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Global numerical analysis of elasto-plastic frames[1]

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We discuss a global, iteration-free numerical scheme (based on the Piecewise Linear algorithm), with special respect to the computation of elasto-plastic frames. The plastic deformations are concentrated in plastic hinges which may appear at both ends of the bars, while the inner parts of the bars can have only elastic deformations but without limation on their magnitude.

Our method is tested on a classical example and the results show very good match with those known from the literature. We discuss advantages and disadvantages and point out other, related applications.

1. Introduction

In this paper we investigate the global behaviour of elasto-plastic frames, applying an iteration-free path-following scheme which is based on the PL (Piecewise Linear) algorithm [1]. Our method for elastic frames was introduced in two earlier papers [7, 8], here we merely outline the main concept and then turn our attention to the extension of the method for the analysis of elasto-plastic frames.

As opposed to "local" (or, "approximate") methods with built-in limitations, global numerical techniques set the bold goal of following branches for arbitrary intervals, with arbitrary precision, or, to find *all* solutions of an equation system in a given domain. In structural mechanics these equation systems are often large and highly nonlinear. The reduction of the problem to an algebraic equation system is non-trivial, either. This paper is restricted to the analysis of bar structures, where the individual bars can be modeled by Boundary-Value Problems (BVPs) corresponding to (nonlinear) Ordinary Differential Equations (ODEs). The bar structure is referred to as a frame. The material behaviour along the bars is assumed to be elastic. At nodes (intersection of several bars, free ends, location of concentrated loads) plastic behaviour will be also admitted. Traditional global methods rely on iterative-incremental algorithms, hence problems with convergence often arise. Our method, which we call the Path Following Simplex Method (PFSM) — not to be confused with the famous algorithm in linear programming — is free of iteration and relies on an "organized" shooting technique. The BVPs are converted into a parameter-dependent Initial-Value Problems (IVPs), or, as input-output devices: the input data are the initial conditions, the output data are the values of the same variables at the far end of the bar. Using any convergent integrator for the IVP, the output is determined with arbitrary precision as a function of the input. Of course, certain intial conditions are constant, fixed by the boundary conditions at the origin. We will call the non-constant initial conditions variables. The variables can be regarded as unknown quantities, needed to integrate the IVP. If we regard a sequence of IVPs, then the input for the subsequent IVP is inherited partially from the output of the previous one, if necessary, new variables have to be added. The matching conditions at intermediate points and the boundary conditions at the far end can be formulated as functions of the output scalars, thus, as functions of our variables. This means, that by using the numerical integrator, our multi-point BVP can be reduced to a system of nonlinear algebraic equations.

[1]This work is dedicated to Professor Franz Ziegler on the occasion of his 60th birthday

The PFSM solves this one-parameter equation system in the following manner: the space spanned by the variables is subdivided into (small) simplices along the equilibrium path, the function values are computed at the vertices and the equilibrium path is linearly interpolated inside the simplices. This means that the PFSM uses interpolation instead of extrapolation and provides the (approximate) equilibria in a direct recursion. It always delivers one subsequent point, but definitely not more than one. Since the continuity (C^0) of the equilibrium paths is preserved in the piecewise linear approximation, when approaching a bifurcation point on the primary path, the algorithm selects one of the secondary "exit" possibilities practically at random and continues along the selected path. We remark that there is a Global Simplex method which is suitable for a more general task: finding all globally possible equilibria in a given domain, as described by [5, 7].

In order to run the PFSM on a BVP (regarded as a sequence of elementary BVPs) we have to specify

- the variables, i.e. those initial conditions and parameters for all elementary BVPs which are not fixed by the boundary conditions at the initial points and not inherited from the previous segment, in addition, those endpoint locations which are not fixed in advance,
- the functions, i.e. the matching conditions at intermediate points and the boundary conditions at the far end and
- the values of the variables at one equilibrium configuration.

In the next section we will describe how this information can be obtained for elasto-plastic frames. Since we expect one-parameter families of solutions, the number V of variables has to exceed the number F of functions by one.

2. Extension to elasto-plastic analysis

Structural response is characterized not only by geometrical but many times also by physical nonlinearity. An important class of such problems is the elasto-plastic analysis of frames. The plastic behaviour can be modeled conveniently in this case by the so called plastic hinge assumption. This means that plastic flow can be considered as restricted to certain cross sections which are called critical sections. In the case of frames loaded only at their nodes when the deformation of the individual members is not large and thus the bending moment reaches its maxima at the extremities of the members, it is sufficient to consider critical sections only at those places, which is the case in the present study. The plastic hinge assumption means that the yield condition depends only on one component of the internal forces which is the bending moment and that the corresponding generalized plastic deformation is the relative rotation in the yielded cross section. In this case, the yield condition and the associated flow rules for the whole structure are the following (for details see [3]):

$$\boldsymbol{\varphi} = \boldsymbol{Nm} - \boldsymbol{m}_p, \qquad I : \{i|\varphi_i = 0\}, \qquad \dot{\boldsymbol{\vartheta}}_I = \boldsymbol{N}_I^T \dot{\boldsymbol{\lambda}}_I, \qquad \dot{\boldsymbol{\varphi}}_I^T \dot{\boldsymbol{\lambda}}_I = 0,$$

$$\dot{\boldsymbol{\lambda}}_I \geq \boldsymbol{0}, \qquad \dot{\boldsymbol{\lambda}}_{\bar{I}} = \boldsymbol{0}, \qquad \boldsymbol{\varphi} \leq \boldsymbol{0}, \qquad \dot{\boldsymbol{\varphi}}_I \leq \boldsymbol{0},$$

where:

$\boldsymbol{\varphi}$ — vector of plastic potentials of size $2 * nc$ (nc is the number of critical sections).

$\boldsymbol{N}$ — yield matrix of size $(2 * nc) \times nc$. It is a block-diagonal matrix with identical blocks $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\boldsymbol{m}$ — vector of bending moments in the critical sections of size nc.

$\boldsymbol{m}_p$ — vector of positive and negative plastic limit moments in the critical sections of size $2 * nc$.

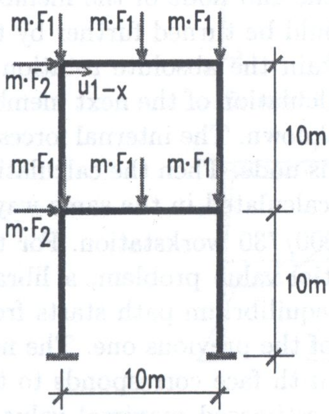
I — index set of active yield locations, $\bar{I}$ is the indexed set of the non active yield locations. When used as a subscript, they refer to the appropriate subvectors and submatrices.

ϑ — vector of relative rotations in the critical sections of size nc.
λ — vector of plastic multipliers in the critical sections of size $2 * nc$.

The dot refers to the derivative of the variables with respect to time. In quasi-static, perfect plasticity time can be replaced by any strictly monotonous variable, e.g. by the arclength of the equilibrium path. Superscript T means transposition.

It can be seen from these relations that there are sign constraints on the governing variables and on their rates. As the simplex method proceeds by small steps on the curve of the equilibrium path, these requirements can be checked at each simplex step and by this way their violation can be detected immediately and kept within the precision of the method. This also means that the validity of set I, which is the index set of the active yield locations, can be examined at every simplex step and the set can be updated if necessary i.e. the opening and the closing of the plastic hinges can be monitored in a convenient way.

The nature of the physical nonlinearity in this problem is different from that of the geometrical one. While the geometrical nonlinearity can be considered as a smooth one, the physical nonlinearity presents a non-smooth characteristic. This comes from the fact that the activation and deactivation of the yield locations changes the number or the nature of the governing variables in the simplex method. This fact means that instead of having a single system of one parameter nonlinear equations there is a series of such equations. The starting solution of each system is the last valid solution of the previous one. The application of the method is shown on an example (Fig. 1) which has been already studied by other authors [9, 2, 4, 10].



$$E = 2.1\text{E}8\ \text{kNm}^2$$
$$A = 2.0\text{E}-3\ \text{m}^2$$
$$I = 2.0\text{E}-5\ \text{m}^4$$
$$M = 50.6\ \text{kNm}$$
$$F_1 = 2.89100\ \text{kN}$$
$$F_2 = 0.72275\ \text{kN}$$

Fig. 1. The investigated frame, geometrical data and physical constants

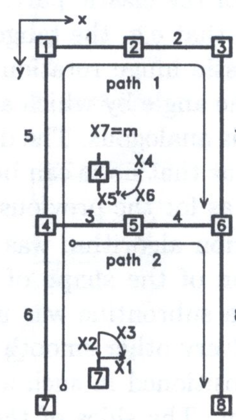


Fig. 2. Integration paths, nodes, members and variables

Where two critical sections refer to the same cross section (e.g.: the end of member 1 and the beginning of member 2 in Fig. 2), the plastic limit moment of one of them was taken greater to avoid meaningless multiplication of variables, which would cause the breakdown of the algorithm. The 7 variables ($V = 7$) for the elastic solution are shown in Fig. 2. The first three are the internal forces at the bottom of member 6, the second three are the internal forces of member 3 at the beginning. They are defined in the global coordinate system of the structure. The seventh variable is the load parameter. The number of the corresponding functions is 6 ($F = 6$). The first 3 functions express the requirement that the displacements of node 6 calculated by the two different paths (1, 2) should be the same. The second 3 functions express the fact that the displacements of node 8 are zero.

The opening of a plastic hinge generally represents one more variable i.e. the relative rotation in the hinge and one more equation i.e. the bending moment in the hinge should coincide with the plastic limit moment of the cross section taken by the appropriate sign. If the hinge is formed in a section where the internal force variables are defined, there is no increase in the number of variables and equations because the internal force variable which corresponds to the bending moment is changed to the variable which represents the rotation in the plastic hinge. The closing of a plastic hinge is treated similarly. The difference is that in this case the number of variables and equations can be decreased.

At a given value of the variables, the displacements and the internal forces of the structure are determined in the following way. The beginning and the end of a member is defined by the sense of the path which contains the member (Fig. 2).

The calculation is done member by member. First members along path 1 (members: 6, 5, 1, 2, 7), then members along path 2 (members: 3, 4, 8) are calculated. Node 7 is fixed, so its displacements are zero. By adding the relative rotation in the critical section at the beginning of member 6 (if any), the displacements of the beginning of the elastic part of the member are known. To determine the orientation of the local coordinate system at the beginning cross section of the elastic part one has also to take into consideration the properties of the initial geometry. It means that for members 5, 6 one has to add $(-\pi/2)$ and for members 7, 8 $(+\pi/2)$ rotational eccentricity to the absolute rotation of the beginning elastic cross section to get the angle of the local coordinate system with respect to the global. (This means that e.g. the direction of the first axis of the local coordinate system at the beginning point of member 6 is obtained by turning the global x axis with the absolute rotation of node 7 and then by turning it further by the eventual plastic hinge rotation at the beginning of member 6 and finally by turning it still by $-\pi/2$ radian.) Knowing the first three internal force variables one can determine the internal forces in the local coordinate system by appropriate transformation. The displacements of the end of the elastic part then can be determined by solving a first order nonlinear initial value problem (cf. [7]). Finally, adding the relative rotation in the end critical section and the appropriate rotational eccentricity $(+\pi/2)$ to the rotation of the end section of the elastic part, one can obtain the displacements of the end node of the member. (This means that e.g. the tangent at the elastic end of member 6 should be turned further by the eventual plastic hinge rotation there and then by $\pi/2$ radian to obtain the absolute rotation of node 4 i.e. the angle by which an axis fixed to the node turns.) The calculation of the next member (member 5) is analogous. The displacements of its beginning node are known. The internal forces of the member at that node can be calculated from the equilibrium of this node. Then the calculation can proceed as for the previous member. The other members can be calculated in the same way.

The solution algorithm was programmed in Fortran on an HP 9000/730 workstation. For the determination of the shape of the members which represent an initial value problem, a library Runge–Kutta subroutine was used. The first smooth interval of the equilibrium path starts from the origin. Every other smooth interval starts from the last solution of the previous one. The new simplex is positioned in such a way that the center of gravity of its n-th face corresponds to the starting point. The sides of the simplex were taken as 1/200 of the estimated maximal value of the corresponding variables. Using these values, the numerical experience showed that violation of the yield condition was limited to 1/10000 of the value of the plastic limit moment. It was more difficult to handle numerically the closing of the plastic hinges. In order to avoid a fluctuation of consecutive closing and opening of a hinge, two constraints were used. A hinge was closed if either of the following conditions were satisfied: the relative rotation in the hinge decreased by 1 percent of its maximal value or decreased by 0.00015. These two constraints were necessary as there are considerable differences in magnitude between the hinge rotations. Figures 3–5 show the results of the analysis.

Figure 3 shows the displacements, the bending moment diagrams of states (A–I) where opening or closing of the plastic hinges happen together with the value of the load parameter (m). The last three states illustrate only the final collapse mechanism of the structure. The hinges are numbered in the order of their formation. If a hinge is closed its number is placed in parentheses. The reopening of a hinge is marked by an exclamation mark. Figure 4 shows the evaluation of the relative rotations in the plastic hinges. The same notation is used in this figure as in Fig. 3 for the numbering of the plastic hinges and the labelling of the states where hinge opening and closing occur. Table 1 shows some main characteristics of those states, namely: the number of variables for the next smooth equilibrium path interval, the number of simplex steps made on this interval, the load parameter and the horizontal displacement at node 1 at the beginning of the interval and finally the status of

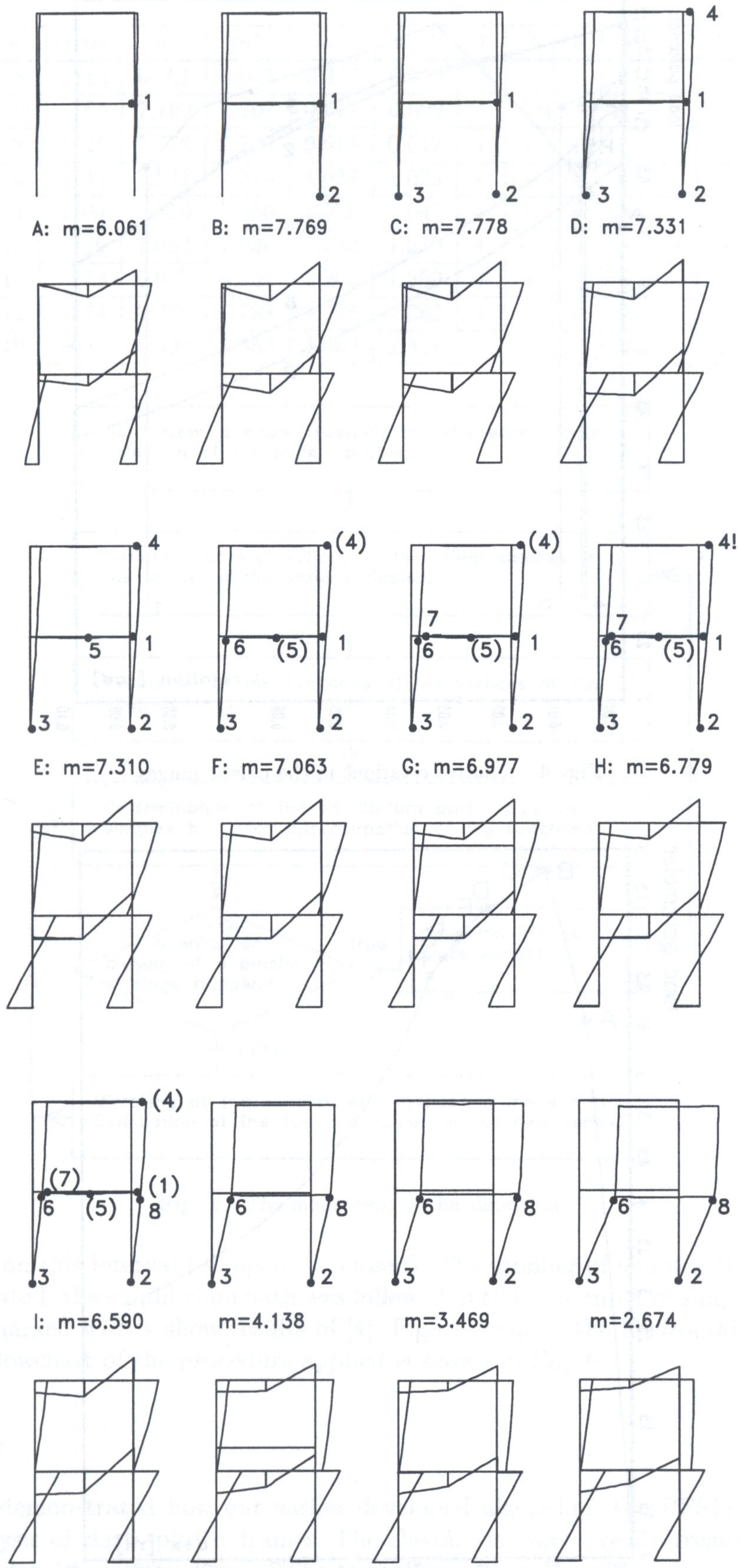


Fig. 3. Displacements, bending moments, location of plastic hinges

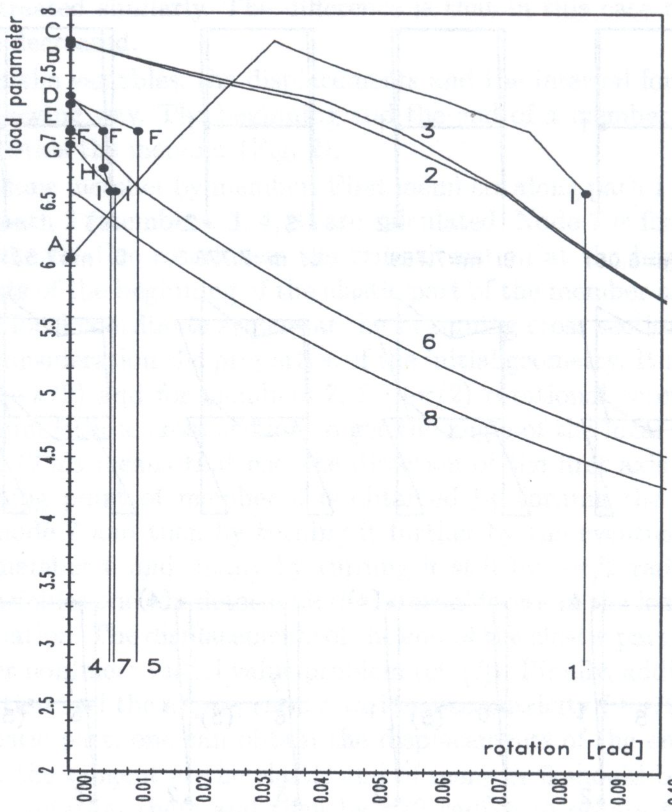


Fig. 4. Relative rotations in the plastic hinges

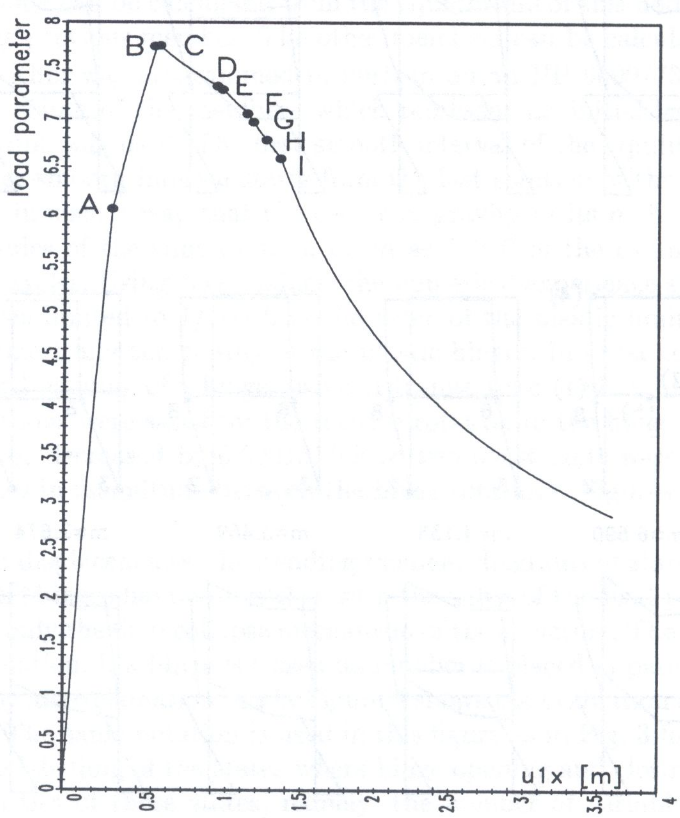


Fig. 5. Load parameter versus horizontal displacement of node 1

Table 1. Characteristics of critical states

state	vars.	steps	m	m^*	u_{1x}	$u_{1x}{}^*$	1	2	3	4	5	6	7	8
A	8	2210	6.061	6.045	0.317	0.316	+							
B	9	65	7.769	7.707	0.612	0.609	+	+						
C	9	1725	7.778	7.746	0.615	0.612	+	+	+					
D	10	143	7.331	7.319	1.033	1.025	+	+	+	+				
E	11	1204	7.310	7.286	1.055	1.047	+	+	+	+	+			
F	10	426	7.063	7.030	1.220	1.219	+	+	+	−	−	+		
G	10	779	6.977	6.947	1.262	1.259	+	+	+	−	−	+	+	
H	11	874	6.778	6.758	1.353	1.362	+	+	+	+	−	+	+	
I	10	9630	6.590	6.585	1.452	1.448	−	+	+	−	−	+	−	+

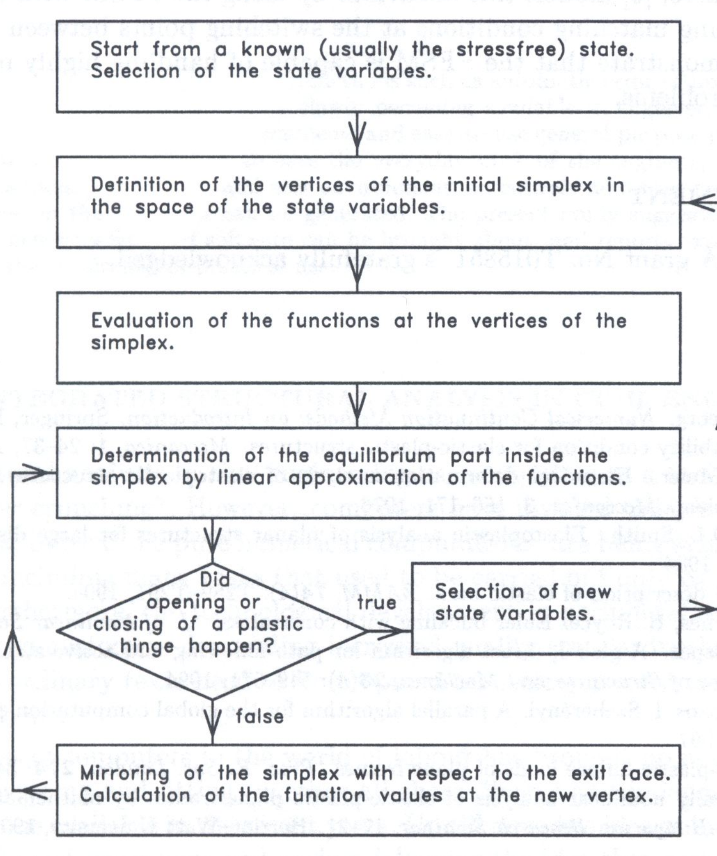


Fig. 6. The main steps of the algorithm

the plastic hinges on this interval (+: open, −: closed). The number of simplex steps until state A was 2944. After state I, the equilibrium path was followed until the number of simplex steps reached 20000. Columns marked with $*$ show results of [4]. Figure 5 shows the horizontal displacement of node 1. A rough flowchart of the procedure applied is shown in Fig. 6.

3. Conclusion

In this paper we demonstrated how our earlier developed algorithm, the PFSM, can be applied to the global analysis of elasto-plastic frames. The elastic case was already treated by [7]. Plastic behaviour was modeled by plastic hinges. The appearance of such a plastic hinge meant from the point of view of the PFSM the introduction of a new variable, or, the physical reinterpretation of an old one. As we followed the equilibrium path, at the formation of each new hinge (also, at the closing of each hinge) a new model was introduced, inheriting its starting configuration from the last configuration of the previous model. In general, the dimension V of these models did not coincide, so the preservation of the continuity of the equilibrium path required special attention. The example illustrated in this paper is a classic one, often used to check new numerical techniques. Our computations show a remarkably good match with all earlier data, moreover, we could follow the equilibrium path further than any of the cited authors.

It is worth mentioning that, although physically rather different, another application of the PFSM shows similar mathematical aspects. [6] investigates the buckling of a slender, elastic rod between parallel constraining walls, under various boundary conditions. Initially the problem is identical with the classical Euler buckling problem, however, as the rod touches one of the constraining walls, a singularity enters. Along the equilibrium path contact is established and lost several times, in complete analogy to the appearance and disappearance of plastic hinges in our present problem. As here, [6] models this behaviour by using the PFSM with different numbers V of variables and meeting matching conditions at the switching points between the models.

Both examples demonstrate that the PFSM is capable of handling highly nonlinear, piece-wise smooth mechanical problems.

4. Acknowledgement

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References

[1] E.L. Allgower, K. Georg. *Numerical Continuation Methods: an Introduction.* Springer, Berlin, 1990.

[2] L. Corradi. On a stability condition for elastic-plastic structures. *Meccanica,* **1**: 24–37, 1977.

[3] O. De Donato, G. Maier. Historical deformation analysis of elastoplastic structures as a parametric linear complementary problem. *Meccanica,* **3**: 166–171, 1976.

[4] J.A.T. de Freitas, D.L. Smith. Elastoplastic analysis of planar structures for large displacements. *J. Struct. Mech.,* **12**: 419–445, 1984.

[5] G. Domokos. Global description of elastic bars. *ZAMM,* **74**(4): T289–T291, 1994.

[6] G. Domokos, P. Holmes, B. Royce. Euler buckling with constraints. *J. of Nonlinear Science,* [submitted].

[7] G. Domokos, Zs. Gáspár. A global, direct algorithm for path-following and active static control of elastic bar structures. *Mechanics of Structures and Machines,* **23**(4): 549–571, 1994.

[8] Zs. Gáspár, G. Domokos, I. Szeberényi. A parallel algorithm for the global computation of elastic bar structures. *CAMES,* **4**: 55–68, 1997.

[9] M.R. Horne. Elastic-plastic failure loads of plane frames. *Proc. R. Soc. London,* **274**: 343, 1963.

[10] P. Nédli. Geometrically nonlinear analysis of elastic-plastic plane frames by mathematical programming. In: *Proc. Scottish-Anglo-Hungarian Research Seminar*, 17–21. Herriot–Watt University, 1994.