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Domain optimisation using Trefftz functions — application to free boundaries

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Trefftz approximations are known to require very few degrees of freedom to give an order of magnitude of the solution. In this paper, we show that it is possible to take advantage of this situation in two ways: (i) we show that it is also possible to get accurate solutions (especially for the pressure) at a reasonable cost and (ii) we show that the low number of degrees of freedom needed for this accuracy allows an easy domain optimisation, which is illustrated here for the free boundary problem in extrusion. A construction of Trefftz polynomials associated with Stokes problem for plane strains is also given with some recurrence properties which is usefull for computing them at a low cost. Moreover a domain decomposition method which has shown to be efficient for compressible elastic material has been extended here to the case of incompressible linear viscous fluids.

1. Introduction

The need for computation becomes more precise in a sens that it becomes now possible to deal with optimisation associated with problems of continuum mechanics. We are far nowaday to solve any real tridimensional industrial nonlinear problem concerning a structure optimisation. And if the finite element methods is certainly the most popular one in structural mechanics, intermediate methods such as Trefftz's approximations could be very helpfull for optimisation. Trefftz approximations are namely known to require very few degrees of freedom to give an order of magnitude of the solution and unable thus to achieve a much cheaper optimisation. This paper takes place in this framework and deals with incompressible linear viscous fluids. We show that it is possible to take advantage of Trefftz's approximations in two ways: (i) that it is possible to get accurate solutions (especially for the pressure) at a reasonable cost and (ii) that the low number of degrees of freedom needed for this accuracy allows an easy domain optimisation, which is illustrated here on the free boundary problem in extrusion. The first section is devoted to the construction of Trefftz polynomials associated with Stokes problem for plane strains and gives also some recurrence properties which are usefull for computing them at a low cost. The second section extends to Stokes equations some variational formulations which have already proved their efficiency in the framework of compressible linear elasticity. The last section is devoted to examples which are discussed in terms of classical well posed problems as well as free boundary problems.

2. Trefftz aproximations for plane strains

Trefftz polynomials are built in order to satisfy both the Stokes equations and the incompressibility condition. For a sake of simplicity, these equations have been adimensionalised here, so that in the following, the pressure must be understood as the real pressure devided by the viscosity.

$$\frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^n}{\partial y^2} = \frac{\partial p^n}{\partial x}, \tag{1}$$

$$\frac{\partial^2 v^n}{\partial x^2} + \frac{\partial^2 v^n}{\partial y^2} = \frac{\partial p^n}{\partial y}, \tag{2}$$

$$\frac{\partial u^n}{\partial x} + \frac{\partial v^n}{\partial y} = 0. \tag{3}$$

In equation (1, 2), if u^n and v^n are polynomials of order n, p^n is a polynomial of order $n-1$. Such functions can be found in a very similar manner as for plate bending problems [3] with use of classical properties of harmonic functions.

2.1. Some properties on harmonic polynomials

Let P_n and Q_n be the two following polynomials.

$$P_n(x,y) = \mathrm{Re}\,[(x+iy)^n], \qquad Q_n(x,y) = \mathrm{Im}\,[(x+iy)^n]. \tag{4}$$

Then the four following properties hold.

Property 1

$$\begin{aligned} &\Delta P_n = 0, \qquad \Delta Q_n = 0, \\ &\deg(P_n) = \deg(Q_n) = n. \end{aligned} \tag{5}$$

Property 2

$$\begin{aligned} &(x+iy)^n = (x+iy)(x+iy)^{n-1}, \\ &P_n = x P_{n-1} - y Q_{n-1}, \\ &Q_n = y P_{n-1} + x Q_{n-1}. \end{aligned} \tag{6}$$

Property 3

$$\begin{aligned} &\Delta(fg) = f\,\Delta(g) + 2\,\underline{\mathrm{grad}\,f}\cdot\underline{\mathrm{grad}\,g} + g\,\Delta(f), \\ &\Delta(x P_n) = 2n P_{n-1}, \qquad \Delta(x Q_n) = 2n Q_{n-1}, \\ &\Delta(y P_n) = -2n Q_{n-1}, \qquad \Delta(y Q_n) = 2n P_{n-1}. \end{aligned} \tag{7}$$

Property 4

$$\begin{aligned} &\frac{\partial P_n}{\partial x} = \frac{\partial Q_n}{\partial y} = n P_{n-1}, \\ &\frac{\partial P_n}{\partial y} = -\frac{\partial Q_n}{\partial x} = -n Q_{n-1}. \end{aligned} \tag{8}$$

2.2. Trefftz polynomials for Stokes equations

Let ψ be the stream function associated with the velocity field.

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{9}$$

ψ is a biharmonic function and according to the above properties, it can be developped on the following polynomials basis:

$$\begin{aligned} \psi_1^n &= \frac{1}{n+1} P_{n+1}, \qquad \psi_2^n = \frac{1}{n+1} Q_{n+1}, \\ \psi_3^n &= -x\,P_n, \qquad \psi_4^n = -y\,Q_n. \end{aligned} \tag{10}$$

A basis of Trefftz functions $\{\underline{u} = (u, v), p\} = V^n$ for which the velocity field $\underline{u}$ is a polynomial of degree n is then straightfully deduced:

$$\begin{aligned} u_1^n &= -Q_n, \quad v_1^n = -P_n, \quad p_1^n = 0, \\ u_2^n &= P_n, \quad v_2^n = -Q_n, \quad p_2^n = 0, \\ u_3^n &= n\,x\,Q_{n-1}, \quad v_3^n = P_n + n\,x\,P_{n-1}, \quad p_3^n = 2\,n\,Q_{n-1}, \\ u_4^n &= n\,x\,P_{n-1}, \quad v_4^n = -Q_n - n\,x\,Q_{n-1}, \quad p_4^n = 2\,n\,P_{n-1}. \end{aligned} \tag{11}$$

With use of the above properties, the associated stress tensors take a particularly simple form for which the derivatives are expressed analytically for any value of n:

$$\sigma_1^n = \begin{pmatrix} -2\,n\,Q_{n-1} & -2\,n\,P_{n-1} & 0 \\ -2\,n\,P_{n-1} & 2\,n\,Q_{n-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \sigma_2^n = \begin{pmatrix} 2\,n\,P_{n-1} & -2\,n\,Q_{n-1} & 0 \\ -2\,n\,Q_{n-1} & -2\,n\,P_{n-1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_3^n = \begin{pmatrix} 2n(n-1)xQ_{n-2} & 2nP_{n-1} + 2n(n-1)xP_{n-2} & 0 \\ 2nP_{n-1} + 2n(n-1)xP_{n-2} & -4nQ_{n-1} - 2n(n-1)xQ_{n-2} & 0 \\ 0 & 0 & -2nQ_{n-1} \end{pmatrix},$$

$$\sigma_4^n = \begin{pmatrix} 2n(n-1)xP_{n-2} & -2nQ_{n-1} - 2n(n-1)xQ_{n-2} & 0 \\ -2nQ_{n-1} - 2n(n-1)xQ_{n-2} & -4nP_{n-1} - 2n(n-1)xP_{n-2} & 0 \\ 0 & 0 & -2nP_{n-1} \end{pmatrix}.$$

It is to be noted that the dimension of V^n is independant of n and that the four basis vectors have been chosen here so that two of them exhibit a zero pressure field.

3. Formulations

Many different strategies [6, 7] and [8] are now available to compute solutions of partial differential equations with Trefftz functions (for a discussion of these methods see for example [6]). The method which is proposed here is straightfully deduced from [4] and will be discussed here in a slightly different way as what is usually done. It will be shown namely, that with a few more degrees of freedom than usual, it is possible to compute a precise solution instead of an approximated one. In a first paragraph, the one domain formulation is derived whereas in a second one, a multidomain approach is proposed.

3.1. One domain formulation

Let us consider the Stokes problem shown in Fig. 1. The velocity field is prescribed to $\underline{u}_p$ on $\partial_2\Omega$ and the stress vector is prescribed to $\underline{F}_p$ on $\partial_1\Omega$. Moreover, the body forces are assumed to be zero.

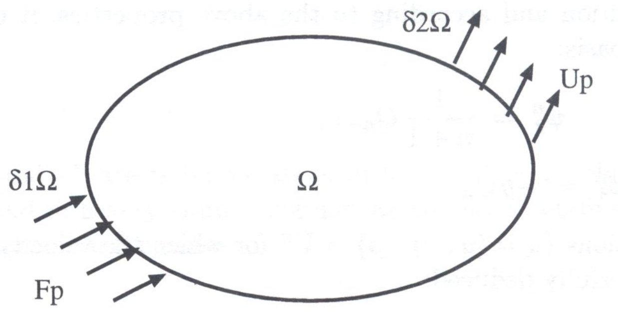

Fig. 1. Stokes problem

If the domain Ω is regular and if $\partial_1\Omega$ is not negligeable, such problem has a solution ($\underline{u}$, p) for which $\underline{u}$ is in $(H^1(\Omega))^2$ and p is in $L^1(\Omega)$. An aproximation of this exact solution can be searched as [4]:

$$\forall(\underline{u}^*,p^*)\in V^n\,,\quad \int_{\partial_2\Omega}(2\underline{\underline{d}}(\underline{u}^*)\cdot\underline{n}-p^*\underline{n})\cdot(\underline{u}-\underline{u}_p)\mathrm{d}S+\int_{\partial_1\Omega}\underline{u}^*\cdot(2\underline{\underline{d}}(\underline{u})\cdot\underline{n}-p\,\underline{n}-\underline{F}_p)\mathrm{d}S=0\,. \qquad (12)$$

As a solution of the Stokes problem (which exists) is solution of the formulation (12), if a solution of (12) is unique, it is the right solution. Let $(\underline{u}_1,p_1)$ and $(\underline{u}_2,p_2)$ be two solutions of (12), the difference $(\delta\underline{u},\delta p) = (\underline{u}_1-\underline{u}_2,p_1-p_2)$ is solution of the homogeneous problem associated with (12).

$$\forall(\underline{u}^*,p^*)\in V^n\,,\qquad \int_{\partial_2\Omega}(2\underline{\underline{d}}(\underline{u}^*)\cdot\underline{n}-p^*\underline{n})\cdot\delta\underline{u}\mathrm{d}S+\int_{\partial_1\Omega}\underline{u}^*\cdot(2\underline{\underline{d}}(\delta\underline{u})\cdot\underline{n}-\delta p\,\underline{n})\mathrm{d}S=0\,. \qquad (13)$$

That is, with $(\underline{u}^*,p^*) = (\delta\underline{u},\delta p)$

$$\int_\Omega 2\,\mathrm{Tr}\,(2\underline{\underline{d}}(\delta\underline{u})^2))\mathrm{d}\Omega \;=\; 0\,. \qquad (14)$$

Because $\partial_1\Omega$ is not negligeable, $\delta\underline{u} = \underline{0}$ and $\delta p = 0$, which shows the validity of the formulation. It is to be noted that no compatibility LBB condition is required as for a finite element calculation between the pressure and the velocity: the pressure functions are naturally associated with the velocity ones.

3.2. Multidomain formulation

Let us consider the Stokes problem posed on a multidomain. The domain Ω is divided into two subdomains Ω^1 and Ω^2. These two subdomains are connected through a surface Σ. The problem can then be written.

$$\underline{\Delta}(\underline{u}^i) \;=\; \underline{\mathrm{grad}}\,p^i \quad \text{on} \quad \Omega^i\,, \qquad (15)$$

$$-p^i\underline{n}^i+2\underline{\underline{d}}(\underline{u}^i)\cdot\underline{n}^i \;=\; \underline{F}_p^i \quad \text{on} \quad \partial_1\Omega^i\,, \qquad (16)$$

$$\underline{u} \;=\; \underline{u}_p^i \quad \text{on} \quad \partial_2\Omega^i\,, \qquad (17)$$

$$\left\{-p^1\underline{n}^1 + 2\underline{\underline{d}}(\underline{u}^1)\cdot\underline{n}^1\right\} + \left\{-p^2\underline{n}^2 + 2\underline{\underline{d}}(\underline{u}^2)\cdot\underline{n}^2\right\} = \underline{0} \quad \text{on} \quad \Sigma, \tag{18}$$

$$\underline{u}^1 = \underline{u}^2 \quad \text{on} \quad \Sigma. \tag{19}$$

Equations (18) and (19) hold for the velocity and stress vector continuity along Σ (with the notation $\underline{n}_1 + \underline{n}_2 = 0$). Following the method proposed in [3], a variational formulation of the problem is written as:

$$\forall(\underline{u}^{*i}, p^{*i}) \in V^n(\Omega^i)$$

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\partial_2\Omega^i}(-p^{*i}\underline{n}^i + 2\underline{\underline{d}}(\underline{u}^{*i})\cdot\underline{n}^i)\cdot(\underline{u}^i - \underline{u}^i_p)\mathrm{d}S\\
&\quad+\sum_{i=1}^{2}\int_{\partial_1\Omega^i}\underline{u}^{*i}\cdot(-p^i\underline{n}^i + 2\underline{\underline{d}}(\underline{u}^i)\cdot\underline{n}^i - \underline{F}^i_p)\mathrm{d}S\\
&\quad+\frac{1}{2}\int_\Sigma\left\{(-p^1\underline{n}^1 + 2\underline{\underline{d}}(\underline{u}^1)\cdot\underline{n}^1) - (-p^2\underline{n}^2 + 2\underline{\underline{d}}(\underline{u}^2)\cdot\underline{n}^2)\right\}\cdot(\underline{u}^{*1} + \underline{u}^{*2})\mathrm{d}S\\
&\quad+\frac{1}{2}\int_\Sigma(\underline{u}^1 - \underline{u}^2)\cdot\left\{(-p^{*1}\underline{n}^1 + 2\underline{\underline{d}}(\underline{u}^{*1})\cdot\underline{n}^1) + (-p^{*2}\underline{n}^2 + 2\underline{\underline{d}}(\underline{u}^{*2})\cdot\underline{n}^2)\right\}\mathrm{d}S = 0.
\end{aligned} \tag{20}$$

It is easily shown that the solution of (3.2) is unique. In order to show this uniqueness let us consider two solutions $(\underline{u}^i_1, p^i_1)$ and $(\underline{u}^i_2, p^i_2)$ for each domain. The difference $(\delta\underline{u}^i, \delta p^i) = (\underline{u}^i_1 - \underline{u}^i_2, p^i_1 - p^i_2)$ solves the homogenous problem associated with (20)

$$\forall(\underline{u}^{*i}, p^{*i}) \in V^n(\Omega^i)$$

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\partial_2\Omega^i}(-p^{*i}\underline{n}^i + 2\underline{\underline{d}}(\underline{u}^{*i})\cdot\underline{n}^i)\cdot\delta\underline{u}^i\mathrm{d}S\\
&\quad+\sum_{i=1}^{2}\int_{\partial_1\Omega^i}\underline{u}^{*i}\cdot(-\delta p^i\underline{n}^i + 2\underline{\underline{d}}(\delta\underline{u}^i)\cdot\underline{n}^i)\mathrm{d}S\\
&\quad+\frac{1}{2}\int_\Sigma\left\{(-\delta p^1\underline{n}^1 + 2\underline{\underline{d}}(\delta\underline{u}^1)\cdot\underline{n}^1) - (-\delta p^2\underline{n}^2 + 2\underline{\underline{d}}(\delta\underline{u}^2)\cdot\underline{n}^2)\right\}\cdot(\underline{u}^{*1} + \underline{u}^{*2})\mathrm{d}S\\
&\quad+\frac{1}{2}\int_\Sigma(\delta\underline{u}^1 - \delta\underline{u}^2)\cdot\left\{(-p^{*1}\underline{n}^1 + 2\underline{\underline{d}}(\underline{u}^{*1})\cdot\underline{n}^1) + (-p^{*2}\underline{n}^2 + 2\underline{\underline{d}}(\underline{u}^{*2})\cdot\underline{n}^2)\right\}\mathrm{d}S = 0.
\end{aligned} \tag{21}$$

For $(\underline{u}^{*i}, p^{*i}) = (\delta\underline{u}^i, \delta p^i)$ and by using Stoke's formula, (21) gives:

$$\sum_{i=1}^{2}\int_{\Omega^i}(-\delta p^i\underline{\underline{I}} + 2\underline{\underline{d}}(\delta\underline{u}^i)) : \underline{\underline{d}}(\delta\underline{u}^i)\mathrm{d}\Omega^i = 0 \Leftrightarrow \sum_{i=1}^{2}\int_{\Omega^i}2\underline{\underline{d}}(\delta\underline{u}^i) : \underline{\underline{d}}(\delta\underline{u}^i)\mathrm{d}\Omega^i = 0.$$

Because $\partial_1\Omega^i$ is not negligeable, $\delta p^i = 0$ and $\delta\underline{u}^i = \underline{0}$, which shows the uniqueness of the solution.

3.3. Free boundary formulation

A stationary free boundary problem is a problem where on a part $\partial_1\Omega$ of the boundary, a stress vector and a normal velocity are simultaneously prescribed. The problem is thus generally ill posed except for a special boundary geometry, which is to be found. There are many numerical strategies to calculate it [5,2, ...]. The easiest one consists in minimizing the functional J, with respect to the geometry, under the constraint that the velocity field is a solution of the Stokes equations [1]:

$$J = \int_{\partial_1\Omega} (\mathbf{v}\cdot\mathbf{n})^2 \mathrm{d}S\,. \tag{22}$$

The domain optimisation is generally difficult to achieve because of the high number of degrees of freedom required to solve the Stokes problem and because of the necessity to have a good evaluation of the pressure field, which is known to be non trivial with the finite element method. On a contrary, with a Trefftz's approximation, there is very few degrees of freedom and the pressure is naturally associated with the velocity, which makes the method to be potentially interesting. Therefore the solution is looked for in order to minimizing J under the constraints (12).

4. Results and discussions

4.1. One domain results

The first example is depicted in Fig. 2. A forging velocity is prescribed on the upper surface which is opposite to the velocity which is prescribed on the lower surface. The problem can thus be

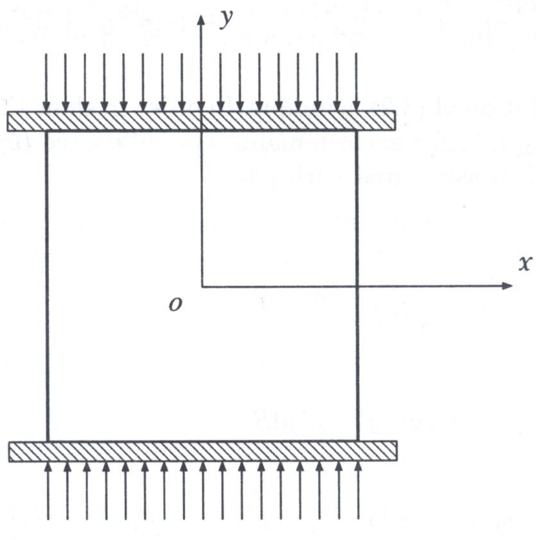

Fig. 2. Plane strain forging

solved on a quarter of the domain only. The magnitude of the prescribed velocity is 1/200 and the viscosity is 2/3. Figure 3 shows the velocity norm computed along the first diagonal $y = x$. Two interesting results have to be noted: (i) the velocity field is correctly computed in the center even for very few Trefftz degrees of freedom and (ii) the accuracy of the method becomes very good when the degree of approximation is increased from 5 to 30. The first conclusion confirms other results already obtained for linear compressible elasticity as well as for plate structure. It is due to both the capability of Trefftz functions to fit the solution as well as the particularly good quality of the variational formulation (12). The second conclusion is new and is confirmed in Fig. 4 where the pressure is plotted along the same diagonal. At the corner $x = y = 1$, the pressure is singular so that with the finite element method 10000 nodes (i.e. 30000 degrees of freedom where required to reach the convergence). With our Trefftz approximation the same level of accuracy is obtained with only 120 d.o.f.'s.

For the extrusion process (Fig. 5), the velocity field is prescribed to $\overline{u}$ at the entrance and at the wall of the die ($\partial_1\Omega$). The stress vector is prescribed to be zero on ($\partial_2\Omega$) as well as far from the die ($\partial_3\Omega$). In this section $\partial_2\Omega$ is not assumed to be a free surface, that is we do not prescribe

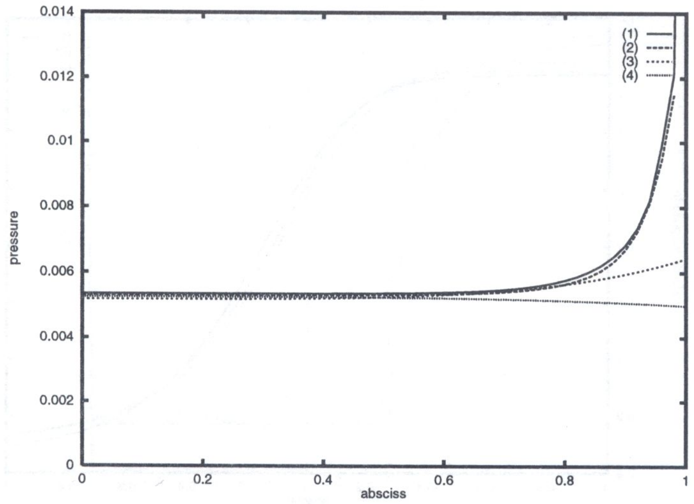

Fig. 3. Forging example: norm of the velocity — (1) Reference finite element calculation, (2) Trefftz approximation of order 30, (3) Trefftz approximation of order 10, (4) Trefftz approximation of order 5

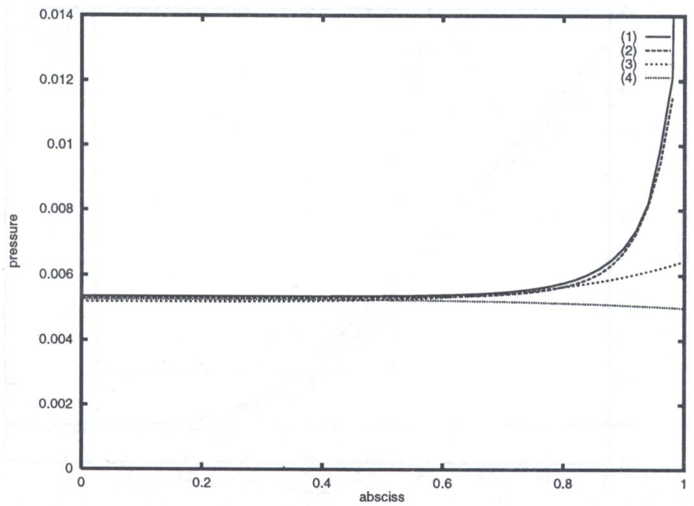

Fig. 4. Forging example: pressure — (1) Reference finite element calculation, (2) Trefftz approximation of order 30, (3) Trefftz approximation of order 10, (4) Trefftz approximation of order 5

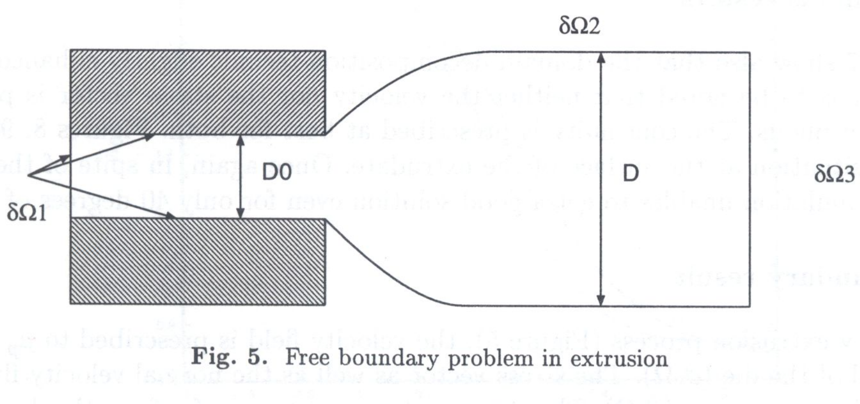

Fig. 5. Free boundary problem in extrusion

any stationary condition for this boundary or in other words no special condition is required on the velocity. Figures 6 and 7 show the norm of the velocity and the pressure along the axis of symetry $y = 0$. It can be seen that the quality of the solution is good with respect to the low number of d.o.f's and to the high shape factor of the domain.

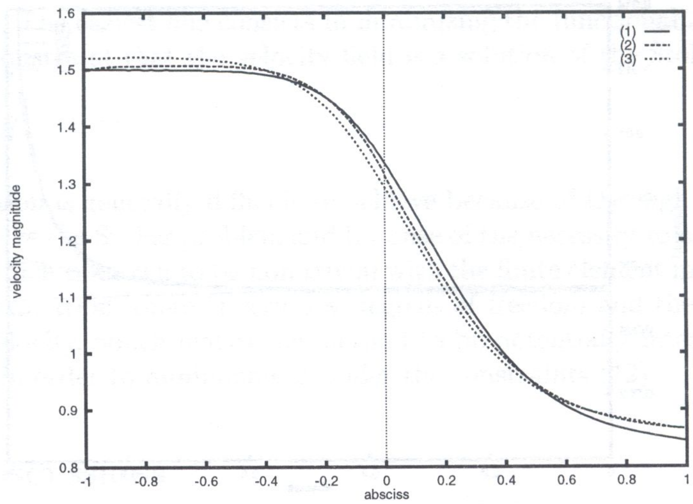

Fig. 6. Extrusion - Velocity norm along the axis of symetry — (1) Reference finite element calculation, (2) Two domains Trefftz approximation of order 15, (3) One domain Trefftz approximation of order 25

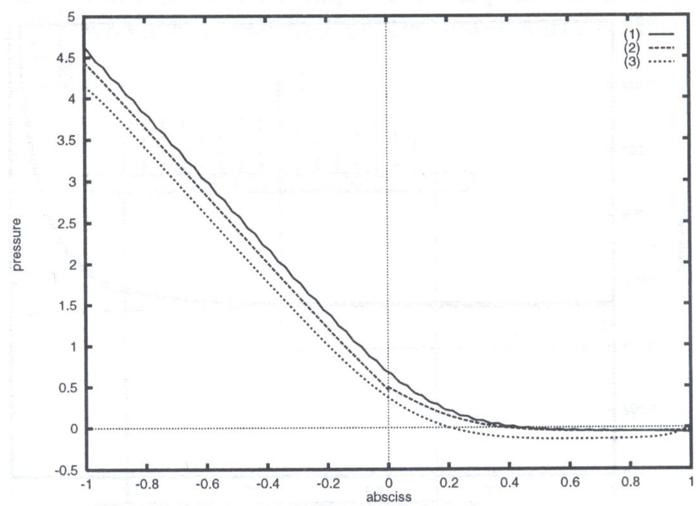

Fig. 7. Extrusion - Pressure along the axis of symetry (1) Reference finite element calculation, (2) Two domains Trefftz approximation of order 15, (3) One domain Trefftz approximation of order 25

4.2. Two domains results

Figures 6 and 7 show also that the domain decomposition technic slightly enhances the quality of the solution. It is to be noted that neither the velocity nor the stress vector is prescribed to be rigourously continuous. The continuity is prescribed at best for both. Figures 8, 9 and 10 show a more difficult situation at the surface of the extrudate. Once again, in spite of the singularity for $x = 0$, the formulation unables to get a good solution even for only 40 degrees of freedom.

4.3. Free boundary result

For a stationary extrusion process (Figure 5), the velocity field is prescribed to $\underline{u}_p$ at the entrance and at the wall of the die ($\partial_1\Omega$). The stress vector as well as the normal velocity flux is prescribed to be zero on the free surface ($\partial_2\Omega$). The stress vector is set to zero far from the die ($\partial_3\Omega$). This free boundary has been computed as a case study to see the potentiality of Trefftz aproximations within the framework of domain optimisation. Figure 11 shows the value of the die swell as a function of the degree of approximation: the convergence of the method is reached for only 50 degrees of freedom, whereas almost 5000 are needed with the FEM.

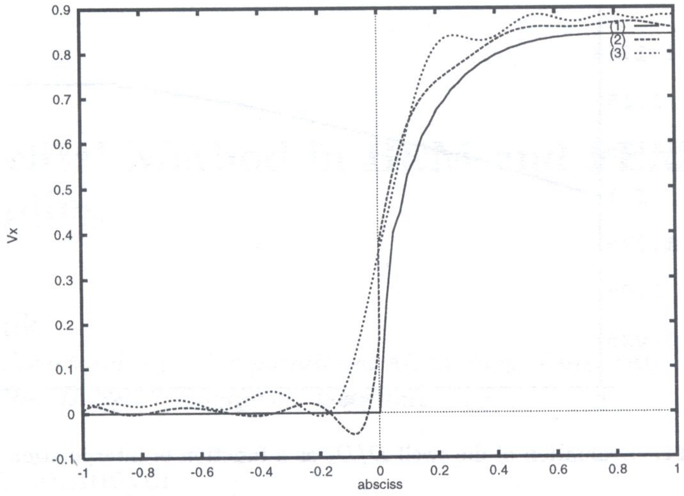

Fig. 8. Extrusion - Tangent velocity along the surface of the extrudate — (1) Reference finite element calculation, (2) Two domains Trefftz approximation of order 15, (3) One domain Trefftz approximation of order 25

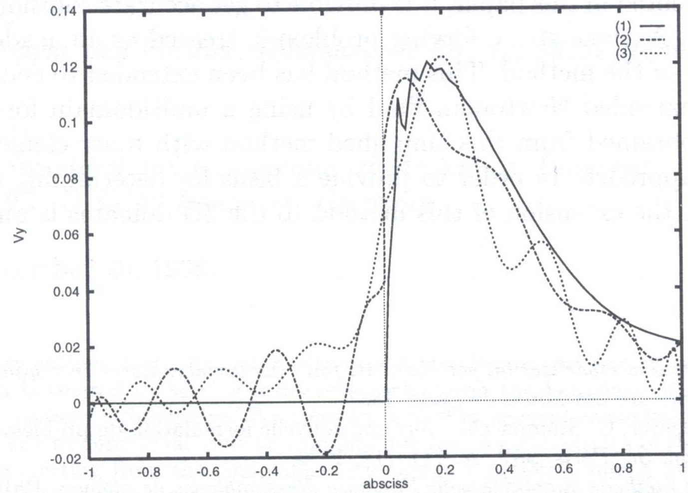

Fig. 9. Extrusion - Normal velocity along the surface of the extrudate — (1) Reference finite element calculation, (2) Two domains Trefftz approximation of order 15, (3) One domain Trefftz approximation of order 25

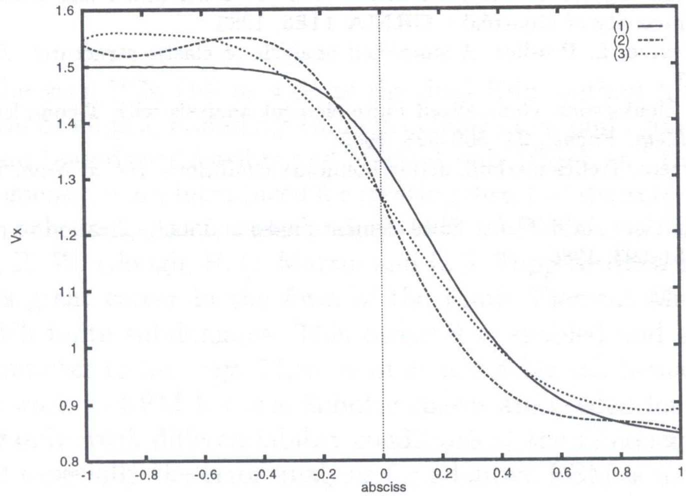

Fig. 10. Extrusion - Tangent velocity along the surface of the extrudate — (1) Reference finite element calculation, (2) Two domains Trefftz approximation of order 5, (3) One domain Trefftz approximation of order 10

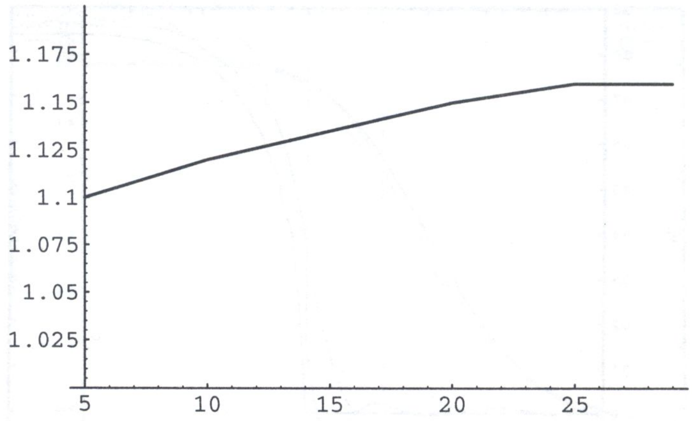

Fig. 11. Evaluation of die swell D/D_0 as a function of interpolation degree

5. Conclusion

With the method presented in this paper, it is possible to get accurate solutions with a lower number of degrees of freedom. A plane strain forging problem is treated as an academic example in order to show the effeciency of the method. This method has been extended to compute the pressure and velocity fields of an extruded Newtonian fluid by using a multidomain formulation. Comparison of numerical results obtained from this simplified method with finite element calculations proves the accuracy of this approach. In order to provide a basis for determining the solution of certain free-surface problems, the extension of this method to the 3D domains is under study.

References

[1] A. Bern. *Contribution à la modélisation par éléments finis des surfaces libres en régime permament.* PhD Thesis, Ecole Nationale Supérieure des Mines de Paris, 1987.

[2] H. Brezis, D. Kinderlehrer, G. Stampacchia. Sur une nouvelle formulation du problème de l'écoulement à travers une digue. *C. R. Acad. Sc. Paris*, **287, A**: 711–714, 1978.

[3] C. Hochard. *Sur une méthode simplifiée pour l'analyse d'assemblages de plaques.* PhD Thesis, Université Pierre et Marie Curie, France, 1991.

[4] P. Ladevèze. *R.D.M. tridimensionnelle.* Internal Report, LMT ENS de Cachan, France, 42, 53–61, 1983.

[5] J.P. Zolesio. *Numerical algorithms and existence results for Bernouilli's like steady free boundary problems.* Internal Report – University of Montréal – **CRMA 1183**, 1983.

[6] C. Hochard, P. Ladeveze, L. Proslier. A simplified analysis of elastic structures. *Eur. Journal. Num. Mech.*, **12**(4): 509–535, 1993.

[7] A.P. Zieliński, O.C. Zienkiewicz. Generalized finite element analysis with T-complete boundary solution functions. *Int. J. Num. Meth. Engng.*, **21**: 509–528, 1985.

[8] A.P. Zieliński, I. Herrera. Trefftz method: fitting boundary conditions. *Int. J. Num. Meth. Engng.*, **24**: 871–891, 1987.

[9] J. Jirousek, L. Guex. The hybrid-Trefftz finite element model and its application to plate bending. *Int. J. Num. Meth. Engng.*, **23**: 651–693, 1986.