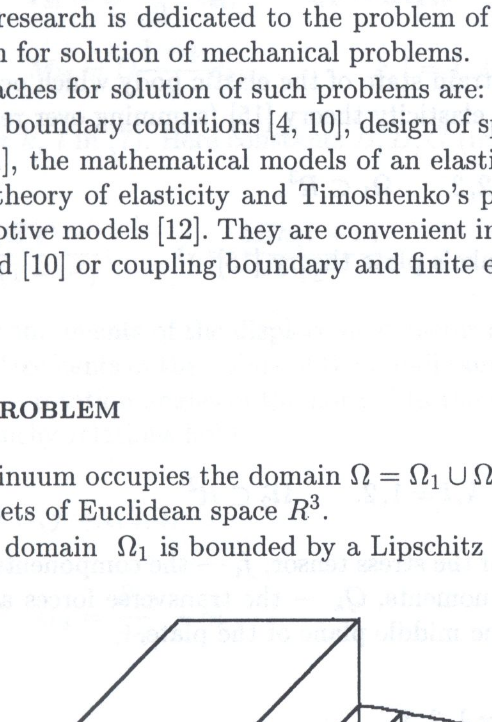

Fig. 1. The domain of D-adaptive model for the theory of elasticity

Ω_2^* is bounded by two parallel planes and a cylindrical side surface perpendicular to these planes. The distance between the planes is h. (Here and in the rest of the paper the terms "plane" and "surface" mean simple parts of planes and surfaces.)

Let us assume that the size h of the domain Ω_2^*, is much smaller than other characteristic dimensions of Ω_2^* in the sense of the paper [15]. The middle plane Ω_2 is between the parallel planes and it is equidistant from them. The cylindrical side surface intersects the plane Ω_2 along the line Γ_2, that is Γ_2 is the boundary of the plane domain Ω_2

$$\Gamma_2 = \Gamma_2^{(1)} \cup \Gamma_2^{(2)} \cup \Gamma_2^{(3)}, \qquad \Gamma_2^{(1)} \cap \Gamma_2^{(2)} \cap \Gamma_2^{(3)} = \emptyset,$$

where $\Gamma_2^{(1)}, \Gamma_2^{(2)}, \Gamma_2^{(3)}$, — piecewise smooth curves. Let x_1, x_2, x_3 be the Cartesian coordinates system of the elastic body, $\bar{n}_1, \bar{n}_2, \bar{n}_3$ – are an orthogonal right triple of unit-length vectors defined on Γ_1, where $\bar{n}_3$ is the outer normal to Γ_1. Let $\alpha_1, \alpha_2, \alpha_3$ be the Cartesian coordinate system on the middle surface (the direction of α_3 coincides with the normal to the middle plane). Let $\bar{t}_1, \bar{t}_2$ be a pair of orthogonal unit vectors on Γ_2 where $\bar{t}_1$ is the outer normal to the boundary, $\bar{t}_2$ — the tangent vector corresponding to the positive direction along the curve Γ_2.

Assume that the part of the boundary $\Gamma_1^{(3)}$ is common to both domains Ω and Ω_2^*,

$$\Gamma_1^{(3)} = \Gamma_2^{(1)} \times \left[-\frac{h}{2}, \frac{h}{2}\right]$$

and the following hold

$$\bar{t}_1 = -\bar{n}_3, \qquad \bar{t}_2 = -\bar{n}_1.$$

Let us describe the stress-strain state of the elastic body which occupies the domain Ω_1 by the differential equations of linear elasticity theory [15] (summing over repeating indices)

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \qquad i, j = 1, 2, 3 \qquad \Omega_1 \subset R^3, \tag{1}$$

and the equations of Timoshenko's plate theory [15]

$$\begin{aligned} &\frac{\partial T_{kl}}{\partial \alpha_l} + p_k = 0, \\ &\frac{\partial Q_k}{\partial \alpha_k} + p_3 = 0, \\ &\frac{\partial M_{kl}}{\partial \alpha_l} - Q_k + m_k = 0, \qquad k, l = 1, 2, \qquad \Omega_2^* \subset R^2 \end{aligned} \tag{2}$$

Here, σ_{ij} are the components of the stress tensor; f_i — the components of the vector of body forces; T_{kl} — the forces, M_{kl} — the moments, Q_k — the transverse forces arising in the plate; p_i, m_k — the surface loads reduced to the middle plane of the plate.

It is known [15] that

$$\begin{aligned} p_i &= \sigma_{i3}^+ + \sigma_{i3}^- + \rho_i, \qquad i = 1, 2, 3, \\ m_k &= \frac{h}{2}(\sigma_{k3}^+ - \sigma_{k3}^-) + \mu_k, \qquad k = 1, 2, \end{aligned} \tag{3}$$

where σ_{i3}^+, σ_{i3}^- are the components of the vector of surface loading on the surfaces of the plate $\alpha_i = \pm\frac{h}{2}$, respectively in the α_i coordinates; ρ_i, μ_k — the components of the plate body forces and moments reduced to the middle surface which are evaluated by formulas

$$\rho_i = \int_{-h/2}^{h/2} q_i \, d\alpha_3, \qquad \mu_k = \int_{-h/2}^{h/2} q_k \, \alpha_3 \, d\alpha_3; \tag{4}$$

q_i — the components of the body force vector in the coordinate system α_i on the middle surface of the plate.

In linear elasticity theory the components of the stress tensor are expressed in terms of the components of the strain tensor e_{kl} by the physical law

$$\sigma_{ij} = C_{ijkl}\, e_{kl}, \qquad i,j,k,l = 1,2,3, \tag{5}$$

where C_{ijkl} — elastic constants; the nonzeros among them for the case of isotropic homogeneous body are

$$\begin{aligned} C_{iiii} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \\ C_{iikk} &= \frac{E\nu}{(1+\nu)(1-2\nu)}, \\ C_{ikik} &= \frac{E}{2(1+\nu)} \qquad i,k = 1,2,3; \quad i \neq k. \end{aligned} \tag{6}$$

Here E — Young modulus, ν — Poisson ratio of the elastic body.

The forces and moments acting in plate are also expressed in terms of deformation characteristics ϵ_{ij} and κ_{ij} by means of the relations of the physical law of Timoshenko's plate theory

$$\begin{aligned} T_{kk} &= B(\epsilon_{kk} + \nu\epsilon_{ll}), \qquad T_{kl} = B\frac{1-\nu}{2}\epsilon_{kl}, \qquad Q_k = G\epsilon_{k3}, \\ M_{kk} &= D(\kappa_{kk} + \nu\kappa_{ll}), \qquad M_{kl} = D\frac{1-\nu}{2}\kappa_{kl}, \qquad k,l = 1,2; k \neq l. \end{aligned} \tag{7}$$

Note that we do not sum over k, l in (7). Here constants B, D, G (in the case of isotropic material) are given by

$$B = \frac{Eh}{1-\nu^2}, \qquad D = \frac{Eh^3}{12\,(1-\nu^2)}, \qquad G = \frac{5Eh(1+\nu)}{12}. \tag{8}$$

Let u_i $(i = 1,2,3)$ be the components of the displacement vector of the elastic body in x_1, x_2, x_3 coordinate system; v_i — displacements of the points of the middle surface in the direction of axes α_i $(i = 1,2,3)$ and γ_l $(l = 1,2)$ — rotation angles of the normal to the middle surface in the direction of axes α_l. The following Cauchy relations hold

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i,j = 1,2,3, \tag{9}$$

$$\begin{aligned} \epsilon_{kl} &= \frac{1}{2}\left(\frac{\partial v_k}{\partial \alpha_l} + \frac{\partial v_l}{\partial \alpha_k}\right), \qquad \epsilon_{k3} = \frac{\partial v_3}{\partial \alpha_k} + \gamma_k, \\ \kappa_{kl} &= \frac{1}{2}\left(\frac{\partial \gamma_k}{\partial \alpha_l} + \frac{\partial \gamma_l}{\partial \alpha_k}\right), \end{aligned} \tag{10}$$

To the equations (1), (2) we add boundary conditions and conditions of continuity of the medium and equilibrium of the body and the plate. Boundary conditions on the boundary of Ω_1:

$$u_{ni} = 0, \qquad i = 1,2,3, \quad x \in \Gamma_1^{(1)}, \tag{11}$$

$$\sigma_{ni,n3} = 0, \qquad i = 1,2,3, \quad x \in \Gamma_1^{(2)}, \tag{12}$$

where

$$u_{ni} = u_k n_{ik}, \qquad \sigma_{ni,n3} = \sigma_{kl}\, n_{ik}\, n_{3l}, \qquad k,l = 1,2,3,$$

$n_{ij} = \cos(n_i, x_j)$ — direction cosines of the triple $\bar{n}_j$. Boundary conditions on the boundary of Ω_2:

$$v_{tk} = 0, \quad v_3 = 0, \quad \gamma_{tk} = 0, \qquad k = 1,2 \quad x \in \Gamma_2^{(1)}; \tag{13}$$

$$T_{tk,tl} = 0, \quad Q_{ti} = 0, \quad M_{tk,tl} = 0, \qquad i = 1, \quad k = 1,2 \quad x \in \Gamma_2^{(2)}; \tag{14}$$

Here

$$v_{tk} = v_l t_{kl}, \qquad \gamma_{tk} = \gamma_l t_{kl}, \qquad l = 1,2;$$

$$T_{tk,tl} = T_{ij}\, t_{ki}\, t_{lj}, \qquad M_{tk,tl} = M_{ij}\, t_{ki}\, t_{lj}, \qquad i,j = 1,2,$$

where t_{kl} — the direction cosines of the vectors $\bar{t}_k$ in the coordinate system $\alpha_l : t_{kl} = \cos(t_k, \alpha_l)$.

Conditions of continuity of the medium and equilibrium of the body and plate on

$$\Gamma_1^{(3)} = \Gamma_2^{(3)} \times \left[-\frac{h}{2} \times \frac{h}{2}\right]$$

are

$$u_{n_1} = -(v_2 + \alpha_3\gamma_2), \qquad u_{n_2} = v_3, \qquad u_{n_3} = -(v_1 + \alpha_3\gamma_1), \tag{15}$$

and

$$\begin{gathered}
\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n_3n_3}\, d\alpha_3 = T_{t_1t_1}, \qquad \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n_1n_3}\, d\alpha_3 = T_{t_2t_1}, \qquad \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n_2n_3}\, d\alpha_3 = -Q_{t_1}, \\
\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n_3n_3}\alpha_3\, d\alpha_3 = M_{t_1t_1}, \qquad \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n_1n_3}\alpha_3\, d\alpha_3 = M_{t_1t_2}.
\end{gathered} \tag{16}$$

Thus, the D-adaptive mathematical model of the elastic body and the Timoshenko's plate in terms of displacements and rotation angles that describes the stress-strain state of the system made up of a massive and thin-layer element, is made up of the equilibrium equations (1), (2) written in terms of unknowns u_i, v_i, $i = 1,2,3$ and γ_k, $k = 1,2$ by means of formulas (5), (7), (9), (10). These equations form a closed system by means of relations (15), (16) and boundary conditions (11)–(14).

2. Variational statement of the problem

Let us write the BVP described in Section 1 using operator notation as

$$\mathbf{A}\,\mathbf{Z} = \mathbf{f}, \qquad \mathbf{f} \in \mathbf{H} \tag{17}$$

where

$$\mathbf{H} = [L_2(\Omega_1)]^3 \times [L_2(\Omega_2)]^5,$$

$$\mathbf{Z} = (u_1, u_2, u_3, v_1, v_2, v_3, \gamma_1, \gamma_2, \gamma_3)^T,$$

$$\mathbf{f} = \left(f_1, f_2, f_3, \sigma_{13}^+ + \rho_1, \sigma_{23}^+ + \rho_2, \sigma_{33}^+ + \rho_3, \frac{h}{2}\sigma_{13}^+ + \mu_1, \frac{h}{2}\sigma_{23}^+ + \mu_2\right)^T.$$

The operator of the problem (17) is defined on

$$\mathbf{D}_A = \{u_i, v_i, \gamma_k : \ i = 1,2,3; \quad k = 1,2;$$

$$u_i \in \left[\mathbf{W}_2^{(2)}(\Omega_1)\right]^3; \quad v_i \in \left[\mathbf{W}_2^{(2)}(\Omega_2)\right]^3; \quad \gamma_k \in \left[\mathbf{W}_2^{(2)}(\Omega_2)\right]^2;$$

conditions (11), (15)}.

Let us define a scalar product of the vector functions over the lineal $\mathbf{D}_A$ as

$$(u,\hat{v}) = \int_{\Omega_1} u_i\hat{v}_i \, d\Omega_1 + \int_{\Omega_2} (u_i\hat{v}_i + \gamma_k\hat{\gamma}_k)\, d\Omega_2 \,.$$

The following lemma holds.

Lemma. The operator of the problem (17) is symmetrical in the space $\mathbf{H}$.

Proof. Note that since $C_0^{(\infty)} \subset \mathbf{D}_A$ the lineal $\mathbf{D}_A$ is a dense set in the space $\mathbf{H}$ [8].

Consider the bilinear form $(\mathbf{A\,Z},\ \hat{\mathbf{Z}})$ where $\mathbf{Z}$, $\hat{\mathbf{Z}}$ are arbitrary elements of $\mathbf{D}_A$. Applying the Ostrogradsky formula and accounting for the boundary conditions (11)–(14) we will obtain after simple but unwieldy transformations

$$(\mathbf{A\,Z},\ \hat{\mathbf{Z}}) = \int_{\Omega_1} \sigma_{ij}(u_1,u_2,u_3)\, e_{ij}(\hat{u}_1,\hat{u}_2,\hat{u}_3)\, \mathrm{d}\Omega_1$$

$$+ \int_{\Omega_2} [T_{kl}(v_1,v_2)\, \epsilon_{kl}(\hat{v}_1,\hat{v}_2) + Q_k(v_3,\gamma_1,\gamma_2)\, \epsilon_{k3}(\hat{v}_3,\hat{\gamma}_1,\hat{\gamma}_2) + M_{kl}(\gamma_1,\gamma_2)\, \kappa_{kl}(\hat{\gamma}_1,\hat{\gamma}_2)]\, \mathrm{d}\Omega_2$$

$$- \int_{\Gamma_1^{(3)}} \sigma_{n_i n_3}\, \hat{u}_{n_i}\, \mathrm{d}\Gamma - \int_{\Gamma_2^{(3)}} (T_{t_k t_l}\, \hat{v}_{t_k} - M\ {}_{t_k t_l}\hat{\gamma}_k)\, \mathrm{d}\Gamma\,, \qquad i,j=1,2,3; \quad k,l=1,2.$$

Note that the sum of the last two integrals in the previous formula is equal to zero according to (15), (16).

Let us express stresses, forces and moments in terms of strains in the previous formula using (15), (17). We will obtain

$$(\mathbf{A\,Z},\ \hat{\mathbf{Z}}) = \int_{\Omega_1} C_{ijmn}\, e_{mn}\, \hat{e}_{ij}\, \mathrm{d}\Omega_1$$

$$+ \int_{\Omega_2} \Big[B(\epsilon_{kk}+\nu\epsilon_{ll})\hat{\epsilon}_{kk} + B\tfrac{1-\nu}{2}\epsilon_{kl}\hat{\epsilon}_{kl} + G\epsilon_{k3}\hat{\epsilon}_{k3} + D(\kappa_{kk}+\nu\kappa_{ll})\hat{\kappa}_{kk} + D\tfrac{1-\nu}{2}\kappa_{kl}\hat{\kappa}_{kl}\Big]\, \mathrm{d}\Omega_2\,, \tag{18}$$

where $i,j,m,n = 1,2,3$; $k,l = 1,2$; $k \neq l$; $\hat{e}_{ij}$, $\hat{\epsilon}_{kl}$, $\hat{\epsilon}_{k3}$, $\hat{\kappa}_{kk}$ — deformations characteristics of which are based on functions from $\hat{\mathbf{Z}}$.

Taking into account the symmetry of the matrix $\mathbf{C}_{ijmn}$ it is clear that the expression (18) is symmetrical with respect to functions $\mathbf{Z}$, $\hat{\mathbf{Z}}$. Hence,

$$(\mathbf{A\,Z},\ \hat{\mathbf{Z}}) = (\mathbf{Z},\ \mathbf{A}\ \hat{\mathbf{Z}}).$$

Theorem 1. The operator of problem (17) is positive.

Proof. In view of the lemma it is sufficient to show that $(\mathbf{A\,Z},\ \mathbf{Z}) \geq 0$, and if $(\mathbf{A\,Z},\ \mathbf{Z}) = 0$ then $\mathbf{Z} \equiv 0$.

Using (18) we write the expression for $(\mathbf{A\,Z},\ \mathbf{Z})$. It is known in the theory of elasticity that the integrand of the first integral is a positive definite quadratic form of the strain tensor components (it can be verified by direct calculations in the case of an isotropic material). The integrand of the second integral in the formula for $(\mathbf{A\,Z},\ \mathbf{Z})$ is also a positive definite quadratic form of the plate deformations characteristics (it can be verified by simple calculations). Thus we can write the inequality

$$(\mathbf{A\,Z},\ \mathbf{Z}) \geq 0.$$

Next, if $(\mathbf{A\,Z},\ \mathbf{Z})=0$, then each addend must necessarily be zero, because $(\mathbf{A\,Z},\ \mathbf{Z})$ is a sum of positive definite quadratic forms of the deformations of the body and plate. Besides, they will equal zero too. Then functions u_i, v_i, $i=1,2,3$; γ_k, $k=1,2$ are constants. But taking into account the boundary conditions (11) and (13) we obtain $\mathbf{Z}=0$.

Uniqueness of the weak solution of the problem (17) and possibility to give its variational statement as the problem of minimization of quadratic functional follow from the proved theorem. To write this functional let us introduce spaces

$$U=\{u_1,u_2,u_3:\quad u_i\in \mathbf{W}_2^{(1)}(\Omega_1),\ u_i=0\ \text{ on }\ \Gamma_1^{(1)}\};$$

$$V=\{v_1,v_2,v_3,\gamma_1,\gamma_2:\quad u_i,\gamma_k\in \mathbf{W}_2^{(1)}(\Omega_2),\ v_i=0,\ \gamma_k=0\ \text{ on }\ \Gamma_2^{(1)}\};$$

$$Y=\Big\{U\times V:\quad u_{n_1}=-(v_2+\alpha_3\gamma_2),\ u_{n_2}=v_3,\ u_{n_3}=-(v_1+\alpha_3\gamma_1),\quad \text{on}\ \ \Gamma_1^{(3)}=\Gamma_2^{(3)}\times\Big[-\frac{h}{2}\times\frac{h}{2}\Big]\Big\}. \tag{19}$$

According to the energy functional theorem the problem (17) is equivalent to functional minimization problem [8]

$$\mathbf{F}(\mathbf{Z})=\int\limits_{\Omega_1}\sigma_{ij}\,e_{ij}\,\mathrm{d}\Omega_1+\int\limits_{\Omega_2}(T_{kl}\,\epsilon_{kl}+Q_k\,\epsilon_{k3}+M_{kl}\,\kappa_{kl})\,\mathrm{d}\Omega_2 -2\int\limits_{\Omega_1}f_iu_i\,\mathrm{d}\Omega_1-2\int\limits_{\Omega_2}(p_iv_i+m_k\gamma_k)\,\mathrm{d}\Omega_2,\qquad \mathbf{Z}\in Y, \tag{20}$$

where $i,j=1,2,3$; $k,l=1,2$; p_i, m_k — are given by formulas (3), where we set $\sigma_{i3}^{-}=0$. Note that conditions (12), (14) and (16) are natural for the given functional.

3. Existence of the solution

It is well known [8] that the generalized solution of (17) exists provided $\mathbf{A}$ is a positive definite operator. Thus, the following theorem answers the question of existence of the generalized solution of (17).

Theorem 2. The operator of problem (17) is positive definite.

Proof. In order to estimate from below the bilinear form (18) we will use positive definiteness of the bilinear form of displacement functions corresponding to the first integral [8], positive definiteness of the bilinear form of the deformations characteristics of the plate, Korn's inequality for both deformations characteristics $\epsilon_{kk}, \kappa_{kl}$, $k,l=1,2$, and inequalities

$$2v_3'\gamma_k\geq-(\lambda_k v_3'^2+\frac{1}{\lambda_k}\gamma_k^2),\qquad \forall\lambda_k>0,\quad k=1,2, \tag{21}$$

and also the Friedrichs' inequality [8]. We will obtain

$$(\mathbf{A\,Z},\ \mathbf{Z})\geq C_1^2\,\|\mathbf{Z}\|^2 \tag{22}$$

where

$$C_1\in R, C_1>0,$$

$$\|\mathbf{Z}\|^2=\int\limits_{\Omega_1}u_iu_i\,\mathrm{d}\Omega_1+\int\limits_{\Omega_2}(v_iv_i+\gamma_k\gamma_k)\,\mathrm{d}\Omega_2,\qquad i=1,2,3,\quad k=1,2.$$

The last inequality means that the operator of (17) is positive definite.

Thus, the problem (17) has a unique solution $\mathbf{Z}\in Y$.

4. Numerical implementation

For numerical implementation of the D-adaptive model we will use a hybrid method based on simultaneous use of the direct boundary element method (DBEM) for the problem of the theory of elasticity and finite element method (FEM) for the problem of Timoshenko plate theory.

The employment of such method enables the use of approximations of the same order in the whole domain.

A numerical solution of the problem of elasticity theory (1) in Ω_1 is constructed by means of the DBEM using Galerkin method. Unknown displacements and surface forces as well as the domain boundary are approximated by quadratic "bubble" functions [14]. The values of displacements and forces on the boundary can be obtained from the system of linear algebraic equations that can be written as

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{1c} \\ \mathbf{A}_{c1} & \mathbf{A}_{cc} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_c \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{0} \end{bmatrix} \tag{23}$$

where $\mathbf{x}_1$ is a vector of unknown boundary terms on the boundary $\Omega_1 \backslash \Gamma_1^{(3)}$ and $\mathbf{x}_c$ — unknown boundary values on the interface boundary $\Gamma_1^{(3)}$.

The solution of the problem of Timoshenko plates theory is determined on the base of Lagrangian functional minimization using isoparametric quadratic approximations of displacements. For determination of displacements and angles we obtain a system of linear algebraic equations

$$\begin{bmatrix} \mathbf{A}_{22} & \mathbf{A}_{2c} \\ \mathbf{A}_{c2} & \mathbf{A}_{cc} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_c \end{bmatrix} = \begin{bmatrix} \mathbf{b}_2 \\ \mathbf{0} \end{bmatrix} \tag{24}$$

where $\mathbf{y}_1$ — unknown values of displacements inside the domain $\Omega_2^* \backslash \Gamma_1^{(3)}$, and $\mathbf{y}_c$ — unknown boundary values on the interface boundary $\Gamma_1^{(3)}$.

Note that the matrix of this system as well as the matrix of (23) is rectangular. These equations are added to by the relations on the interface boundary, which can be obtained from (15) and (16):

$$\mathbf{x}_c = \mathbf{S} * \mathbf{y}_c\,. \tag{25}$$

The solution of the system of linear algebraic equations in case of large dimensions of matrices can be a formidable task. First of all, this is due to the complexity of the matrix structure, which is entailed by the presence of interface conditions (25) and principally different structures of the boundary element and finite element matrices. Thus, the matrix arising from boundary element approximations (23) is nonsymmetric and dense, while the matrix (24) — banded symmetric.

There are well-known some schemes for coupling BEM and FEM [3, 5]. Therefore to obtain the solution we will employ the algorithm [10] which enables us to separate the process of solution of (23) and (24). This algorithm is efficient when the number of points in interface boundary is not large. To do so we use the following decomposition of the solutions

$$\begin{aligned} \mathbf{x} &= \mathbf{x}^0 + \mathbf{X}^1\,\mathbf{c}, \\ \mathbf{y} &= \mathbf{y}^0 + \mathbf{Y}^1\,\mathbf{d}. \end{aligned} \tag{26}$$

Here the entries with zero superscript correspond to the solutions of the problems with given zero displacements on the interface boundary and a given loading, and the columns of matrices $\mathbf{X}^{1j}$ and $\mathbf{Y}^{1j}$ — solutions of (23), (24) with given unit displacements in the j-th node on the interface boundary and without external loading. The components of unknown vectors $\mathbf{c}, \mathbf{d}$ can be determined from (25). Note that in the case when there are three nodes of boundary element mesh and one node of finite element mesh on the interface boundary, then $i = 1, \ldots, 6$; $j = 1, 2, 3$.

5. Example

Let us consider the plane strain problem of layer with piece-constant of thickness, cross-section of which can be seen in Fig. 2. It is clamped on the left side and uniformly loaded by distributed external pressure with intensity P. The geometrical and physical parameters are: $a/l = b/l = 1/3$, $h/l = 1/9$, $c = 0.5h$, $E/P = 10^4$, $\nu = 0.3$.

The solution of the problem was obtained by proposed combined model based on the theory of elasticity for the left side of construction and theory of Timoshenko's plates for another one.

The quadratic approximations were used for the construction in FEM and DBEM for combined model. The results based on theory of elasticity for 2-D model were obtained by FEM on a grid

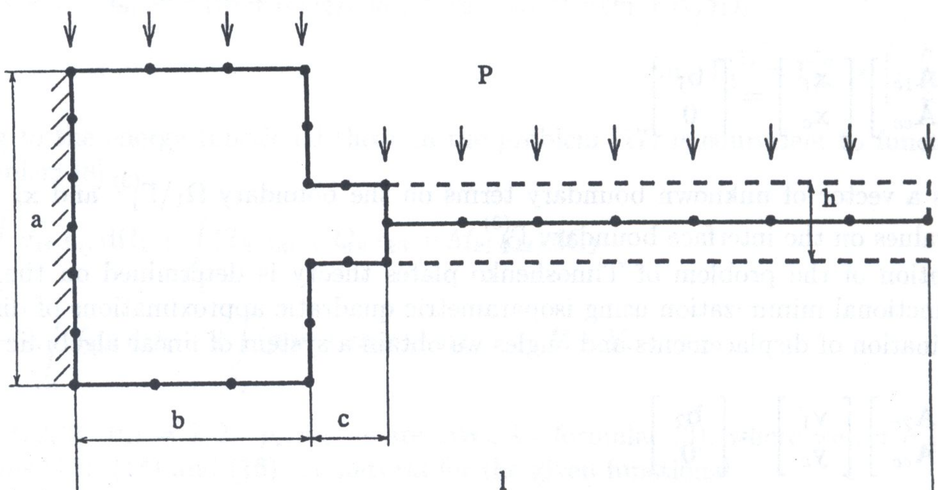

Fig. 2. Cross-section of body for the plane strain problem

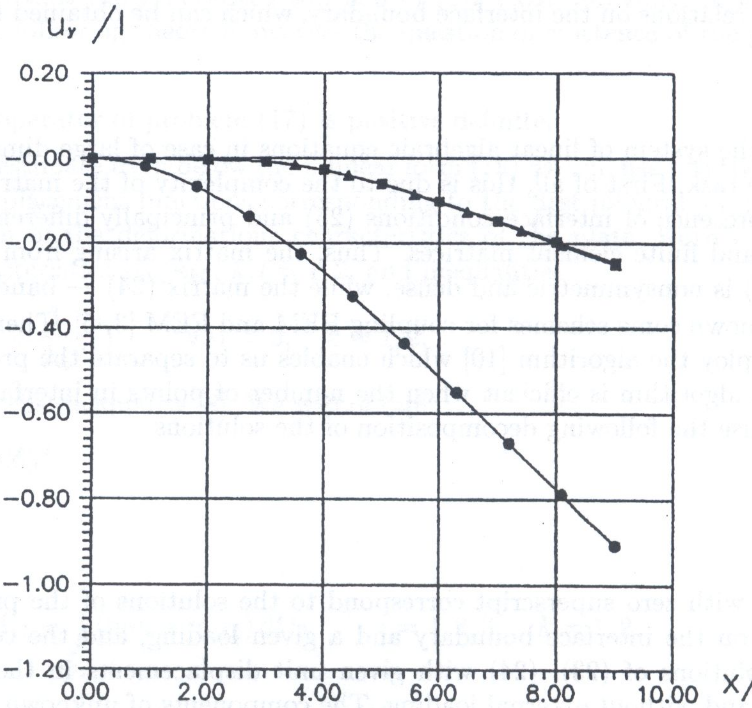

Fig. 3. The displacements of the middle of the construction obtained on the base of diffirent model and algorithms

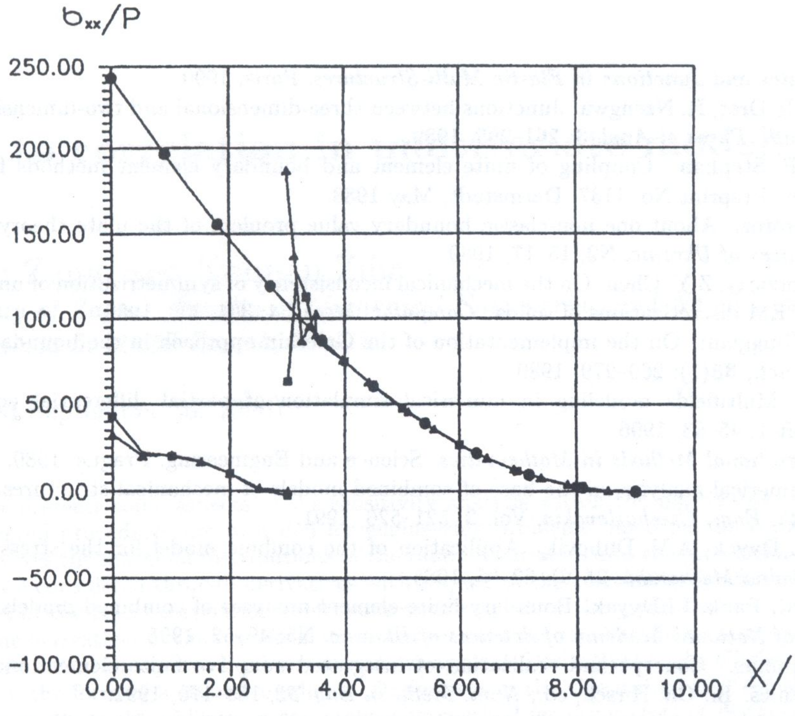

Fig. 4. The stresses σ_{xx}/P on the upper surface of the construction

9×9 and 3×18 elements with 8 nodes and Timoshenko's plates theory was obtained by FEM with quadratic approximations of displacements and angles. Figures 3 and 4 represent deflections of middle surface of construction and stresses on exterior surface of it, respectively. (For the results it introduces the following notations: a curve with triangles represents results, which are obtained by proposed schemes, a curve with circle section — the results, which are obtained by FEM for Timoshenko's plates theory; a curve with squares represents results, which are obtained by FEM for the theory of elasticity.) The deflections calculated by means of combined and 2-D model yield one curve in scales used in Fig. 3. Figure 4 shows coincidence of approximate solution for stresses based on combined model with results from theory of elasticity in all points, except those in zone of thickness's drop.

Note, that only when parameter $0.5h \le c \le 2h$ (length of zone of plate where the theory of elasticity is used) the really stress-strain picture is obtained.

6. Conclusion

In conclusion we can say that the D-adaptive model together with coupled FEM ande BEM methods is effective instrument for analyzing of boundary value problem of the theory of elasticity. Indeed, they allow us to implement the intrinsic features of FEM and BEM in a very natural way.

We note also that it could be interesting to consider any other algorithms of coupling FEM and BEM methods, in particular iterative ones.

One must be careful, however, because implementation of D-adaptive model imposes very rigid conditions on the coupling boundary. It may cause local disturbing of stress–strain state.

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