Computer Assisted Mechanics and Engineering Sciences, 5: 75–84, 1998.
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# Sensitivity to correlation in multivariate models

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(Received September 30, 1996)

The paper presents some aspects of the sensitivity analysis within the multivariate distribution models. The presented procedures are provided for engineering problems based on the Nataf model. The Nataf model involves the marginal distributions of the random variables and the correlation between them. Sensitivities are considered through derivatives with respect to the correlation coefficients. The terms for the derivatives of the Nataf correlation coefficients with respect to the given correlation coefficients are presented. The derivatives of the transformations between the random variables are given next. The Cholesky decomposition and the spectral decomposition are applied. Derivatives of the Cholesky decomposition are obtained in the form of a recursive scheme. Derivatives of the eigenvalues and eigenvectors are obtained using perturbations. In addition, a comprehensive method for derivatives of distances and derivatives of angles between the directions is given. Finally, numerical examples are attached to illustrate the presented procedures.

## 1. Introduction

Application of the finite difference method in sensitivity analysis is useful but usually numerically inefficient. Therefore, analytical methods convenient for numerical solutions using computers, without recalculation of the mathematical model, are suggested.

The multivariate distribution approach consistent with prescribed marginal distributions and correlations, based on the Nataf model, is applied, [3, 4].

The problem is defined by $n$ random variables $\mathbf{X} = \{x_k\}$, $k = 1, 2, \ldots, n$. Each of the random variables is defined by its marginal cumulative distribution function (CDF) denoted as $F_x$ and by its marginal probability density function (PDF) denoted as $f_x$. The $n$ by $n$ correlation matrix $\mathbf{R} = [\rho_{km}]$ is a function of the correlation coefficients $\rho_{km}$, $k, m = 1, 2, \ldots, n$ or the parameters associated with them.

Sensitivity analysis with respect to correlation matrix implies construction of sensitivity matrices $\mathbf{S} = [\frac{\partial g}{\partial \rho_{km}} s_{km}]$, where $g$ can in general represent coordinates, distances, angles etc., associated with random variables, while $s_{km}$ are the appropriate multiplicators. The sensitivity matrices can be used to assess influence of correlation on calculated probabilities as well as accuracy needed for specifying particular correlation coefficients.

## 2. The Nataf model

Three principal steps when applying the Nataf model are as follows:

(1) The correlation matrix $\mathbf{R}$ is transformed to matrix $\mathbf{R}'$ with elements $\rho'_{km}$, $k, m = 1, 2, \ldots, n$. The relation of $\rho'$ and $\rho$ is uniquely expressed as $\rho'_{km} = \rho_{km} F_{km}$, see [3] for commonly used two-parametric distributions.

(2) Standard normal variables $\mathbf{Y} = \{y_k\}$, $k = 1, 2, \ldots, n$, are obtained by using marginal transformations of $\mathbf{X}$, defined as $F(x_i) = \Phi(y_i)$, $i = 1, 2, \ldots, n$, where $\Phi(y_i)$ is the standard normal CDF.

The joint PDF $f(\mathbf{X})$ of the random vector $\mathbf{X}$ is expressed on the basis of the marginal PDFs $f_{x_i}$, as follows:

$$f(\mathbf{X}) = \frac{\phi_n(\mathbf{Y}, \mathbf{R}')}{\phi_n(\mathbf{Y})} \prod_{i=1}^{n} f_{x_i}(x_i) . \tag{1}$$

The $n$-dimensional joint normal PDF of zero mean, unit standard deviation and correlation matrix $\mathbf{R}'$ in Eq. (1), is defined by using the quadratic form $Q_c = \mathbf{Y}^T \mathbf{R}'^{-1} \mathbf{Y}$ as follows:

$$\phi_n(\mathbf{Y}, \mathbf{R}') = \frac{1}{(2\pi)^{n/2} |\mathbf{R}'|^{1/2}} e^{-\frac{1}{2} Q_c} . \tag{2}$$

The joint standard normal PDF $\phi_n(\mathbf{Y})$ for independent random variables in Eq. (1) is defined using the quadratic form $Q_y = \mathbf{Y}^T \mathbf{Y}$ as shown:

$$\phi_n(\mathbf{Y}) \equiv \phi_n(\mathbf{Y}, \mathbf{I}) = \prod_{i=1}^{n} \phi(y_i) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2} Q_y} . \tag{3}$$

(3) Variables $\mathbf{Y}$ can be related to independent standard normal variables $\mathbf{U} = \{u_k\}$, $k = 1, 2, \ldots, n$, by transformation $\mathbf{Y} = \mathbf{AU}$. The matrix $\mathbf{A}$ can be obtained by applying two widely used methods, the Cholesky decomposition and the spectral decomposition, to the Nataf correlation matrix $\mathbf{R}' = \mathbf{A}\mathbf{A}^T$.

## 3. THE DERIVATIVES IN THE FIRST TWO STEPS OF THE NATAF MODEL

Considering step (1) in the Nataf model, the derivatives of the Nataf correlation matrix can be in general expressed as shown:

$$\frac{\partial \rho'_{km}}{\partial \rho_{km}} = F_{km} + \rho_{km} \frac{\partial F_{km}}{\partial \rho_{km}}, \qquad k, m = 1, 2, \ldots, n; \tag{4}$$

The expressions for the Nataf correlation matrix derivatives are given in Appendix A. Considering step (2) in the Nataf model, the derivatives of the marginal transformation $x_i = F_{x_i}^{-1}[\Phi(y_i)]$ can be represented in a diagonal matrix $\mathbf{J}$. The elements of $\mathbf{J}$ are easily obtained for $i = 1, 2, \ldots, n$ in the form:

$$\mathbf{J} = \left[\frac{\partial x_i}{\partial y_i}\right] = \left[\frac{\phi(y_i)}{f_{x_i}(x_i)}\right] . \tag{5}$$

## 4. DERIVATIVES OF THE TRANSFORMATION Y=AU IN THE THIRD STEP OF THE NATAF MODEL

Considering step (3) in the Nataf model, and since $\mathbf{Y}$ is not a function of $\rho_{km}$, the derivatives of the random variables $\mathbf{U}$ with respect to correlation coefficients can be in general expressed as shown:

$$\frac{\partial \mathbf{U}}{\partial \rho_{km}} = \frac{\partial \mathbf{A}^{-1}}{\partial \rho_{km}} \mathbf{Y} = \frac{\partial \mathbf{A}^{-1}}{\partial \rho_{km}} \mathbf{AU} = -\mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial \rho_{km}} \mathbf{U} . \tag{6}$$

Two methods are used for calculation of the expressions in Eq. (6).

### 4.1. Sensitivity analysis using the Cholesky decomposition

The transformation $\mathbf{Y} = \mathbf{LU}$ , obtained by using the Cholesky decomposition $\mathbf{R}' = \mathbf{LL}^T$, transforms the quadratic form $Q_c$ to the form $Q_u = \mathbf{U}^T\mathbf{U}$. $\mathbf{L}$ is a lower-triangular $n$ by $n$ matrix with elements $\lambda_{ij}$ , whose inverse is denoted as $\mathbf{M}$, with elements $\mu_{ij}$ , $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, i$, see Appendix B. $\mathbf{L}$ and $\mathbf{M}$ always exist since $\mathbf{R}'$ is symmetric and positive definite.

Analytically, the Nataf transformation and its inverse based on the Cholesky decomposition, for $j = 1, 2, \ldots, n$, are expressed as follows:

$$x_j = F_{x_j}^{-1}\left[\Phi(y_j)\right] = F_{x_j}^{-1}\left[\Phi\left(\sum_{i=1}^{j} \lambda_{ji} u_i\right)\right], \tag{7}$$

$$u_j = \sum_{i=1}^{j} \mu_{ji} y_i = \sum_{i=1}^{j} \mu_{ji} \Phi^{-1}\left[F_{x_i}(x_i)\right]. \tag{8}$$

The differentiation applied on Eqs. (7)–(8) locally in the point $\mathbf{U}^*$ and corresponding $\mathbf{Y}^*$ and $\mathbf{X}^*$ points gives:

$$\left.\frac{\partial x_j}{\partial \rho_{km}}\right|_{U^*} = \frac{\partial \rho'_{km}}{\partial \rho_{km}} \frac{\phi(y_j^*)}{f_{x_j}(x_j^*)} \sum_{i=1}^{j} u_i^* \frac{\partial \lambda_{ji}}{\partial \rho'_{km}}, \tag{9}$$

$$\left.\frac{\partial u_j}{\partial \rho_{km}}\right|_{X^*} = \frac{\partial \rho'_{km}}{\partial \rho_{km}} \sum_{i=1}^{j} \Phi^{-1}\left[F_{x_i}(x_i^*)\right] \frac{\partial \mu_{ji}}{\partial \rho'_{km}}. \tag{10}$$

General expressions for the derivatives of the terms in the Cholesky decomposition in Eqs. (9)–(10) are given in Appendix B.

### 4.2. Sensitivity analysis using the spectral decomposition

The transformation of quadratic form $Q_c$ to principal axes can be performed by the spectral decomposition in the form $\mathbf{R}' = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$ .

In this case $\mathbf{Y} = \mathbf{V}\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{U}$, where $\mathbf{V}$ is the $n$ by $n$ orthonormal matrix, with elements $v_{ji}$ , containing the eigenvectors $\mathbf{v}_i = \{v_j\}_i$ , $j = 1, 2, \ldots, n$, i.e. unit base vectors in principal directions of PDF. Eigenvalues $\lambda_i$ represent variances for each principal axis, and are the elements of a diagonal matrix $\mathbf{\Lambda}$. Eigenvalues and eigenvectors can be obtained using standard computer library routines.

Derivatives of the eigenvalues and of the eigenvectors could be calculated in a number of ways [1, 2, 5]. Approach selected uses perturbed correlation matrix $\overline{\mathbf{R}'} = \overline{\mathbf{V}\mathbf{\Lambda}\mathbf{V}^T}$ with assumption $\overline{\mathbf{V}} = \mathbf{VZ}$ where $\overline{\mathbf{V}}$ is the matrix of eigenvectors of perturbed matrix $\overline{\mathbf{R}'}$ and $\mathbf{Z}$ is the matrix of coefficients. $\overline{\mathbf{\Lambda}}$ denotes matrix of eigenvalues of perturbed correlation matrix. Small perturbation, $z_{ii} = 1$, $|z_{ij}| \ll 1$ are assumed and approximate solution corresponding to given $\mathbf{V}$ is obtained:

$$\frac{\partial \mathbf{\Lambda}}{\partial \rho'_{km}} = \left[\frac{\partial \lambda_i}{\partial \rho'_{km}}\right] = \left[2\,(v_k)_i\,(v_m)_i\right], \tag{11}$$

$$\frac{\partial \mathbf{V}}{\partial \rho'_{km}} = \left[\frac{\partial \mathbf{v}_i}{\partial \rho'_{km}}\right] = \mathbf{VN}, \tag{12}$$

where $(v_k)_i$ is the $k$-th element of the $i$-th eigenvector $\mathbf{v}_i$ and $\mathbf{N} = [n_{ij}]$ is a matrix with terms:

$$n_{ii} = 0, \qquad n_{ij} = \frac{(v_m)_i\,(v_k)_j + (v_k)_i\,(v_m)_j}{\lambda_j - \lambda_i + \delta\lambda_j},$$

where $\delta\lambda_j$ may be calculated using Eq. (11) when $\lambda_i = \lambda_j$ .

Analytically the Nataf transformation and its inverse based on spectral decomposition, for $j = 1, 2, \dots, n$, are expressed as follows:

$$x_j = F_{x_j}^{-1}[\Phi(y_j)] = F_{x_j}^{-1}\left[\Phi(\sum_{i=1}^{n} v_{ji}\lambda_i^{1/2}u_i)\right], \tag{13}$$

$$u_j = \lambda_j^{-1/2}\sum_{i=1}^{n} v_{ij}y_i = \lambda_j^{-1/2}\sum_{i=1}^{n} v_{ij}\Phi^{-1}[F_{x_i}(x_i)]. \tag{14}$$

Derivatives of random variables in a given point denoted by *, are as shown:

$$\left.\frac{\partial \mathbf{U}}{\partial \rho_{km}}\right|_{Y^*} = -\frac{\partial \rho'_{km}}{\partial \rho_{km}}\mathbf{\Lambda}^{-1/2}\left(\mathbf{V}^T\frac{\partial \mathbf{V}}{\partial \rho'_{km}} + \frac{1}{2}\mathbf{\Lambda}^{-1}\frac{\partial \mathbf{\Lambda}}{\partial \rho'_{km}}\right)\mathbf{V}^T\mathbf{Y}^*, \tag{15}$$

$$\left.\frac{\partial \mathbf{U}}{\partial \rho_{km}}\right|_{U^*} = -\frac{\partial \rho'_{km}}{\partial \rho_{km}}\mathbf{\Lambda}^{-1/2}\left(\mathbf{N} + \frac{1}{2}\mathbf{\Lambda}^{-1}\frac{\partial \mathbf{\Lambda}}{\partial \rho'_{km}}\right)\mathbf{\Lambda}^{1/2}\mathbf{U}^* = \frac{\partial \rho'_{km}}{\partial \rho_{km}}\mathbf{W}\mathbf{U}^*, \tag{16}$$

where $\mathbf{W} = [w_{ij}]$ and $w_{ij} = -n_{ij}(\lambda_j/\lambda_i)^{1/2}$ and $w_{ii} = -(v_k v_m/\lambda)_i$.

## 5. Sensitivity of distance between points and of angle between the directions in the Nataf model

The derivatives of distances and angles defined in standard normal space, can be directly related to the sensitivities of probabilities, particularly in problems with bounding hyperplanes. To avoid the calculation of the derivatives of transformation matrix $\mathbf{A}$ for each $\rho'_{km}$, the following calculations are performed, using $Y$ coordinates. In this case, only the derivatives of the correlation matrix $\mathbf{R}'$ are required w.r.t. its own elements $\rho'_{km}$.

(1) The distance between points $\mathbf{U}_2$ and $\mathbf{U}_1$ is defined as $d = (\mathbf{u}^T\mathbf{u})^{1/2} = (\mathbf{y}^T\mathbf{R}'^{-1}\mathbf{y})^{1/2}$ (where $\mathbf{u} = \mathbf{U}_2 - \mathbf{U}_1 = \mathbf{A}^{-1}\mathbf{y}$ ). The derivative of $d$ w.r.t. correlation coefficients $\rho'_{km}$, where $\mathbf{y}$ is not a function $\rho'_{km}$, is given by

$$\frac{\partial d}{\partial \rho'_{km}} = -\frac{1}{2d}\mathbf{y}^T\mathbf{R}'^{-1}\frac{\partial \mathbf{R}'}{\partial \rho'_{km}}\mathbf{R}'^{-1}\mathbf{y}. \tag{17}$$

If the auxiliary vector $\tilde{\mathbf{y}} = \{\tilde{y}_k\} = \mathbf{R}'^{-1}\mathbf{y}$ is introduced, the derivative reads

$$\frac{\partial d}{\partial \rho'_{km}} = -\frac{1}{2d}\tilde{\mathbf{y}}^T\mathbf{r}_{KM}\tilde{\mathbf{y}}. \tag{18}$$

The only non-zero terms in the derivative pointer matrix $\mathbf{r}_{KM} = \partial \mathbf{R}'/\partial \rho'_{km}$ are at the positions $(k, m)$ and $(m, k)$ due to $\rho'_{km} = \rho'_{mk}$. Finally, the simple expression for the sensitivity of distance between points in the Nataf model can be obtained as shown:

$$\frac{\partial d}{\partial \rho_{km}} = -\frac{1}{d}\tilde{y}_k\tilde{y}_m\frac{\partial \rho'_{km}}{\partial \rho_{km}}. \tag{19}$$

(2) The sensitivity of an angle between two vectors $\mathbf{u}_i$ and $\mathbf{u}_j$ is calculated in the sequel. The cosine of angle, $\gamma_{ij}$, is defined as shown:

$$\gamma_{ij} = \cos\Theta = \frac{\mathbf{u}_i^T}{d_i}\cdot\frac{\mathbf{u}_j}{d_j} = \frac{\mathbf{y}_i^T\mathbf{R}'^{-1}\mathbf{y}_j}{(\mathbf{y}_i^T\mathbf{R}'^{-1}\mathbf{y}_i)^{1/2}\cdot(\mathbf{y}_j^T\mathbf{R}'^{-1}\mathbf{y}_j)^{1/2}}. \tag{20}$$

The derivative of the cosine is obtained from the following relation:

$$\frac{\partial \gamma_{ij}}{\partial \rho'_{km}} = \frac{\mathbf{y}_i^T \frac{\partial \mathbf{R}'^{-1}}{\partial \rho'_{km}} \mathbf{y}_j}{d_i d_j} - \gamma_{ij} \left[ \frac{\partial d_i}{\partial \rho'_{km}} \cdot \frac{1}{d_i} + \frac{\partial d_j}{\partial \rho'_{km}} \cdot \frac{1}{d_j} \right]. \tag{21}$$

Using auxiliary vector $\widetilde{\mathbf{y}} = \{\widetilde{y}_k\}$, the derivative in Eq. (21) reads:

$$\left.\frac{\partial \gamma_{ij}}{\partial \rho_{km}}\right|_{R'} = \left\{ \gamma_{ij} \left[ \left( \frac{\widetilde{y}_k \widetilde{y}_m}{d^2} \right)_i + \left( \frac{\widetilde{y}_k \widetilde{y}_m}{d^2} \right)_j \right] - \frac{(\widetilde{y}_k)_i \cdot (\widetilde{y}_m)_j + (\widetilde{y}_m)_i \cdot (\widetilde{y}_k)_j}{d_i d_j} \right\} \frac{\partial \rho'_{km}}{\partial \rho_{km}}. \tag{22}$$

(3) For the gradient vectors the transformation between $U$ and $Y$ coordinates reads $\bar{\mathbf{u}} = \mathbf{A}^T \bar{\mathbf{y}}$ and the corresponding length is $\bar{d} = (\bar{\mathbf{y}}^T \mathbf{R}' \bar{\mathbf{y}})^{1/2}$. The cosine of the angle is given by the expression

$$\bar{\gamma}_{ij} = \frac{\bar{\mathbf{u}}_i^T}{\bar{d}_i} \cdot \frac{\bar{\mathbf{u}}_j}{\bar{d}_j} = \frac{\bar{\mathbf{y}}_i^T \mathbf{R}' \bar{\mathbf{y}}_j}{(\bar{\mathbf{y}}_i^T \mathbf{R}' \bar{\mathbf{y}}_i)^{1/2} (\bar{\mathbf{y}}_j^T \mathbf{R}' \bar{\mathbf{y}}_j)^{1/2}}. \tag{23}$$

Since $\partial \mathbf{R}' / \partial \rho'_{km} = \mathbf{r}_{KM}$, the expression for the derivative of Eq. (23) is

$$\left.\frac{\partial \bar{\gamma}_{ij}}{\partial \rho_{km}}\right|_{R'} = \left\{ \frac{(\bar{y}_k)_i \cdot (\bar{y}_m)_j + (\bar{y}_m)_i \cdot (\bar{y}_k)_j}{\bar{d}_i \bar{d}_j} - \bar{\gamma}_{ij} \left[ \left( \frac{\bar{y}_k \bar{y}_m}{\bar{d}^2} \right)_i + \left( \frac{\bar{y}_k \bar{y}_m}{\bar{d}^2} \right)_j \right] \right\} \frac{\partial \rho'_{km}}{\partial \rho_{km}}. \tag{24}$$

These procedures will be sufficient for construction of sensitivity matrices

$$\mathbf{D} = \left[ \frac{\partial d}{\partial \rho_{km}} s_{km} \right], \qquad \mathbf{G} = \left[ \frac{\partial \gamma_{ij}}{\partial \rho_{km}} s_{km} \right], \qquad \bar{\mathbf{D}} = \left[ \frac{\partial \bar{d}}{\partial \rho_{km}} s_{km} \right] \quad \text{and} \quad \bar{\mathbf{G}} = \left[ \frac{\partial \bar{\gamma}_{ij}}{\partial \rho_{km}} s_{km} \right]$$

where $\mathbf{s} = [s_{km}]$ is the matrix of multiplicators (e.g. $s_{km} = \frac{\partial \rho_{km}(p)}{\partial p} \Delta p$ and $p$ is a parameter used in the sensitivity study).

## 6. Example

An illustrative example of a multivariate model with three random variables is presented, (e.g. if random variables are correlated loads, the sensitivity of failure probability w.r.t. correlation coefficients can be calculated). Marginal distributions of random variables are given in Table 1.

**Table 1.** Random variables in example

| Variable | $\mu$ | $\sigma$ | Distribution |
|---|---|---|---|
| X1 | 500 | 100 | Log-normal |
| X2 | 2000 | 400 | Log-normal |
| X3 | 5 | 0.5 | Uniform |

The given correlation matrix $R$ and the Nataf correlation matrix $\mathbf{R}'$ are:

$$\mathbf{R} = \begin{bmatrix} 1.0 & 0.3 & 0.5 \\ 0.3 & 1.0 & 0.8 \\ 0.5 & 0.8 & 1.0 \end{bmatrix}, \qquad \mathbf{R}' = \begin{bmatrix} 1.0 & 0.304 & 0.517 \\ 0.304 & 1.0 & 0.830 \\ 0.517 & 0.830 & 1.0 \end{bmatrix}.$$

The inverse $\mathbf{R}'^{-1}$ of the Nataf correlation matrix is as follows:

$$\mathbf{R}'^{-1} = \begin{bmatrix} 1.466 & 0.592 & -1.250 \\ 0.592 & 3.463 & -3.182 \\ -1.250 & -3.182 & 4.290 \end{bmatrix}.$$

The derivative of the Nataf correlation matrix reads:

$$\frac{\partial \mathbf{R}'}{\partial \rho_{km}} = \left[\frac{\partial \rho'_{km}}{\partial \rho_{km}}\right] = \begin{bmatrix} 0.0 & 1.0078 & 1.0393 \\ 1.0078 & 0.0 & 1.0510 \\ 1.0393 & 1.0510 & 0.0 \end{bmatrix}.$$

## 6.1. The Cholesky decomposition

Results of the Cholesky decomposition of the Nataf correlation matrix are

$$\mathbf{L} = \begin{bmatrix} 1.0 & & \\ 0.304 & 0.953 & \\ 0.517 & 0.706 & 0.483 \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} 1.0 & & \\ -0.319 & 1.050 & \\ -0.604 & -1.536 & 2.071 \end{bmatrix}.$$

The derivatives of the Cholesky decomposition matrix $\mathbf{L}$ are as shown:

$$\frac{\partial \mathbf{L}}{\partial \rho_{21}} = \begin{bmatrix} 0 & & \\ 1 & -0.319 & \\ 0 & -0.306 & 0.448 \end{bmatrix}, \qquad \frac{\partial \mathbf{L}}{\partial \rho_{31}} = \begin{bmatrix} 0 & & \\ 0 & 0.0 & \\ 1 & -0.319 & -0.604 \end{bmatrix},$$

$$\frac{\partial \mathbf{L}}{\partial \rho_{32}} = \begin{bmatrix} 0 & & \\ 0 & 0.0 & \\ 0 & 1.050 & -1.536 \end{bmatrix}.$$

The derivatives of the Cholesky decomposition matrix $\mathbf{M}$ are as follows:

$$\frac{\partial \mathbf{M}}{\partial \rho_{21}} = \begin{bmatrix} 0 & & \\ -1.157 & 0.352 & \\ 2.050 & 1.575 & -1.921 \end{bmatrix}, \qquad \frac{\partial \mathbf{M}}{\partial \rho_{31}} = \begin{bmatrix} 0 & & \\ 0 & 0 & \\ -3.037 & -1.227 & 2.588 \end{bmatrix},$$

$$\frac{\partial \mathbf{M}}{\partial \rho_{32}} = \begin{bmatrix} 0 & & \\ 0 & 0 & \\ -1.227 & -7.172 & 6.521 \end{bmatrix}.$$

## 6.2. The spectral decomposition

Eigenvalues and eigenvectors of the Nataf correlation matrix are

$$\mathbf{\Lambda} = \begin{bmatrix} 0.733 & & \\ & 0.135 & \\ & & 2.132 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0.860 & 0.220 & -0.460 \\ -0.484 & 0.634 & -0.602 \\ -0.159 & -0.741 & -0.652 \end{bmatrix};$$

The derivatives of the eigenvalues are as shown:

$$\frac{\partial \mathbf{\Lambda}}{\partial \rho_{21}} = \begin{bmatrix} -0.833 & & \\ & 0.279 & \\ & & 0.554 \end{bmatrix}, \qquad \frac{\partial \mathbf{\Lambda}}{\partial \rho_{31}} = \begin{bmatrix} -0.274 & & \\ & -0.326 & \\ & & 0.600 \end{bmatrix},$$

$$\frac{\partial \mathbf{\Lambda}}{\partial \rho_{32}} = \begin{bmatrix} 0.154 & & \\ & -0.940 & \\ & & 0.785 \end{bmatrix}.$$

The derivatives of the eigenvectors are as shown:

$$\frac{\partial \mathbf{V}}{\partial \rho_{21}} = \begin{bmatrix} 0.064 & -0.730 & -0.228 \\ 0.339 & 0.228 & -0.032 \\ -0.682 & -0.022 & 0.191 \end{bmatrix}, \quad \frac{\partial \mathbf{V}}{\partial \rho_{31}} = \begin{bmatrix} -0.408 & 1.013 & -0.278 \\ -0.924 & -0.485 & 0.232 \\ -0.606 & -0.115 & -0.018 \end{bmatrix},$$

$$\frac{\partial \mathbf{V}}{\partial \rho_{32}} = \begin{bmatrix} 0.230 & -0.363 & 0.256 \\ 0.451 & 0.218 & -0.132 \\ -0.127 & 0.079 & -0.059 \end{bmatrix}.$$

## 6.3. The derivatives of random variables

The point

$$\mathbf{X} = \begin{bmatrix} 745.06 \\ 1801.29 \\ 4.73 \end{bmatrix}$$

and the corresponding point

$$\mathbf{Y} = \Phi^{-1}\left[F_{x_i}(\mathbf{X}_i)\right] = \begin{bmatrix} 2.11 \\ -0.43 \\ -0.39 \end{bmatrix}$$

are considered. The derivatives of the marginal transformation are as follows:

$$\mathbf{J} = \left[\frac{\partial x_i}{\partial y_i}\right] = \begin{bmatrix} 147.6 & & \\ & 356.7 & \\ & & 0.6 \end{bmatrix}.$$

The corresponding U-coordinates when using the Cholesky decomposition are

$$\mathbf{U} = \mathbf{MY} = \begin{bmatrix} 2.11 \\ -1.12 \\ -1.44 \end{bmatrix}.$$

The corresponding derivatives are as shown:

$$\frac{\partial \mathbf{U}}{\partial \rho_{21}} = \begin{bmatrix} 0 \\ -2.61 \\ 4.45 \end{bmatrix}, \quad \frac{\partial \mathbf{U}}{\partial \rho_{31}} = \begin{bmatrix} 0 \\ 0 \\ -7.19 \end{bmatrix}, \quad \frac{\partial \mathbf{U}}{\partial \rho_{32}} = \begin{bmatrix} 0 \\ 0 \\ -2.23 \end{bmatrix};$$

$$\frac{\partial \mathbf{X}}{\partial \rho_{21}} = \begin{bmatrix} 0 \\ 888.82 \\ -0.19 \end{bmatrix}, \quad \frac{\partial \mathbf{X}}{\partial \rho_{31}} = \begin{bmatrix} 0 \\ 0 \\ 2.21 \end{bmatrix}, \quad \frac{\partial \mathbf{X}}{\partial \rho_{32}} = \begin{bmatrix} 0 \\ 0 \\ 0.69 \end{bmatrix}.$$

The corresponding U-coordinates when using the spectral decomposition are

$$\mathbf{U} = \mathbf{\Lambda}^{-1/2}\mathbf{V}^T\mathbf{Y} = \begin{bmatrix} 2.44 \\ 1.32 \\ -0.31 \end{bmatrix}$$

and the corresponding derivatives are

$$\frac{\partial \mathbf{U}}{\partial \rho_{21}} = \begin{bmatrix} 1.70 \\ -5.84 \\ -0.33 \end{bmatrix}, \quad \frac{\partial \mathbf{U}}{\partial \rho_{31}} = \begin{bmatrix} -0.38 \\ 8.42 \\ -0.44 \end{bmatrix}, \quad \frac{\partial \mathbf{U}}{\partial \rho_{32}} = \begin{bmatrix} 0.15 \\ 2.27 \\ 0.51 \end{bmatrix}.$$

### 6.4. The derivatives of the Euclidean norm

The Euclidean norm of the considered vector reads: $d = (\mathbf{U}^T\mathbf{U})^{1/2} = 2.792$. Auxiliary vector reads:

$$\widetilde{\mathbf{Y}} = \mathbf{R}^{-1}\mathbf{Y} = \begin{bmatrix} 3.334 \\ 1.023 \\ -2.971 \end{bmatrix}$$

The sensitivity matrix with multiplicator $s_{km} = 1$, contains all derivatives of the Euclidean norm, as presented next:

$$\mathbf{D}' = \left[\frac{\partial d}{\partial \rho'_{km}}\right] = -\frac{1}{d}\widetilde{\mathbf{Y}}\widetilde{\mathbf{Y}}^T;$$

$$\mathbf{D} = \left[\frac{\partial d}{\partial \rho'_{km}}\frac{\partial \rho'_{km}}{\partial \rho_{km}}\right] = \begin{bmatrix} 0 & & \text{sym} \\ -1.236 & 0 & \\ 3.698 & 1.148 & 0 \end{bmatrix}.$$

## 7. Conclusion

The Nataf correlation matrix derivatives with respect to the input correlation matrix and the derivatives of the marginal transformations involved in the multivariate distribution model definition, are given first. It has been demonstrated that the derivative of the recursive scheme of the Cholesky decomposition with respect to correlation coefficients, is also a recursive scheme. Exact derivatives in the spectral decomposition are possible but numerically inefficient. The approach based on perturbations is faster and quite accurate. The application of the Cholesky decomposition in sensitivity analysis could be more efficient but the spectral decomposition renders more information about the sensitivity of variances and principal directions of PDF. Efficient procedures not involving derivatives of the transformation matrices are developed for calculation of sensitivity matrices for distances and angles. Illustrative examples demonstrate the applicability of the given methods. The presented methods for sensitivity analysis in multivariate distribution models are intended for application in engineering problems involving stochastic systems, such as reliability analysis and optimization.

## Acknowledgement

The research was supported by the Scientific Fund of the Republic of Croatia under grant no. 2-09-298.

## Appendix A

The relation of $\rho$ and $\rho'$ is uniquely expressed as $\rho'_{ij} = \rho_{ij} F_{ij}$, see [3] for commonly used two-parametric distributions. The derivatives of the terms in the Nataf correlation matrix can be expressed in general as shown:

$$\frac{\partial \rho'_{ij}}{\partial \rho_{ij}} = F_{ij} + \rho_{ij}\frac{\partial F_{ij}}{\partial \rho_{ij}}, \qquad i,j = 1,2,\ldots,n;$$

Two groups of two prametric distributions are considered.

| **Group 1** | | **Group 2** | |
|---|---|---|---|
| U | – Uniform | LN | – Lognormal |
| SE | – Shifted exponential | GM | – Gamma |
| SR | – Shifted Rayleigh | T2L | – Type-II largest value |
| T1L | – Type-I largest value | T3S | – Type-II smallest value |
| T2S | – Type-II smallest value | | |

There are five categories of formulae for $F$. In some cases, the function $F$ depends on the correlation coefficient $\rho$ and also on the coefficient of variation $\delta$. The values for $\frac{\partial F_{ij}}{\partial \rho_{ij}}$ are given for each of the categories.

(I Cat.) $F_{ij}$ =const. for $x_j$ belonging to group 1 and $x_i$ normal: $\frac{\partial F_{ij}}{\partial \rho_{ij}} = 0$.

(II Cat.) $F_{ij} = F(\delta_{ij})$ for $x_j$ belonging to group 2 and $x_i$ normal: $\frac{\partial F_{ij}}{\partial \rho_{ij}} = 0$.

(III Cat.) $F_{ij} = F(\rho_{ij})$ for both $x_i$ and $x_j$ belonging to group 1:

| $\downarrow x_j$ $\rightarrow x_i$ | U | SE | SR | T1L | T1S |
|---|---|---|---|---|---|
| U | $-.094\rho$ | | | | |
| SE | $+.058\rho$ | $-.367 + .306\rho$ | | | |
| SR | $-.016\rho$ | $-.100 + .042\rho$ | $-.029$ | | |
| T1S | $+.030\rho$ | $+.154 + .062\rho$ | $+.045 + .012\rho$ | $+.069 + .010\rho$ | |
| T1S | $+.030\rho$ | $+.154 + .062\rho$ | $+.045 + .012\rho$ | $+.069 + .010\rho$ | $-.069 + .010\rho$ |

(IVa Cat.) $F_{ij} = F(\rho_{ij}, \delta_j)$ for $x_i$ belonging to group 1 and $x_j$ to group 2:

| $\downarrow x_j$ $\rightarrow x_i$ | U | SE | SR |
|---|---|---|---|
| LN | $+.020\rho$ | $+.003 + .050\rho - .437\delta_j$ | $+.001 + .008\rho - .130\delta_j$ |
| GM | $+.004\rho$ | $+.003 + .028\rho - .296\delta_j$ | $+.001 + .004\rho - .090\delta_j$ |
| T2L | $+.148\rho$ | $-.152 + .260\rho - .728\delta_j$ | $-.038 + .056\rho - .229\delta_j$ |
| T3S | $-.010\rho$ | $+.145 + .020\rho - .467\delta_j$ | $+.042 - .136\delta_j$ |

(IVb Cat.) $F_{ij} = F(\rho_{ij}, \delta_j)$ for $x_i$ belonging to group 1 and $x_j$ to group 2:

| $\downarrow x_j$ $\rightarrow x_i$ | T1L | T1S |
|---|---|---|
| LN | $+.001 + .008\rho - .197\delta_j$ | $-.001 + .086\rho + .197\delta_j$ |
| GM | $+.001 + .006\rho - .132\delta_j$ | $-.001 + .006\rho + .132\delta_j$ |
| T2L | $-.060 + .040\rho - .332\delta_j$ | $-.001 + .040\rho + .332\delta_j$ |
| T3S | $+.065 + .006\rho - .211\delta_j$ | $-.065 + .006\rho + .211\delta_j$ |

(Va Cat.) $F_{ij} = F(\rho_{ij}, \delta_i, \delta_j)$ for both $x_i$ and $x_j$ belonging to group 2:

| $\downarrow x_j$ $\rightarrow x_i$ | LN | GM |
|---|---|---|
| GM | $+.033 + .004\rho - .104\delta_i - .119\delta_j$ | $+.022 + .002\rho - .077(\delta_i + \delta_j)$ |
| T2L | $+.082 + .036\rho - .441\delta_i - .277\delta_j$ | $+.056 + .024\rho - .313\delta_i - .182\delta_j$ |
| T3S | $+.052\rho$ | $+.034 + .006\delta_i - .111\delta_j$ |

(Vb Cat.) $F_{ij} = F(\rho_{ij}, \delta_i, \delta_j)$ for both $x_i$ and $x_j$ belonging to group 2:

| $\downarrow x_j$ $\rightarrow x_i$ | T2L | T3S |
|---|---|---|
| T2L | $+.054 - .110\rho - .0602\rho^2 - .570(\delta_i + \delta_j) +$ $+ .514\rho(\delta_i + \delta_j) - .371(\delta_i^2 + \delta_j^2)$ | |
| T3S | $+.146 + .026\rho + .005\delta_i - .481\delta_j$ | $-.004 - .002\rho - .005(\delta_i + \delta_j)$ |

For both $x_i$ and $x_j$ lognormally distributed:

$$\frac{\partial F_{ij}(\rho,\delta_i,\delta_j)}{\partial \rho} = F_{ij}\cdot\left[\frac{\delta_i\delta_j}{(1+\rho\delta_i\delta_j)\cdot \ell n(1+\rho\delta_i\delta_j)} - \frac{1}{\rho}\right]$$

## APPENDIX B

The Cholesky decomposition is of the form $\mathbf{R} = \mathbf{L}\mathbf{L}^T$ or $\mathbf{R}^{-1} = \mathbf{M}^T\mathbf{M}$. The elements $\lambda_{ij}$ , $i = 1,2,\dots,n$; $j = 1,2,\dots,i$, of the lower-triangular matrix $\mathbf{L}$ are as follows:

$$\lambda_{ii} = \left(\rho_{ii} - \sum_{r=1}^{i-1}\lambda_{ir}^2\right)^{\frac{1}{2}} \tag{B-1}$$

$$\lambda_{ij} = \frac{1}{\lambda_{jj}}\left(\rho_{ij} - \sum_{r=1}^{j-1}\lambda_{ir}\lambda_{jr}\right) \quad \text{for } i>j;\ (\lambda_{ij}=0 \text{ for } i<j.) \tag{B-2}$$

The elements $\mu_{ij}$ , $i = 1,2,\dots,n$; $j = 1,2,\dots,i$, of the matrix $\mathbf{M} = \mathbf{L}^{-1}$ can be determined as follows:

$$\mu_{ii} = \frac{1}{\lambda_{ii}} \tag{B-3}$$

$$\mu_{ij} = -\frac{1}{\lambda_{ii}}\sum_{r=1}^{i-1}\lambda_{ir}\mu_{rj} \quad \text{for } i>j;\ (\mu_{ij}=0 \text{ for } i<j.) \tag{B-4}$$

The derivatives of the elements of the matrix $\mathbf{L}$ can be determined in recursion as follows:

$$\frac{\partial \lambda_{ii}}{\partial \rho_{km}} = \frac{1}{\lambda_{ii}}\sum_{r=1}^{i-1}\lambda_{ir}\frac{\partial \lambda_{ir}}{\partial \rho_{km}} \tag{B-5}$$

$$\frac{\partial \lambda_{ij}}{\partial \rho_{km}} = \frac{1}{\lambda_{jj}}\left[\frac{\partial \rho_{ij}}{\partial \rho_{km}} - \frac{\rho_{ij}}{\lambda_{jj}}\frac{\partial \lambda_{jj}}{\partial \rho_{km}} - \sum_{r=1}^{i-1}\left(\lambda_{jr}\frac{\partial \lambda_{ir}}{\partial \rho_{km}} + \lambda_{ir}\frac{\partial \lambda_{jr}}{\partial \rho_{km}} - \frac{\lambda_{ir}\lambda_{jr}}{\lambda_{jj}}\frac{\partial \lambda_{jj}}{\partial \rho_{km}}\right)\right]. \tag{B-6}$$

The derivatives of the elements of the matrix $\mathbf{M}$ can be determined in recursion as follows:

$$\frac{\partial \mu_{ii}}{\partial \rho_{km}} = -\frac{1}{\lambda_{ii}^2}\frac{\partial \lambda_{ii}}{\partial \rho_{km}} \tag{B-7}$$

$$\frac{\partial \mu_{ij}}{\partial \rho_{km}} = -\frac{1}{\lambda_{ii}}\left[\sum_{r=j}^{i-1}\left(\mu_{rj}\frac{\partial \lambda_{ir}}{\partial \rho_{km}} + \lambda_{ir}\frac{\partial \mu_{rj}}{\partial \rho_{km}} - \frac{\lambda_{ir}\mu_{rj}}{\lambda_{ii}}\frac{\partial \lambda_{ii}}{\partial \rho_{km}}\right)\right], \quad i>j. \tag{B-8}$$

The derivatives of the elements of the matrix $\mathbf{R}^{-1}$ can be obtained as follows:

$$\frac{\partial \rho_{km}^{-1}}{\partial \rho_{km}} = \sum_{i=k}^{n}\left(\mu_{ik}\frac{\partial \mu_{im}}{\partial \rho_{km}} + \mu_{im}\frac{\partial \mu_{ik}}{\partial \rho_{km}}\right). \tag{B-9}$$

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