

Hybrid-Trefftz displacement and stress elements for elastodynamic analysis in the frequency domain

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Two alternative models of the hybrid-Trefftz finite element formulation to solve linear elastodynamic problems are presented. In the displacement model, the displacements are approximated in the domain of the element and the tractions are approximated on its boundary. The fields selected for approximation in the complementary stress model are the stresses in the domain and the displacements on the boundary of the element. In both models the domain approximation functions are constrained to satisfy locally the governing wave equation. A Fourier time approximation is used to uncouple the solving system in the frequency domain. The formulations are derived from the fundamental relations of elastodynamics and the associated energy statements are obtained *a posteriori*. Sufficient conditions for the existence and uniqueness of statically and kinematically admissible solutions are presented. Numerical implementation is briefly discussed.

1. INTRODUCTION

The displacement and stress models for the hybrid-Trefftz finite element formulation previously developed for quasi-static structural analysis problems [1-6] are here extended into elastodynamic analysis in the frequency domain. The finite element formulation is said to be *hybrid* because a domain and a boundary field are simultaneously approximated and the *Trefftz* specification derives from the constraint placed on the domain approximation functions, which are required to satisfy locally the differential equations governing the boundary value problem under analysis [7]. Two models are suggested, depending on the boundary field chosen to implement inter-element continuity: the tractions in the *stress model* and the displacements in the *displacement model*.

In the displacement model, the displacements are explicitly approximated in the domain of the element and the displacement approximation functions are used as weighing functions in the Galerkin weighed residual enforcement of the dynamic equilibrium condition. The resulting average enforcement of the equilibrium condition is integrated by parts to force the emergence of the static boundary condition and thus combine in a single statement the finite element static admissibility requirements. It is found that the tractions on the boundary of the element must also be explicitly approximated but that no assumptions need to be made on the stress distribution.

As the strain field is computed directly from the compatibility condition for the assumed displacements, in order to ensure that kinematically admissible finite element solutions are obtained, in the displacement model the kinematic boundary conditions are explicitly enforced on average, using the traction approximation functions as weighing functions.

Conversely, in the stress model, it is the stress field which is explicitly approximated in the domain of the element and the approximation functions are used as weighing functions in the average enforcement of the compatibility condition. The resulting average enforcement of the compatibility condition is integrated by parts to force the emergence of the kinematic boundary condition and thus obtain a single finite element kinematic admissibility statement. It is found that the displacements

on the boundary of the element must also be explicitly approximated but that no assumptions need to be made on the distribution of the displacements in the domain of the element.

In the stress model, the stress field is required to satisfy locally the dynamic equilibrium condition. In order to ensure that (weak) statically admissible finite element solutions are obtained, in this model the static boundary conditions are explicitly enforced on average, using the boundary displacement approximation functions as weighing functions.

In both the displacement and stress models, the elasticity relations are also enforced on average. The most interesting feature of the hybrid-Trefftz finite element formulation is revealed when the properties of the field approximation functions are implemented. In consequence of requiring these functions to satisfy the governing wave equation, it is found that the finite element system of equations combines the most attractive properties of the finite element and boundary element methods: it is symmetric and sparse, as the conventional finite element solving systems are, and all the intervening arrays have boundary integral definitions, as it is typical of the conventional boundary element method. Moreover, the kernel functions are not necessarily singular.

In the first part of the paper, the fundamental relations of linear elastodynamics and the finite element approximation criteria are summarised to facilitate the direct comparison of features of the displacement and stress models of the hybrid-Trefftz formulation. The time discretisation criterion is so chosen as to uncouple the fundamental relations in an infinite sequence of problems in the frequency domain, as it is typical of the method of spectral analysis. The alternative space discretisation criteria adopted here lead to the two distinct hybrid-Trefftz finite element models described above.

In order to justify clearly the different approximation criteria from which the two models develop, the finite element equations for the displacement and stress models are derived separately in the second part of the paper. These equations are obtained directly from the fundamental conditions of elastodynamics, namely the local equilibrium, compatibility and elasticity conditions and the boundary and initial conditions of the problem under analysis. They are not obtained from existing or specially derived statements on energy functionals. The second part of the paper closes with the enforcement of the Trefftz constraints on the domain approximation functions used to develop the displacement and stress models of the hybrid-Trefftz finite element formulation.

In the third part of the paper the finite element equations for equilibrium, compatibility and elasticity are combined to obtain the governing systems for the displacement and stress models and the numerical implementation of the hybrid-Trefftz models is discussed. Besides commenting on the choice of the approximation bases, the phases of pre- and post-processing, computation, storage and solution of the finite element systems are briefly reviewed. Attention is drawn to the fact that nodeless, hierarchical approximation functions leading to adaptive procedures are adopted. It is also stressed that accurate solutions for local effects, for instance those associated with fracture, can be easily incorporated, and that coarse meshes of rich, super elements can be advantageously implemented in a parallel processing mode.

The fourth and last part of the paper opens with the characterization of the finite element solutions in terms of the static and kinematic admissibility conditions. Mathematical programming theory is called upon to establish the energy statements associated with the hybrid-Trefftz displacement and stress models. It is found that the quadratic programs equivalent to the linear solving systems encode the necessary stationary statements on the functionals for the potential energy and co-energy and, also, for the functionals of Hu–Washizu and Hellinger–Reissner, extended to linear elastodynamic analysis in the frequency domain. The paper closes with the application of mathematical programming theorems to the quadratic programs of the hybrid-Trefftz models to establish the sufficient conditions for the existence and uniqueness of the finite element solutions.

The formulations used here are, naturally, related with alternative Trefftz finite element formulations reported in the literature, in particular with those supporting the HT-D and HT-T elements suggested by Jirousek and his co-workers [8,9]. As it is shown in Ref. [10], the approach adopted in the development of the hybrid-Trefftz displacement and stress elements differs from Jirousek's in three basic aspects, namely the selection of the alternative approximation criteria, the enforce-

ment of the inter-element continuity criteria and the identification of the associated variational statements. Moreover, the nodal variable concept is not called upon in the present approach, all approximation bases are hierarchical and the finite element governing system is solved in explicit form using sparse matrix manipulation algorithms, meaning that the internal field variables are not eliminated by partial condensation at element level, thus enhancing the implementation of p -adaptive procedures.

2. FUNDAMENTAL RELATIONS

Let V represent the domain of the element and Γ the enveloping surface, referred to Cartesian system \mathbf{x} , and t define the time parameter. The fundamental relations governing the linear elastodynamic response of the structure are summarised in Table 1.

Table 1. Fundamental conditions for linear elastodynamics.

Equilibrium (1)	Compatibility (2)	Elasticity (3)
$\mathbf{D}\boldsymbol{\sigma} + \mathbf{b} = \mathbf{d}\dot{\mathbf{u}} + \boldsymbol{\rho}\ddot{\mathbf{u}}$ in V	$\boldsymbol{\varepsilon} = \mathbf{D}^* \mathbf{u} \in V$	$\boldsymbol{\sigma} = \mathbf{k}\boldsymbol{\varepsilon} + \mathbf{c}\dot{\boldsymbol{\varepsilon}} + (\boldsymbol{\sigma}_\theta - \mathbf{k}\boldsymbol{\varepsilon}_\theta)$ in V
$\mathbf{N}\boldsymbol{\sigma} = \mathbf{t}_\Gamma$ on Γ_σ	$\mathbf{u} = \mathbf{u}_\Gamma$ on Γ_u	$\mathbf{u} = \mathbf{u}_0$ and $\dot{\mathbf{u}} = \dot{\mathbf{u}}_0$ in V , $t = 0$
Static b.c. (4)	Kinematic b.c. (5)	Initial conditions (6)

In the conditions of dynamic equilibrium and compatibility (1) and (2), vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ collect the independent components of the stress and strain tensors, respectively, \mathbf{b} is the body force vector, and the symmetric matrices \mathbf{d} and $\boldsymbol{\rho}$ collect, respectively, the relevant structural damping and specific mass coefficients. Vectors \mathbf{u} , $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$, collect the displacement, velocity and acceleration components, respectively. As a geometrically linear model is assumed, the differential equilibrium and compatibility operators \mathbf{D} and \mathbf{D}^* are linear and adjoint.

In description (3) for the elasticity conditions, the stiffness matrix, \mathbf{k} , is symmetric and has constant entries, as it is used to model the response of linear elastic materials. Similar properties are assumed for the material damping matrix \mathbf{c} . Vectors $\boldsymbol{\sigma}_\theta$ and $\boldsymbol{\varepsilon}_\theta$ are used to represent residual stresses and strains, respectively.

In the static boundary condition (4), vector \mathbf{t}_Γ defines the tractions prescribed on portion Γ_σ of the boundary and the boundary equilibrium matrix \mathbf{N} collects the components of the unit outward normal vector associated with the differential operators present in the domain equilibrium matrix \mathbf{D} . In the kinematic boundary conditions (5), vector \mathbf{u}_Γ defines the displacements prescribed on the complementary portion of the boundary, Γ_u . Mixed boundary conditions are assumed to be accounted for in the usual notation for geometric complementarity: $\Gamma = \Gamma_\sigma \cup \Gamma_u$; $\emptyset = \Gamma_\sigma \cap \Gamma_u$.

The initial conditions (6) are expressed in terms of initial displacements and velocities, \mathbf{u}_0 and $\dot{\mathbf{u}}_0$; alternative initial conditions can be easily accommodated. The equations summarised in Table 1 are constrained to geometrically and physically linear problems but they can be applied to the analysis of inhomogeneous and multiphase media.

It should be stressed that the variables, arrays and operators are identified above in the generalized sense, meaning that the fundamental conditions stated in Table 1 hold for alternative structural models. They may also be used to represent non-structural problems, for instance the standard potential problems associated with the Poisson equation [10].

3. TIME DISCRETISATION

Let the generic variable, say \mathbf{v} , be expanded in the time series (7), and use the time approximation function as the weighing function in the average enforcement (8) of the dynamic equilibrium condition (1):

$$\mathbf{v}(\mathbf{x}, t) = \sum_{n=-\infty}^{+\infty} \mathbf{v}_n(\mathbf{x}) e^{i\omega_n t}, \quad (7)$$

$$\int_0^T e^{-i\omega_n t} (\mathbf{D}\boldsymbol{\sigma} + \mathbf{b}) dt = \int_0^T e^{-i\omega_n t} (\mathbf{d}\dot{\mathbf{u}} + \boldsymbol{\rho}\ddot{\mathbf{u}}) dt. \quad (8)$$

To force the emergence of the initial conditions, the right hand side of the equation above is integrated by parts to yield result (9), where $\boldsymbol{\rho}_n$ is the generalized mass matrix defined by equation (10), which combines the inertia and structural damping effects:

$$\int_0^T e^{-i\omega_n t} (\mathbf{D}\boldsymbol{\sigma} + \boldsymbol{\rho}_n \omega_n^2 \mathbf{u} + \mathbf{b}) dt = \int_0^T e^{-i\omega_n t} (\mathbf{d}\mathbf{u} + \boldsymbol{\rho}\dot{\mathbf{u}} + i\omega_n \boldsymbol{\rho}\mathbf{u}) dt, \quad (9)$$

$$\boldsymbol{\rho}_n = \boldsymbol{\rho} - \frac{i}{\omega_n} \mathbf{d}. \quad (10)$$

For a forced vibration problem, equation (7) defines the Fourier expansion in the time period T and the following identifications hold:

$$\omega_n = \frac{2n\pi}{T}, \quad (11)$$

$$\mathbf{v}_n = \frac{1}{T} \int_0^T \mathbf{v} e^{-i\omega_n t} dt. \quad (12)$$

The initial condition terms in equation (9) become trivial because, in the frequency domain analysis, T is assumed to represent the actual or the fictitious period of the forcing load:

$$\mathbf{u}_1(x, 0) = \mathbf{u}_1(x, T) \quad \text{in } V, \quad (13)$$

$$\dot{\mathbf{u}}_1(x, 0) = \dot{\mathbf{u}}_1(x, T) \quad \text{in } V. \quad (14)$$

Description (15) for the equilibrium condition in the frequency domain is recovered by enforcing in equation (9) definitions (11) and (12). The remaining equations for the frequency domain analysis given in Table 2 are obtained in a similar manner. These equations are also valid for the free vibration problem, where n represents now the n -th natural frequency of the structure under analysis.

Table 2. Fundamental conditions for linear elastodynamics in the frequency domain

Equilibrium (15)	Compatibility (16)	Elasticity (17)
$D\sigma_n + \rho_n \omega_n^2 u_n + b_n = 0$ in V	$\varepsilon_n = D^* u_n$ in V	$\sigma_n = k_n \varepsilon_n + \sigma_{rn}$ in V
$N\sigma_n = t_{\Gamma_n}$ on Γ_σ	$u_n = u_{\Gamma_n}$ on Γ_u	<i>Arbitrary</i>
Static b.c. (18)	Kinematic b.c. (19)	Initial conditions

Let \mathbf{v}_1 represent the forced vibration solution for arbitrary initial conditions and \mathbf{v}_2 denote the free vibration modes. The response of the structure in the time domain is described by equation

(20) and the weights γ on the contribution of the eigenmodes are computed by enforcing the initial conditions (6) in form (21) and (22):

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}_1(\mathbf{x}, t) + \gamma \mathbf{v}_2(\mathbf{x}, t) \quad \text{in } V \quad \text{or} \quad \text{on } \Gamma, \tag{20}$$

$$\mathbf{u}_1(\mathbf{x}, t = 0) + \gamma \mathbf{u}_2(\mathbf{x}, t = 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } V, \tag{21}$$

$$\dot{\mathbf{u}}_1(\mathbf{x}, t = 0) + \gamma \dot{\mathbf{u}}_2(\mathbf{x}, t = 0) = \dot{\mathbf{u}}_0(\mathbf{x}) \quad \text{in } V. \tag{22}$$

In the elasticity equation (17), the following notation is used for the generalized stiffness matrix and for the generalized residual stress vector:

$$\mathbf{k}_n = \mathbf{k} + i\omega_n \mathbf{c}, \tag{23}$$

$$\boldsymbol{\sigma}_{rn} = \boldsymbol{\sigma}_{\theta n} - \mathbf{k}\boldsymbol{\varepsilon}_{\theta n}. \tag{24}$$

For later use, the elasticity relation (17) is also written in the alternative flexibility format (25), where \mathbf{f}_n is the flexibility matrix and n denotes equivalent generalized residual strains:

$$\boldsymbol{\varepsilon}_n = \mathbf{f}_n \boldsymbol{\sigma}_n + \boldsymbol{\varepsilon}_{rn} \quad \text{in } V, \tag{25}$$

$$\mathbf{f}_n = \mathbf{k}_n^{-1}, \tag{26}$$

$$\boldsymbol{\varepsilon}_{rn} = -\mathbf{f}_n \boldsymbol{\sigma}_{rn}. \tag{27}$$

The procedure described above is, in essence, the procedure usually adopted to map a transient problem into the frequency domain.

4. SPACE DISCRETISATION

Let V^e and Γ^e denote the domain and the envelope of the typical element e , respectively. No constraints are placed on the geometry of this element, which may not be simply connected or occupy a finite domain. Let further Γ_i represent the set of internal, inter-element boundaries in the finite element mesh; Γ_i is the empty set for a single element mesh discretisation.

As it is shown in Table 3, in the displacement model the displacements in the domain of the element and the tractions on its kinematic boundary are simultaneously approximated. In definitions (28) and (31), matrices \mathbf{U}_V and \mathbf{S}_Γ collect the displacement and traction approximation functions. As they are selected from hierarchical (non-nodal) bases, the weighing vectors \mathbf{q}_V and \mathbf{X}_Γ represent *generalized displacements* and *generalized tractions*, respectively. Vector \mathbf{u}_p is used to model particular solutions, for instance those associated with body forces, specific boundary conditions or residual stresses and strains. The question on whether the displacement approximation (28) should, or not, contain the rigid body modes is addressed below in the text.

Table 3. Approximations in the hybrid-Trefftz displacement model.

Approximation	Dual Transformation	Energy Balance
$\mathbf{u}_n = \mathbf{U}_{V_n} \mathbf{q}_{V_n} + \mathbf{u}_{pn}$ in V^e (28)	$\mathbf{X}_{bn} = \int \mathbf{U}_{V_n}^t \mathbf{b}_n \, dV^e$ (29)	$\mathbf{q}_{V_n}^t \mathbf{X}_{bn} = \int (\mathbf{u}_n - \mathbf{u}_{pn})^t \mathbf{b}_n \, dV^e$ (30)
$\mathbf{t}_n = \mathbf{S}_{\Gamma_n} \mathbf{X}_{\Gamma_n}$ on Γ_u^e (31)	$\mathbf{v}_{un} = \int \mathbf{S}_{\Gamma_n}^t \mathbf{u}_{\Gamma_n} \, d\Gamma_u^e$ (32)	$\mathbf{X}_{\Gamma_n}^t \mathbf{v}_{un} = \int \mathbf{t}_n^t \mathbf{u}_{\Gamma_n} \, d\Gamma_u^e$ (33)

Complementary, in the stress model of the hybrid-Trefftz formulation the fields that are directly approximated are the stresses in the domain of the element and the displacements on its static boundary, as stated by equations (34) and (37) in Table 4. Matrices \mathbf{S}_V and \mathbf{U}_Γ collect the corresponding (hierarchical) stress and boundary displacement functions and the weighing vectors \mathbf{X}_V and \mathbf{q}_Γ list the associated *generalized stresses* and *generalized boundary displacements*, respectively. In the stress approximation (34) the particular solution vector σ_p plays a role identical to that of vector \mathbf{u}_p in the displacement model.

Table 4. Approximations in the hybrid-Trefftz stress model.

Approximation	Dual Transformation	Energy Balance
$\sigma_n = \mathbf{S}_{Vn} \mathbf{X}_{Vn} + \sigma_{pn}$ in V^e (34)	$\mathbf{e}_n = \int \mathbf{S}_{Vn}^t \varepsilon_n dV^e$ (35)	$\mathbf{X}_{Vn}^t \mathbf{e}_n = \int (\sigma_n - \sigma_{Pn})^t \varepsilon_n dV^e$ (36)
$\mathbf{u}_n = \mathbf{U}_{\Gamma n} \mathbf{q}_{\Gamma n}$ on Γ_σ^e (37)	$\mathbf{Q}_{tn} = \mathbf{q}_{\Gamma n}^t \mathbf{Q}_{tn} = \int \mathbf{U}_{\Gamma n}^t \mathbf{t}_{\Gamma n} d\Gamma_\sigma^e$ (38)	$\mathbf{q}_{\Gamma n}^t \mathbf{Q}_{tn} = \int \mathbf{u}_n^t \mathbf{t}_{\Gamma n} d\Gamma_u^e$ (39)

The dual transformations of the basic approximations enforced in the displacement model are also presented in Table 3. They define the *generalized body forces*, \mathbf{X}_b , and the *generalized boundary displacements*, \mathbf{v}_u , that dissipate on the generalized displacements and tractions the same energy as the continuum fields they are associated with, as stated by equations (29) and (33). The corresponding transformations for the stress model are given in Table 4. Equations (35) and (38) define the *generalized strains*, \mathbf{e} , and the *generalized tractions*, \mathbf{Q}_t , that ensure the necessary energy balance between the discrete and continuum fields.

As it will become apparent below, the inter-element continuity conditions and the system boundary conditions are enforced in different ways in the displacement and stress models, which are dictated by the consistency of the domain approximation criteria they develop from, as stated by equations (28) and (34), respectively. Consequent upon this, the static and kinematic boundaries of a typical element, e , are defined differently for the two alternative elements, as it is stated in Table 5: the static boundary of the displacement {stress} element, Γ_σ^e , contains the static boundary of the system under analysis {and the inter-element boundaries} shared by the element; conversely, the kinematic boundary of the stress {displacement} element, Γ_u^e , contains the kinematic boundary of the system under analysis {and the inter-element boundaries} shared by the element.

Table 5. Definition of the finite element static and kinematic boundaries.

Boundary	Displacement model	Stress model
Static, Γ_σ^e	$\{\Gamma^e \cap \Gamma_\sigma\}$	$\{\Gamma^e \cap \Gamma_\sigma\} \cup \{\Gamma^e \cap \Gamma_i\}$
Kinematic, Γ_u^e	$\{\Gamma^e \cap \Gamma_u\} \cup \{\Gamma^e \cap \Gamma_i\}$	$\{\Gamma^e \cap \Gamma_u\}$

The hybrid displacement and stress models and the particular forms they present when the Trefftz constraint is enforced are presented next. They are derived for the non-trivial, forced frequency solution modes, $n > 0$ and $\omega_n > 0$, that typify the elastodynamic analysis in the frequency domain. The models thus obtained are subsequently specialised for the elastostatic problem, characterised by $n = 0$ and $\omega_0 = 0$, which must be solved when the analyst wishes to use the frequency domain solution to build the time domain representation (20) of the system response.

5. HYBRID DISPLACEMENT ELEMENT

In the hybrid displacement model the compatibility condition (16) is used to compute the strain field associated with the displacement approximation (28):

$$\boldsymbol{\varepsilon}_n = \mathbf{E}_{Vn} \mathbf{q}_{Vn} + \boldsymbol{\varepsilon}_{pn} \quad \text{in } V^e, \quad (40)$$

$$\mathbf{E}_{Vn} = \mathbf{D}^* \mathbf{U}_{Vn}, \quad (41)$$

$$\boldsymbol{\varepsilon}_{pn} = \mathbf{D}^* \mathbf{u}_{pn}. \quad (42)$$

The dual of approximation (40) defines the generalized stresses (43) that ensure the energy balance condition (44):

$$\mathbf{X}_n = \int \mathbf{E}_{Vn}^t \boldsymbol{\sigma}_n dV^e, \quad (43)$$

$$\mathbf{q}_{Vn}^t \mathbf{X}_n = \int (\boldsymbol{\varepsilon}_n - \boldsymbol{\varepsilon}_{pn})^t \boldsymbol{\sigma}_n dV^e. \quad (44)$$

The finite element equations found for the hybrid displacement model are collected in Table 6.

Table 6. Finite element equations for the elastodynamic hybrid displacement model.

Kinematics (45)	Statics (46)	Elasticity (47)
$\mathbf{B}_{\Gamma n}^t \mathbf{q}_{Vn} = \mathbf{v}_{u\Gamma n} - \mathbf{v}_{p\Gamma n}$	$\mathbf{X}_n = \mathbf{B}_{\Gamma n} \mathbf{X}_{\Gamma n} + \omega_n^2 \mathbf{M}_{Vn} \mathbf{q}_{Vn} + \mathbf{X}_{pn}$	$\mathbf{X}_n = \mathbf{K}_{Vn} \mathbf{q}_{Vn} + \mathbf{X}_{0n}$

The properties summarised in the following statements emerge from their development below:

- (S1) The equilibrium matrix present in the description of Statics, \mathbf{B}_{Γ} , is defined by a boundary integral expression.
- (S2) Static-kinematic duality holds, as the compatibility matrix in the description of Kinematics is the transpose of the equilibrium matrix.
- (S3) The generalized mass matrix, \mathbf{M}_V , which combines inertia and structural damping effects, is symmetric and defined by a domain integral expression.
- (S4) The generalized stiffness matrix, \mathbf{K}_V , which combines elasticity and material damping effects, is symmetric and defined by a domain integral expression.

5.1. Finite element description of Statics

Definition (29) for the generalized body forces is used to enforce on average the local equilibrium condition (15) in form (48), which is then integrated by parts to this statement to the enforcement of the static boundary condition:

$$\int \mathbf{U}_{Vn}^t (\mathbf{D} \boldsymbol{\sigma}_n + \rho_n \omega_n^2 \mathbf{u}_n + \mathbf{b}_n) dV^e = 0, \quad (48)$$

$$\begin{aligned} & - \int (\mathbf{D}^* \mathbf{U}_{Vn})^t \boldsymbol{\sigma}_n dV^e + \int \mathbf{U}_{Vn}^t \mathbf{N} \boldsymbol{\sigma}_n d\Gamma_u^e \\ & + \int \mathbf{U}_{Vn}^t \mathbf{N} \boldsymbol{\sigma}_n d\Gamma_\sigma^e + \omega_n^2 \int \mathbf{U}_{Vn}^t \rho_n \mathbf{u}_n dV^e + \int \mathbf{U}_{Vn}^t \mathbf{b}_n dV^e = 0. \end{aligned} \quad (49)$$

Using results (41) and (43) and enforcing above the static boundary condition (18), stated in Table 2, the following expression is found for the generalized stresses:

$$\mathbf{X}_n = \int \mathbf{U}_{Vn}^t \mathbf{t}_n d\Gamma_u^e + \int \mathbf{U}_{Vn}^t \mathbf{t}_{\Gamma n} d\Gamma_\sigma^e + \omega_n^2 \int \mathbf{U}_{Vn}^t \rho_n \mathbf{u}_n dV^e + \int \mathbf{U}_{Vn}^t \mathbf{b}_n dV^e. \quad (50)$$

Equation (50) shows that the boundary tractions and the displacements in the domain can be independently approximated. The resulting finite element description (46) for Statics is obtained enforcing approximations (28) and (31) and definition (29) for the generalized body forces. The following identifications are found for the equilibrium matrix, $\mathbf{B}_{\Gamma n}$, the generalized mass matrix, \mathbf{M}_{Vn} , and for the residual term \mathbf{X}_{Pn} that combines the contributions of the body forces, \mathbf{X}_{bn} , the prescribed tractions, \mathbf{X}_{tn} , and the particular solution for the displacement approximation, \mathbf{X}_{un} :

$$\mathbf{B}_{\Gamma n} = \int \mathbf{U}_{Vn}^t \mathbf{S}_{\Gamma n} d\Gamma_u^e, \quad (51)$$

$$\mathbf{M}_{Vn} = \int \mathbf{U}_{Vn}^t \rho_n \mathbf{U}_{Vn} dV^e, \quad (52)$$

$$\mathbf{X}_{t\Gamma n} = \int \mathbf{U}_{Vn}^t \mathbf{t}_{\Gamma n} d\Gamma_\sigma^e, \quad (53)$$

$$\mathbf{X}_{un} = \omega_n^2 \int \mathbf{U}_{Vn}^t \rho_n \mathbf{u}_{pn} dV^e, \quad (54)$$

$$\mathbf{X}_{Pn} = \mathbf{X}_{bn} + \mathbf{X}_{un} + \mathbf{X}_{t\Gamma n}. \quad (55)$$

5.2. Finite element description of Kinematics

As the compatibility condition (16) is locally satisfied by the strain estimate (40), the finite element kinematic admissibility condition reduces to the enforcement of the boundary condition (22). According to definition (32) for the generalized boundary displacements, given in Table 3, this kinematic admissibility condition is stated by the average enforcement on the displacement approximation (28):

$$\int \mathbf{S}_{\Gamma n}^t (\mathbf{U}_{Vn} \mathbf{q}_{Vn} + \mathbf{u}_{pn}) d\Gamma = \int \mathbf{S}_{\Gamma n}^t \mathbf{u}_{\Gamma n} d\Gamma_n^e. \quad (56)$$

The resulting description for Kinematics is given in Table 6, where the following notation is used:

$$\mathbf{v}_{u\Gamma n} = \int \mathbf{S}_{\Gamma n}^t \mathbf{u}_{\Gamma n} d\Gamma_u^e, \quad (57)$$

$$\mathbf{v}_{p\Gamma n} = \int \mathbf{S}_{\Gamma n}^t \mathbf{u}_{pn} d\Gamma_u^e. \quad (58)$$

5.3. Finite element description of Elasticity

The Elasticity description for the displacement model, stated by equation (47) in Table 6, is obtained enforcing the local elasticity condition (17) for the assumed strain distribution (40) in definition (43) for the generalized stresses. The following expressions are found for the generalized stiffness matrix, \mathbf{K}_{Vn} , and for the residual term, \mathbf{X}_{0n} , that combines the contributions of the residual stresses and of the particular solution for the strain distribution:

$$\mathbf{K}_{Vn} = \int \mathbf{E}_{Vn}^t \mathbf{k}_n \mathbf{E}_{Vn} dV^e, \quad (59)$$

$$\mathbf{X}_{0n} = \int \mathbf{E}_{Vn}^t (\boldsymbol{\sigma}_{rn} + \mathbf{k}_n \boldsymbol{\varepsilon}_{pn}) dV^e. \quad (60)$$

6. HYBRID-TREFFTZ DISPLACEMENT ELEMENT

The equations summarised in Table 6 characterise the displacement model of the hybrid finite element formulation presented in Ref. 11. The equations for the corresponding model of the hybrid-Trefftz formulation are obtained by constraining the domain approximation functions present in definition (28), given in Table 3, to satisfy locally all the field conditions. This is equivalent to selecting displacement approximation functions that solve the Helmholtz equation (61), obtained by combining the fundamental conditions (15) to (17), to yield the Trefftz constraints (62) and (63) on the displacement approximation (28):

$$(\mathbf{D}\mathbf{k}_n\mathbf{D}^* + \omega_n\rho_n)\mathbf{u}_n + \mathbf{D}\sigma_{rn} + \mathbf{b}_n = \mathbf{0} \quad \text{in } V^e, \quad (61)$$

$$(\mathbf{D}\mathbf{k}_n\mathbf{D}^* + \omega_n^2\rho_n)\mathbf{U}_{Vn} = \mathbf{0} \quad \text{in } V^e, \quad (62)$$

$$(\mathbf{D}\mathbf{k}_n\mathbf{D}^* + \omega_n\rho_n)\mathbf{u}_{pn} + \mathbf{D}\sigma_{rn} + \mathbf{b}_n = \mathbf{0} \quad \text{in } V^e. \quad (63)$$

Consequently, the corresponding strain fields, defined by equations (41) and (42), are now assumed to be associated with the elastic stress field (64), which satisfies the equilibrium and elasticity conditions (15) and (17), as stated by constraints (65) to (68):

$$\sigma_n = \mathbf{S}_{Vn}\mathbf{q}_{Vn} + \sigma_{pn} \quad \text{in } V^e, \quad (64)$$

$$\mathbf{D}\mathbf{S}_{Vn} = -\omega_n^2\rho_n\mathbf{U}_{Vn}, \quad (65)$$

$$\mathbf{D}\sigma_{pn} = -\omega_n^2\rho_n\mathbf{u}_{pn} - \mathbf{b}_n, \quad (66)$$

$$\mathbf{S}_{Vn} = \mathbf{k}_n\mathbf{E}_{Vn}, \quad (67)$$

$$\sigma_{pn} = \mathbf{k}_n\epsilon_{pn} + \sigma_{rn}. \quad (68)$$

It is recalled that the rigid body displacement modes are not contained in the solution set of the Helmholtz equation (61) for non-trivial forced frequencies, $n > 0$ and $\omega_n > 0$.

Let constraints (41) and (67) be inserted in definition (59) for the finite element stiffness matrix and integrate by parts the resulting equation to yield:

$$\mathbf{K}_{Vn} = - \int \mathbf{U}_{Vn}^t \mathbf{D}\mathbf{S}_{Vn} dV^e + \int \mathbf{U}_{Vn}^t \mathbf{N}\mathbf{S}_{Vn} d\Gamma^e. \quad (69)$$

Inserting condition (65) in equation (69) and recalling definition (52) for the finite element mass matrix, the following result is found, where \mathbf{D}_n is the elasticity matrix and matrix \mathbf{T}_{Vn} collects traction fields associated with the stress definition (64):

$$\mathbf{K}_{Vn} = \omega_n^2\mathbf{M}_{Vn} + \mathbf{D}_{\Gamma n}, \quad (70)$$

$$\mathbf{D}_{\Gamma n} = \int \mathbf{U}_{Vn}^t \mathbf{T}_{Vn} d\Gamma^e, \quad (71)$$

$$\mathbf{T}_{Vn} = \mathbf{N}\mathbf{S}_{Vn}. \quad (72)$$

Definitions (70) and (71) together with statements $S3$ and $S4$ show that:

(S9) Matrix \mathbf{D}_{Γ} is symmetric and defined by a boundary integral expression.

Using results (29), (41), (66) and (68) and processing in a similar way definition (60) for the generalised residual stresses, the following alternative description is obtained:

$$\mathbf{X}_{0n} = \mathbf{X}_{bn} + \mathbf{X}_{un} + \mathbf{X}_{p\Gamma n}, \quad (73)$$

$$\mathbf{X}_{p\Gamma n} = \int \mathbf{U}_{Vn}^t \mathbf{N}\sigma_{pn} d\Gamma^e. \quad (74)$$

7. HYBRID STRESS ELEMENT

The hybrid stress model develops [11] from the requirement that the stress approximation (34), stated in Table 4, satisfies locally the equilibrium condition (15). This implies the existence of an associated displacement field (75) satisfying the local equilibrium constraints (65) and (66):

$$\mathbf{u}_n = \mathbf{U}_{V_n} \mathbf{X}_{V_n} + \mathbf{u}_{pn} \quad \text{in } V^e. \tag{75}$$

The elasticity condition (17) is not involved in the definition above. It is recalled that, as it is typical of the hybrid stress elements, there is no explicit requirement on the consistency of the two alternative strain fields that can be computed from the compatibility condition (16) using the displacement estimate (75), and from the elasticity relation (25) with the assumed stresses (34). This linkage develops only after enforcing the Trefftz constraints and therefore it is a distinguishing feature of the hybrid-Trefftz stress elements.

The equations governing the response of the hybrid stress element are derived below. It is shown that they can be encoded in the forms (76) to (78), collected in Table 7, which are later specialised for the hybrid-Trefftz stress element.

Table 7. Finite element equations for the elastodynamic hybrid stress model.

Statics (76)	Kinematics (77)	Elasticity (78)
$\mathbf{A}_{\Gamma_n}^t \mathbf{X}_{V_n} = \mathbf{Q}_{t\Gamma_n} - \mathbf{Q}_{p\Gamma_n}$	$\mathbf{e}_n = \omega_n^{-2} \tilde{\mathbf{M}}_{V_n} \mathbf{X}_{V_n} + \mathbf{A}_{\Gamma_n} \mathbf{q}_{\Gamma_n} + \mathbf{e}_{pn}$	$\mathbf{e}_n = \mathbf{F}_{V_n} \mathbf{X}_{V_n} + \mathbf{e}_{0n}$

The following properties hold for both the hybrid and hybrid-Trefftz stress finite element arrays:

- (S5) *The compatibility matrix present in the description of Kinematics, A, is defined by a boundary integral expression.*
- (S6) *Static-kinematic duality holds, as the equilibrium matrix in the description of Statics is the transpose of the compatibility matrix.*
- (S7) *The generalized mobility matrix, $\tilde{\mathbf{M}}_V$, which combines inertia and structural damping effects, is symmetric and defined by a domain integral expression.*
- (S8) *The generalized flexibility matrix, \mathbf{F}_V , which combines elasticity and material damping effects, is symmetric and defined by a domain integral expression.*

7.1. Finite element description of Kinematics

Definition (35) for the generalized strains, given in Table 4, is used to enforce on average the local compatibility condition (16):

$$\int \mathbf{S}_{V_n}^t \boldsymbol{\varepsilon}_n dV^e = \int \mathbf{S}_{V_n}^t (\mathbf{D}^* \mathbf{u}_n) dV^e. \tag{79}$$

The equation above is integrated by parts to combine in a single statement the kinematic admissibility conditions (16) and (19) on the generalized strains:

$$\mathbf{e}_n = - \int (\mathbf{D}\mathbf{S}_{V_n})^t \mathbf{u}_n dV^e + \int (\mathbf{N}\mathbf{S}_{V_n})^t \mathbf{u}_n d\Gamma_\sigma^e + \int (\mathbf{N}\mathbf{S}_{V_n})^t \mathbf{u}_{\Gamma_n} d\Gamma_u^e. \tag{80}$$

Using results (65) and (75), as well as definition (72), and enforcing independently the boundary displacement approximation (37) the following expression is found:

$$\mathbf{e}_n = \int (\omega_n^2 \boldsymbol{\rho}_n \mathbf{U}_{V_n})^t (\mathbf{U}_{V_n} \mathbf{X}_{V_n} + \mathbf{u}_{pn}) dV^e + \int \mathbf{T}_{V_n}^t (\mathbf{U}_{\Gamma_n} \mathbf{q}_{\Gamma_n}) d\Gamma_\sigma^e + \int \mathbf{T}_{V_n}^t \mathbf{u}_{\Gamma_n} d\Gamma_u^e. \tag{81}$$

The resulting finite element description (77) for Kinematics is given in Table 7. The following expressions are found for the compatibility matrix, $\mathbf{A}_{\Gamma n}$, for the generalized mobility matrix, $\tilde{\mathbf{M}}_{Vn}$, and for the residual term \mathbf{e}_{Pn} that combines the contributions of the body forces, \mathbf{e}_{bn} , the prescribed displacements, \mathbf{e}_{un} , and the particular solution for the stress approximation, $\mathbf{e}_{\sigma n}$:

$$\mathbf{A}_{\Gamma n} = \int \mathbf{T}_{Vn}^t \mathbf{U}_{\Gamma n} d\Gamma_{\sigma}^e, \quad (82)$$

$$\tilde{\mathbf{M}}_{Vn} = \int (\mathbf{D}\mathbf{S}_{Vn})^t \rho_n^{-1} (\mathbf{D}\mathbf{S}_{Vn}) dV^e, \quad (83)$$

$$\mathbf{e}_{bn} = \omega_n^{-2} \int (\mathbf{D}\mathbf{S}_{Vn})^t \rho_n^{-1} \mathbf{b}_n dV^e, \quad (84)$$

$$\mathbf{e}_{\sigma n} = \omega_n^{-2} \int (\mathbf{D}\mathbf{S}_{Vn})^t \rho_n^{-1} (\mathbf{D}\sigma_{pn}) dV^e, \quad (85)$$

$$\mathbf{e}_{u\Gamma n} = \int \mathbf{T}_{Vn}^t \mathbf{u}_{\Gamma n} d\Gamma_u^e, \quad (86)$$

$$\mathbf{e}_{Pn} = \mathbf{e}_{bn} + \mathbf{e}_{\sigma n} + \mathbf{e}_{u\Gamma n}. \quad (87)$$

7.2. Finite element description of Statics

As the equilibrium condition (15) is locally satisfied by the stress estimate (34), the finite element static admissibility condition reduces to the enforcement of the boundary condition (18). According to definition (38) for the generalized boundary tractions, given in Table 4, this static admissibility condition is stated by the following average enforcement on the stress approximation (34):

$$\int \mathbf{U}_{\Gamma n}^t \mathbf{N} (\mathbf{S}_{Vn} \mathbf{X}_{Vn} + \sigma_{pn}) d\Gamma_{\sigma}^e = \int \mathbf{U}_{\Gamma n}^t \mathbf{t}_{\Gamma n} d\Gamma_{\sigma}^e. \quad (88)$$

The resulting description for Statics is given in Table 7, where the following notation is used:

$$\mathbf{Q}_{t\Gamma n} = \int \mathbf{U}_{\Gamma n}^t \mathbf{t}_{\Gamma n} d\Gamma_{\sigma}^e, \quad (89)$$

$$\mathbf{Q}_{p\Gamma n} = \int \mathbf{U}_{\Gamma n}^t \mathbf{N} \sigma_{pn} d\Gamma_{\sigma}^e. \quad (90)$$

7.3. Finite element description of Elasticity

The Elasticity description for the stress model, stated by equation (78), is obtained enforcing the local elasticity condition in the flexibility format (25) for the assumed stress distribution (34) in definition (35) for the generalized strains. The following expressions are found for the generalized flexibility matrix, \mathbf{F}_{Vn} , and for the residual term, \mathbf{e}_{0n} , that combines the contributions of the residual stresses and of the particular solution for the stress distribution:

$$\mathbf{F}_{Vn} = \int \mathbf{S}_{Vn}^t \mathbf{f}_n \mathbf{S}_{Vn} dV^e, \quad (91)$$

$$\mathbf{e}_{0n} = \int \mathbf{S}_{Vn}^t (\varepsilon_{rn} + \mathbf{f}_n \sigma_{pn}) dV^e. \quad (92)$$

8. HYBRID-TREFFTZ STRESS ELEMENT

To obtain the hybrid-Trefftz stress element from the hybrid stress model presented above it suffices to assume that the displacement approximation (75) solves the Helmholtz equation (61). The Trefftz constraints (62), (63), and (65) to (68) are now assumed to hold for the stress model. It is convenient, however, to rewrite the two latter constraints in the following equivalent flexibility formats:

$$\mathbf{E}_{Vn} = \mathbf{f}_n \mathbf{S}_{Vn}, \quad (93)$$

$$\boldsymbol{\varepsilon}_{pn} = \mathbf{f}_n \boldsymbol{\sigma}_{pn} + \boldsymbol{\varepsilon}_{rn}. \quad (94)$$

It is stressed that the estimate for the strain field is now uniquely defined by equation (95) as the local compatibility conditions (41) and (42) are also holding in consequence of assuming that the governing Helmholtz equation is solved by the finite element domain approximation:

$$\boldsymbol{\varepsilon}_n = \mathbf{E}_{Vn} \mathbf{X}_{Vn} + \boldsymbol{\varepsilon}_{pn} \quad \text{in } V^e. \quad (95)$$

Inserting in definition (91) the elasticity and compatibility conditions (93) and (41), respectively, integrating by parts and enforcing the local equilibrium condition (65), the following alternative description is found for the finite element flexibility matrix, provided that results (71), (72) and (83) are taken into account:

$$\mathbf{F}_{Vn} = \omega_n^{-2} \tilde{\mathbf{M}}_{Vn} + \mathbf{D}_{\Gamma n}. \quad (96)$$

Treating in a similar way definition (92) for the generalized residual strains, the following alternative expression is obtained:

$$\mathbf{e}_{0n} = \mathbf{e}_{bn} + \mathbf{e}_{\sigma un} + \mathbf{e}_{p\Gamma n}, \quad (97)$$

$$\mathbf{e}_{p\Gamma n} = \int \mathbf{T}_{Vn}^t \mathbf{u}_{pn} \, d\Gamma. \quad (98)$$

9. ELASTOSTATIC SOLUTION MODE

The static phase, $n = 0$, in the dynamic response of the structure is modelled by the zero-th order solution of the governing system (15) to (19), in Table 2. The expressions presented above for the finite element equations are still fundamentally valid but they must be conveniently manipulated to account for the specialisation induced by conditions $\omega_0 = 0$ and $\mathbf{k}_0 = \mathbf{k}$, in consequence of definition (23).

The explicit study of the elastostatic solution mode presented below is also instrumental to disclose the fundamental reasons that justify the approximation criteria stated in Tables 3 and 4, respectively, for the displacement and stress models of the hybrid finite element formulation.

9.1. Hybrid displacement elements for elastostatics

In the elastostatic solution mode, the governing Helmholtz equation (61) collapses into the Navier equation (99), which includes the rigid body displacements in its solution set:

$$(\mathbf{Dk}_0 \mathbf{D}^*) \mathbf{u}_0 + \mathbf{D}\boldsymbol{\sigma}_{r0} + \mathbf{b}_0 = \mathbf{0} \quad \text{in } V^e. \quad (99)$$

Hence, completeness demands the extension of the domain displacement approximation (28), given in Table 3, to include the rigid body modes. This is stated in equations (100) and (101),

where \mathbf{U}_R is the matrix that collects the description of the alternative rigid body modes and \mathbf{q}_R is the vector that lists the associated generalized displacements:

$$\mathbf{u}_0 = \mathbf{U}_{V0}\mathbf{q}_{V0} + \mathbf{U}_R\mathbf{q}_R + \mathbf{u}_{p0} \quad \text{in } V^e, \tag{100}$$

$$\mathbf{D}^*\mathbf{U}_R = \mathbf{0}. \tag{101}$$

The finite element kinematic admissibility condition (56) is now written in form (102), yielding the compatibility equation (103), which replaces the forced frequency form (45) stated in Table 6:

$$\int \mathbf{S}_{\Gamma 0}^t (\mathbf{U}_{V0}\mathbf{q}_{V0} + \mathbf{U}_R\mathbf{q}_R + \mathbf{u}_{p0}) \, d\Gamma_u^e = \int \mathbf{S}_{\Gamma 0}^t \mathbf{u}_{\Gamma 0} \, d\Gamma_u^e. \tag{102}$$

Definitions (51), (57) and (58) still hold, now with $n = 0$, where definitions (29), (51) and (53) still hold, with $n = 0$:

Table 8. Finite element equations for the elastostatic hybrid displacement model.

Kinematics (103)	Statics (104)	Elasticity (105)
$[\mathbf{B}_{\Gamma 0}^t \quad \mathbf{B}_{\Gamma R}^t] \begin{Bmatrix} \mathbf{q}_{V0} \\ \mathbf{q}_{VR} \end{Bmatrix} = \mathbf{v}_{u\Gamma 0} - \mathbf{v}_{p\Gamma 0}$	$\begin{Bmatrix} \mathbf{X}_0 \\ \mathbf{0} \end{Bmatrix} = \begin{bmatrix} \mathbf{B}_{\Gamma 0} \\ \mathbf{B}_{\Gamma R} \end{bmatrix} \mathbf{X}_{\Gamma 0} + \begin{Bmatrix} \mathbf{X}_{p0} \\ \mathbf{X}_R \end{Bmatrix}$	$\mathbf{X}_0 = \mathbf{K}_{V0}\mathbf{q}_{V0} + \mathbf{X}_{00}$

The elastodynamic equilibrium condition (46) is stated in Table 8 for the elastostatic specialisation $n = 0$ and $\omega_0 = 0$, to yield:

$$\mathbf{X}_{P0} = \mathbf{X}_{b0} + \mathbf{X}_{t\Gamma 0}. \tag{106}$$

However, besides the generalized stress definition (29), the dual of transformation (100) also produces definition (107) for the body force resultants, which is used to enforce the global elastostatic equilibrium condition of the element in the Galerkin weighed residual form (108):

$$\mathbf{X}_b = \int \mathbf{U}_R^t \mathbf{b}_0 \, dV^e, \tag{107}$$

$$\int \mathbf{U}_R^t (\mathbf{D}\boldsymbol{\sigma}_0 + \mathbf{b}_0) \, dV^e = \mathbf{0}. \tag{108}$$

The additional finite element equilibrium condition in system (104), given in Table 8, is obtained after implementing the integration by parts based procedure described above, now with the enforcement of equation (101), to yield:

$$\mathbf{B}_{\Gamma R} = \int \mathbf{U}_R^t \mathbf{S}_{\Gamma 0} \, d\Gamma_u^e, \tag{109}$$

$$\mathbf{X}_{t\Gamma R} = \int \mathbf{U}_R^t \mathbf{t}_{\Gamma 0} \, d\Gamma_\sigma^e, \tag{110}$$

$$\mathbf{X}_R = \mathbf{X}_b + \mathbf{X}_{t\Gamma R}. \tag{111}$$

The finite element description of Elasticity, equation (105) in Table 8, is the direct specialisation of the elastodynamic equation (47) with $n = 0$. Definitions (51), (53) and (57) to (60) also stand for the elastostatic solution mode with $n = 0$.

Hence, equation (103) for compatibility, equation (104) for equilibrium and equation (105) for elasticity, are the finite element equations used to solve elastostatic problems with the hybrid displacement element [5, 6]. Static-kinematic duality and material reciprocity still hold.

9.2. Hybrid stress elements for elastostatics

The elastostatic hybrid stress element develops [1–4] from the basic assumptions that the stress field is directly approximated, as stated by equation (34) in Table 4, and is explicitly constrained to satisfy locally the equilibrium condition (18), now with $n = 0$ and $\omega_0 = 0$. To satisfy this constraint, it suffices to require the stress approximation functions to represent self equilibrated fields, yielding condition (112) instead of (65), and to select a particular stress approximation (113) that equilibrates the body forces, instead of condition (66):

$$D\mathbf{S}_{V0} = \mathbf{0}, \tag{112}$$

$$D\boldsymbol{\sigma}_{p0} = -\mathbf{b}_0. \tag{113}$$

Consequent upon the conditions above, the elastodynamic assumption (75) on the existence of a supporting equilibrating inertial field is never called upon. Enforcement of condition (112) in the Galerkin weighed residual enforcement (80) of the compatibility and kinematic boundary conditions (16) and (19) shows clearly that in the development of the elastostatic hybrid stress element no assumptions are made on the displacements in the domain of the element. The compatibility condition (77) for the hybrid stress element simplifies into form (115), given in Table 9, which depends only on the independent boundary displacement approximation (37) stated in Table 4.

Table 9. Finite element equations for the elastostatic hybrid stress model.

Statics (114)	Kinematics (115)	Elasticity (116)
$\mathbf{A}_{\Gamma 0}^t \mathbf{X}_{V0} = \mathbf{Q}_{i\Gamma 0} - \mathbf{Q}_{p\Gamma 0}$	$\mathbf{e}_0 = \mathbf{A}_{\Gamma 0} \mathbf{q}_{\Gamma 0} + \mathbf{e}_{u\Gamma 0}$	$\mathbf{e}_0 = \mathbf{F}_{V0} \mathbf{X}_{V0} + \mathbf{e}_{00}$

The remaining finite element equations in Table 9 can be obtained by direct specialisation of the elastodynamic equilibrium and elasticity conditions (76) and (78). Definitions (82), (86) and (89) to (92) still hold, now with $n = 0$.

9.3. Hybrid-Trefftz elements for elastostatics

The Trefftz constraints (62) and (63) when specialised for the governing Navier equation (99) reduce to the following:

$$(D\mathbf{k}_0 D^*) \mathbf{U}_{V0} = \mathbf{0} \quad \text{in } V^e, \tag{117}$$

$$(D\mathbf{k}_0 D^*) \mathbf{u}_{p0} + D\boldsymbol{\sigma}_{r0} + \mathbf{b}_0 = \mathbf{0} \quad \text{in } V^e. \tag{118}$$

Equations (40) to (42), (64), (67) and (68) still hold, with $n = 0$, while constraints (65) and (66) are replaced by equations (112) and (113), respectively. Implementing the same procedure described in section 6, it is easily verified that results (70) and (73) simplify into the following forms:

$$\mathbf{K}_{V0} = D\Gamma_0, \tag{119}$$

$$\mathbf{X}_{00} = \mathbf{X}_{b0} + \mathbf{X}_{p\Gamma 0}. \tag{120}$$

Constraints (101) and (113) can also be used to replace definition (107) by the following boundary integral equivalent form:

$$\mathbf{X}_b = \int U_R^t \mathbf{N} \boldsymbol{\sigma}_0 d\Gamma^e. \tag{121}$$

With regard to the elastostatic hybrid-Trefftz stress element, the implementation of the above Navier conditions in the manner suggested in section 8 produces the following boundary integral expressions for the elasticity arrays (96) and (97):

$$\mathbf{F}_{V0} = \mathbf{D}_{\Gamma 0}, \tag{122}$$

$$\mathbf{e}_{00} = \mathbf{e}_{p\Gamma 0}. \tag{123}$$

No further reference is explicitly made in this paper to the elastostatic solution mode. Firstly, because all matters yet to be addressed have already been stated elsewhere [1-6,11] in the context specific to the elastostatic hybrid-Trefftz displacement and stress elements. Secondly, because it is clearly a particular case of the elastodynamic mode, with the exception of the treatment of the rigid body displacement solution, which is specific to the elastostatic mode.

This solution, defined by vector \mathbf{q}_R in the displacement estimate (100), is directly produced by the implementation of the displacement model, together with the deformation inducing displacement modes associated with array \mathbf{q}_{V0} . The associated strain and stress distributions are then directly computed from definitions (40) and (60), respectively.

Conversely, the implementation of the hybrid-Trefftz stress element produces the solution for the generalized stress vector, \mathbf{X}_{V0} , which is used to determine the stress and strain fields using estimate (34) and definition (95), respectively. The associated displacement field is now defined by equation (124), meaning that the generalized rigid-body displacements, \mathbf{q}_R , are computed *a posteriori*, as it is typical of the hybrid stress elements:

$$\mathbf{u}_0 = \mathbf{U}_{V0}\mathbf{X}_{V0} + \mathbf{U}_R\mathbf{q}_R + \mathbf{u}_{p0} \quad \text{in } V^e. \tag{124}$$

10. FINITE ELEMENT SOLVING SYSTEMS

The elementary governing system (125) for the displacement model is obtained by eliminating the generalized stresses, \mathbf{X}_n , as explicit variables by combining the descriptions found for Statics, Kinematics and Elasticity, as stated by equations (45) to (47) in Table 6, and using the Trefftz results (70) and (73). The implementation of a similar operation on equations (76) to (78) to eliminate the generalized strains, \mathbf{e}_n , yields the elementary governing system (126) for the stress model of the hybrid-Trefftz finite element formulation, provided that results (96) and (97) are taken into account.

Table 10. Governing systems for the hybrid-Trefftz formulation.

Displacement Model (125)	Stress Model (126)
$\begin{bmatrix} \mathbf{D}_\Gamma & -\mathbf{B}_\Gamma \\ -\mathbf{B}_\Gamma^t & \mathbf{0} \end{bmatrix}_n \begin{Bmatrix} \mathbf{q}_V \\ \mathbf{X}_\Gamma \end{Bmatrix}_n = \begin{Bmatrix} \mathbf{X}_{t\Gamma} - \mathbf{X}_{p\Gamma} \\ \mathbf{v}_{p\Gamma} - \mathbf{v}_{u\Gamma} \end{Bmatrix}_n$	$\begin{bmatrix} \mathbf{0} & \mathbf{A}_\Gamma^t \\ \mathbf{A}_\Gamma & -\mathbf{D}_\Gamma \end{bmatrix}_n \begin{Bmatrix} \mathbf{q}_\Gamma \\ \mathbf{X}_V \end{Bmatrix}_n = \begin{Bmatrix} \mathbf{Q}_{t\Gamma} - \mathbf{Q}_{p\Gamma} \\ \mathbf{e}_{p\Gamma} - \mathbf{e}_{u\Gamma} \end{Bmatrix}_n$

The structure displayed by systems (125) and (126) combines the essential features of the solving systems of the conventional finite and boundary element formulations:

(S10) *The hybrid-Trefftz finite element solving systems are symmetric, sparse and described by arrays with boundary integral definitions.*

Besides this special structure, the hybrid-Trefftz models described above can be implemented on elements with arbitrary geometry using hierarchical approximation bases to facilitate the development of adaptive procedures. These features must be conveniently exploited in the phase of numerical implementation, in particular when parallel processing is used.

10.1. Selection of the approximation functions

Hierarchical solutions for the wave equation are readily available in the literature, for a wide range of linear and non-linear problems for alternative structural models and for different boundary conditions. For instance, Fourier series in Cartesian coordinates and Bessel series in polar coordinates can be used as approximation bases to solve two-dimensional problems [12,13]. Also, to implement semi-infinite elements, and thus avoid the different corrective techniques the conventional finite element formulations rely on, namely the so-called absorbing boundary conditions, it suffices to include the decaying displacement and stress modes that satisfy the Sommerfeld diffusion condition. There are also available local solutions for particularly relevant boundary conditions, which can be used to enrich the finite element approximation.

These are the functions used to create the domain approximation bases present in definitions (28) and (34), namely the displacement functions \mathbf{U}_{Vn} and \mathbf{u}_{pn} and the stress functions \mathbf{S}_{Vn} and $\boldsymbol{\sigma}_{pn}$. To ensure convergence, the boundary approximation bases $\mathbf{S}_{\Gamma n}$ and $\mathbf{U}_{\Gamma n}$, present in definitions (30) and (37) are chosen to represent complete subsets of the domain approximations:

$$\mathbf{U}_{\Gamma n} \subseteq \mathbf{U}_{Vn} \quad \text{on } \Gamma^e, \quad (127)$$

$$\mathbf{S}_{\Gamma n} \subseteq \mathbf{S}_{Vn} \quad \text{on } \Gamma^e. \quad (128)$$

The static and kinematic indeterminacy numbers of the displacement and stress hybrid-Trefftz elements are instrumental to address the question arising on the relative dimensions of the domain and boundary approximation bases. Let n_{qV} represent the number of approximation functions used in the displacement approximation law \mathbf{U}_{Vn} and $n_{X\Gamma}$ represent the dimension of the traction approximation basis $\mathbf{S}_{\Gamma n}$, from which the displacement model is derived. Description (46) for Statics shows that the static indeterminacy number of the hybrid-Trefftz displacement element is defined by equation (129) and that its kinematic indeterminacy number is given by equation (130), according to definition (45) for Kinematics:

$$\alpha_d = n_{qV} + n_{X\Gamma}, \quad (129)$$

$$\beta_d = n_{qV} - n_{X\Gamma} \geq 0. \quad (130)$$

The corresponding numbers for the stress element are obtained from descriptions (76) and (77) for Statics and Kinematics, respectively. Letting n_{XV} and n_q represent the number of stress and boundary displacement approximation functions used to build the approximation bases \mathbf{S}_{Vn} and \mathbf{U}_n , the following definitions are found:

$$\alpha_s = n_{XV} - n_{q\Gamma} \geq 0, \quad (131)$$

$$\beta_s = n_{XV} + n_{q\Gamma}. \quad (132)$$

For a given dimension of the approximation in the domain of the element, conditions (130) and (131) state the minimum dimension for the boundary approximation basis. The following statements are consequent upon these conditions, under constraints (127) and (128), and exploit the fact that the solutions for the wave equation are naturally linearly independent:

(S11) *The elastodynamic matrices, \mathbf{K}_V and \mathbf{F}_V , the mass and mobility matrices, \mathbf{M}_V and $\bar{\mathbf{M}}_V$, and the elasticity matrix, \mathbf{D}_Γ , are positive definite.*

(S12) *The equilibrium and compatibility matrices, \mathbf{B}_Γ and \mathbf{A}_Γ , are full-rank.*

10.2. Pre-processing

Because the hybrid-Trefftz models are based on hierarchical approximation functions weighed by generalised variables — the node or frame concepts have never been called upon — it is possible to implement the finite element solving systems (125) and (126) using a coarse mesh of few but highly rich elements. The data structure is substantially reduced and dependency on meshing algorithms is greatly diminished, yielding to important gains in the pre-processing phase of the implementation of the finite element program. Moreover, there are no constraints on the geometry of these elements. For instance, they can be multiply connected or semi-infinite, provided the application of the divergence theorem is correctly implemented in the computation of the finite element matrices.

Alternatively, the hybrid-Trefftz finite element models described above can be so defined and programmed to be assembled and processed by a conventional finite element solver. This is achieved simply by using boundary approximation functions that recover the connecting frame functions of the conventional displacement and stress elements, and condense the corresponding systems (125) and (126), at element level, on the relevant nodal variables. This is the technique used to enrich the libraries of the currently available commercial solvers with hybrid-Trefftz finite elements.

10.3. Computation and assemblage of the finite element arrays

The computation of the finite element arrays is greatly simplified by the reduction in the dimension of the problem under analysis, as all the matrices and vectors intervening in the solving system have boundary integral expressions. When the geometry of the elements is not too complex, it is possible to avoid the implementation of numerical integration schemes and derive directly the analytical expressions for the finite element arrays using the currently available symbolic programming codes. A third aspect that is worth exploiting is the fact that the solutions for the wave equation often include families of orthogonal functions. Besides contributing to increase the sparsity index of the solving system, this property allows for the identification of the system coefficients that are known *a priori* to be null.

For the displacement model, the assemblage of the elementary systems (125) to obtain the governing system for the finite element mesh is implemented simply by requiring connecting elements to share the same traction approximation law (30) on the boundaries they share and exploit the dual role played by the prescribed displacements vector, $\mathbf{u}_{\Gamma n}$. It can be used either to represent the displacements prescribed on the boundary of the finite elements sharing the kinematic boundary of the structure or the unknown displacements on the boundary of an adjacent element. In the latter case, the resulting indeterminacy — $\mathbf{v}_{u\Gamma n}$ is unknown — is solved during the finite element mesh assemblage by enforcing inter-element displacement continuity or, denoting by i the boundary shared by elements j and k :

$$\mathbf{v}_{u\Gamma n}^{i,j} = \mathbf{v}_{u\Gamma n}^{i,k} \quad (133)$$

Similarly, to assemble the elementary systems (126) for the stress model, it suffices to enforce the same displacement law (37) on the boundaries shared by connecting elements. The generalised traction vector, $\mathbf{t}_{\Gamma n}$, can be used either to represent the tractions on the boundary of the finite elements sharing the true static boundary of the finite element mesh or the unknown tractions on the boundary of an adjacent element. This indeterminacy is solved by enforcing inter-element traction continuity, by letting:

$$\mathbf{Q}_{t\Gamma n}^{i,j} + \mathbf{Q}_{t\Gamma n}^{i,k} = 0. \quad (134)$$

Therefore, the assembly of the hybrid-Trefftz finite elements is based on simple, direct allocation operations, which do not involve the operation of superimposition of the contribution of connecting elements typical of the conventional finite element formulation.

10.4. Storage and solution of the finite element system

The resulting finite element solving systems present the same structure as that of the elementary systems they develop from. In particular, the elastodynamic matrix is block diagonal and the equilibrium and compatibility matrices have minimum overlapping as the following properties hold:

- (S13) *The generalised displacements and stresses, \mathbf{q}_V and \mathbf{X}_V , are strictly element dependent.*
- (S14) *Inter-element connectivity is solely dependent on the generalised boundary tractions and displacements, \mathbf{X}_Γ and \mathbf{q}_Γ .*

The high sparsity indices that the solving systems (125) and (126) display are consequent upon these properties and enhanced by the fact that the approximation sets are often orthogonal. This low level element linkage allied to the implementation of the hybrid-Trefftz models on coarse meshes of highly rich elements is naturally suited for parallel processing [14], as the flow of communications is substantially reduced. The structure of the solving systems (125) and (126), allied to the high sparsity of the finite element matrices themselves, motivates the combination of sparse matrix storage schemes with the efficient, direct or iterative solvers for large, symmetric and linear sparse systems now available.

10.5. Post-processing

The methods described below to reconstruct the domain and boundary finite element approximations show that the post-processing phase can be implemented very efficiently for the finite element models under discussion. The graphic representation of the finite element solutions is simplified because the field and boundary approximations are strictly element dependent and derive from linear combinations that can be efficiently processed using fast transform techniques, thus enhancing the implementation of parallel processing. Moreover, smoothing techniques need not to be applied as highly accurate solutions can be obtained using high degree, hierarchical approximation functions.

The technique of solving the dynamic problem in the frequency domain is not designed to model the response of the system in the time domain, for particular initial conditions. This response may, however, be recovered by combining the solutions of the n -th order solving systems in any of the alternative formats (125) and (126), according to the discretisation criteria that have been adopted. For instance, for the displacement model definitions (20) and (29) produce the following expression for the estimate found for the displacement field:

$$\mathbf{u} = \mathbf{u}_1 + \sum_{n=0}^k \mathbf{U}_{Vn} \mathbf{q}_{2Vn} \gamma_n e^{i\omega_{2n}t} . \tag{135}$$

In definition (135), \mathbf{u}_1 is the solution of the forced vibration problem, k is the order in the truncation of the time series, \mathbf{q}_{2Vn} is the eigenvector for the free vibration problem and γ_n denotes the corresponding amplitude:

$$\mathbf{u}_1 = \sum_{n=-k}^k \left(\mathbf{U}_{Vn} \mathbf{q}_{1Vn} e^{i\omega_{1n}t} + \mathbf{u}_{pn} \right) , \tag{136}$$

$$\gamma_n = \alpha_n \cos(\omega_{2n}t) + i\beta_n \sin(\omega_{2n}t) . \tag{137}$$

The free vibration displacement amplitudes α_n and β_n are determined using solution (135) to enforce the initial conditions (13) and (14) in the following weighed residual forms, where \mathbf{u}_{10} and $\dot{\mathbf{u}}_{10}$ represent the initial conditions due to the forced vibration solution:

$$\int (\rho_n \mathbf{U}_{Vn} \mathbf{q}_{2Vn})^t \left(\sum_{n=1}^k \mathbf{U}_{Vn} \mathbf{q}_{2Vn} \alpha_n + \mathbf{u}_{10} - \mathbf{u}_0 \right) dV = 0 , \tag{138}$$

$$\int (\rho_n \mathbf{U}_{Vn} \mathbf{q}_{2Vn})^t \left(\sum_{n=1}^k i\omega_{2n} \mathbf{U}_{Vn} \mathbf{q}_{2Vn} \beta_n + \dot{\mathbf{u}}_{10} - \dot{\mathbf{u}}_0 \right) dV = 0. \tag{139}$$

Using definition (52) for the mass matrix and enforcing the \mathbf{M}_{Vn} — orthogonality condition (140) on the eigenvectors \mathbf{q}_{2Vn} , systems (138) and (139) generate definitions (141) and (142) for the amplitudes of the free vibration modes:

$$\mathbf{q}_{2Vm}^t \mathbf{M}_{Vn} \mathbf{q}_{2Vn} = \delta_{mn} A_{mn}, \tag{140}$$

$$A_{mn} \alpha_n = \mathbf{q}_{2Vn}^t \int \mathbf{U}_{Vn}^t \rho (\mathbf{u}_0 - \mathbf{u}_{10}) dV, \tag{141}$$

$$A_{mn} \omega_{2n} (i\beta_b) = \mathbf{q}_{2Vn}^t \int \mathbf{U}_{Vn}^t \rho (\dot{\mathbf{u}}_0 - \dot{\mathbf{u}}_{10}) dV. \tag{142}$$

In the hybrid-Trefftz displacement model, the displacements are determined using equations (135) and (136) together with definitions (141) and (142) for the free vibration amplitudes. The stresses are determined from definitions (143) and (144), based on the relationship previously established between the displacement and stress fields, consequent upon the implementation of the Trefftz constraints:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \sum_{n=0}^k \mathbf{S}_{Vn} \mathbf{q}_{2Vn} \gamma_n e^{i\omega_{2n} t}, \tag{143}$$

$$\boldsymbol{\sigma}_1 = \sum_{n=-k}^{+k} \left(\mathbf{S}_{Vn} \mathbf{q}_{1Vn} e^{i\omega_{1n} t} + \boldsymbol{\sigma}_{pn} \right). \tag{144}$$

A similar procedure can be used for the hybrid-Trefftz stress model. In either case, the solving systems for the free vibration amplitudes, defined by equations (141) and (142), can be written in terms of the elastodynamic matrix to avoid the computation of the generalised mass matrix.

11. STATIC AND KINEMATIC ADMISSIBILITY CONDITIONS

Except for the compatibility condition (16) in the displacement model and the equilibrium condition (15) in the stress model, in the hybrid-Trefftz finite element formulation all the remaining fundamental conditions are enforced on average, in a Galerkin-weighted residual form, as stated in Table 11, where the weighing functions are identified. The information collected in Table 11 is also used to relate the fundamental conditions (17) to (19) with the finite element descriptions found for Statics, Kinematics and Elasticity, summarised in Tables 6 and 7, respectively.

Table 11. Enforcement of the fundamental conditions.

Displacement Model		Condition		Stress Model	
\mathbf{U}_v -weighed residual,	Eq. (46)	Equilibrium,	Eq. (15)	Locally satisfied	
Locally satisfied		Compatibility,	Eq. (16)	\mathbf{S}_v -weighed residual,	Eq. (77)
\mathbf{E}_v -weighed residual,	Eq. (47)	Elasticity,	Eq. (17)	\mathbf{S}_v -weighed residual,	Eq. (78)
\mathbf{U}_v -weighed residual,	Eq. (46)	Static b.c.,	Eq. (18)	\mathbf{U} -weighed residual,	Eq. (76)
\mathbf{S}_Γ -weighed residual,	Eq. (45)	Kinematic b.c.,	Eq. (19)	\mathbf{S}_v -weighed residual,	Eq. (77)

In the displacement {stress} model, description (46) for Statics {(77) for Kinematics} combines in a single statement the average enforcement of the local equilibrium and static boundary conditions (15) and (18) {compatibility and kinematic boundary conditions (16) and (19)}:

(S15) In the displacement {stress} model, equation (46) {(77)} represents the finite element (weak) static {kinematic} admissibility condition.

In the displacement {stress} model, description (45) for Kinematics {(76) for Statics} enforces explicitly the kinematic boundary condition (19) {static boundary condition (18)}:

(S16) In the displacement {stress} model, equation (45) {(76)} represents the finite element (weak) kinematic {static} admissibility condition, as the compatibility condition (16) {equilibrium condition (15)} is locally satisfied.

It is recalled that the implicit stress estimate (64) {displacement estimate (75)} produced by the hybrid-Trefftz displacement {stress} element does satisfy locally the domain equilibrium condition (15) {compatibility condition (16)}. However, in general, this estimate will not be consistent, in the strong sense, neither with the traction approximation (30) {displacement approximation (37)} nor with the prescribed tractions (18) {displacements (19)}.

In the displacement {stress} model, the smaller the kinematic {static} indeterminacy number of the element is the stronger the enforcement of the kinematic {static} admissibility condition will be and, conversely, the weaker the enforcement of the static {kinematic} admissibility condition is. Hence, and considering results (127) to (132), the following statements are advanced:

(S17) The displacement {stress} model may produce kinematically {statically} admissible solutions.

(S18) The displacement {stress} model will, in general, produce weak statically {kinematically} admissible solutions.

12. RELATED ENERGY STATEMENTS

The finite element formulations are frequently derived from alternative energy statements, namely the virtual work equation (145) and stationary conditions on the potential energy (146), the potential co-energy (147) and the Hu–Washizu (148) and the Hellinger–Reissner (149) functionals:

$$\int \sigma_n^t \epsilon_n dV + \int \dot{\mathbf{u}}_n^t \rho_n \dot{\mathbf{u}}_n dV = \int \mathbf{b}_n^t \mathbf{u}_n dV + \int \mathbf{t}_{\Gamma_n}^t \mathbf{u}_n d\Gamma_\sigma + \int \mathbf{u}_{\Gamma_n}^t \mathbf{t}_n d\Gamma_u, \tag{145}$$

$$\Pi^n = \frac{1}{2} \int (\sigma_n + \sigma_{rn})^t (\epsilon_n - \epsilon_{rn}) dV + \frac{1}{2} \int \dot{\mathbf{u}}_n^t \rho_n \dot{\mathbf{u}}_n dV - \int \mathbf{b}_n^t \mathbf{u}_n dV - \int \mathbf{t}_{\Gamma_n}^t \mathbf{u}_n d\Gamma_\sigma, \tag{146}$$

$$\Pi_*^n = \frac{1}{2} \int (\sigma_n - \sigma_{rn})^t (\epsilon_n + \epsilon_{rn}) dV + \frac{1}{2} \int \dot{\mathbf{u}}_n^t \rho_n \dot{\mathbf{u}}_n dV - \int \mathbf{u}_{\Gamma_n}^t \mathbf{t}_n d\Gamma_u, \tag{147}$$

$$\Pi_{\text{HW}}^n = -\Pi^n + \int \mathbf{t}_n^t (\mathbf{u}_n - \mathbf{u}_{\Gamma_n}) d\Gamma_u, \tag{148}$$

$$\Pi_{\text{HR}}^n = -\Pi_*^n + \int \mathbf{u}_n^t (\mathbf{t}_n - \mathbf{t}_{\Gamma_n}) d\Gamma_\sigma. \tag{149}$$

In definitions (146) and (147) the residual stress and strain fields satisfy the following condition:

$$\int \sigma_{rn}^t \epsilon_{rn} dV = 0. \tag{150}$$

Also, definitions (148) and (149) are specialised for solutions that satisfy locally the compatibility and equilibrium conditions (15) and (16), as it is the case for the hybrid-Trefftz displacement and stress models, respectively.

The inner product of equilibrium and compatibility conditions found for the hybrid-Trefftz displacement stress model, respectively defined by equations (46) and (45) (76) and (77) yields the following results:

$$\omega_n^{-2} \mathbf{q}_{Vn}^t \mathbf{M}_{Vn} \mathbf{q}_{Vn} + \mathbf{X}_{\Gamma n}^t (\mathbf{v}_{u\Gamma n} - \mathbf{v}_{p\Gamma n}) = \mathbf{q}_{Vn}^t (\mathbf{X}_n - \mathbf{X}_{Pn}), \tag{151}$$

$$\omega_n^{-2} \mathbf{X}_{Vn}^t \tilde{\mathbf{M}}_{Vn} \mathbf{X}_{Vn} + \mathbf{q}_{\Gamma n}^t (\mathbf{Q}_{t\Gamma n} - \mathbf{Q}_{p\Gamma n}) = \mathbf{X}_{Vn}^t (\mathbf{e}_n - \mathbf{e}_{Pn}). \tag{152}$$

As a consequence of the consistency conditions (29), (33) and (44) for the displacement model and of conditions (36) and (39) for the stress model, the virtual work equation (145) is recovered by enforcing above the definitions given for the intervening arrays:

(S19) For the hybrid-Trefftz displacement and stress models, the inner product of the finite element equilibrium and compatibility conditions recovers the virtual work equation.

The hybrid-Trefftz finite element systems given in Table 10 can be stated in the common form (153), provided that identifications (154) and (155) hold for the displacement and stress models, respectively:

$$\begin{bmatrix} \mathbf{G} & \mathbf{M} \\ \mathbf{M}^t & -\mathbf{H} \end{bmatrix} \begin{Bmatrix} \mathbf{y} \\ \mathbf{x} \end{Bmatrix} = \begin{Bmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 \end{Bmatrix}, \tag{153}$$

$$\mathbf{x} \equiv \mathbf{X}_{\Gamma} \quad \text{and} \quad \mathbf{y} = \mathbf{q}_V, \tag{154}$$

$$\mathbf{x} \equiv \mathbf{X}_V \quad \text{and} \quad \mathbf{y} = \mathbf{q}_{\Gamma}. \tag{155}$$

Under certain sufficiency conditions [15, 16], system (153) is equivalent to the following pair of dual quadratic programs:

$$\min z = \frac{1}{2} \mathbf{x}^t \mathbf{H} \mathbf{x} + \frac{1}{2} \mathbf{y}^t \mathbf{G} \mathbf{y} + \mathbf{x}^t \mathbf{y}_0 \tag{156a}$$

$$\text{subject to: } \mathbf{G} \mathbf{y} + \mathbf{M} \mathbf{x} = \mathbf{x}_0, \tag{156b}$$

$$\min w = \frac{1}{2} \mathbf{x}^t \mathbf{H} \mathbf{x} + \frac{1}{2} \mathbf{y}^t \mathbf{G} \mathbf{y} - \mathbf{y}^t \mathbf{x}_0 \tag{157a}$$

$$\text{subject to: } \mathbf{M}^t \mathbf{y} - \mathbf{H} \mathbf{x} = \mathbf{y}_0. \tag{157b}$$

System (153) is thus uncoupled into sub-systems (156b) and (157b), which define the primal and dual constraint sets, respectively. The solutions of these sub-systems are said to be feasible solutions of the corresponding programs; it can be easily found that:

(S20) For the hybrid-Trefftz displacement and stress models, the primal feasible solutions (156b) {dual feasible solutions (157b)} represent the set of weak statically {kinematically} admissible finite element solutions.

Among all the primal and dual feasible solutions (156b) and (157b), the optimal solution, or solutions of the corresponding program are those that simultaneously minimise the associated objective function, z and w , as defined in statements (156a) and (157a), respectively. Under the equivalence conditions referred above, these optimal solutions are also the solutions of system (153). It is therefore of interest to identify the physical meaning of the objective functionals z and w .

The results given in Table 12 are obtained substituting in their expressions the definitions found for the relevant matrices and vectors intervening in the governing systems (125) and (126) of the displacement and stress models, for the identifications (154) and (155), respectively, and manipulating next the resulting equations using the expressions found for the finite element arrays.

Table 12. Identification of the objective functions of the associated quadratic programs.

Displacement Model	Functional	Stress Model
Π_* + constant	z	Π_* + constant
Π + constant	w	Π + constant
$-\Pi_{HW}$ + constant	z_*	$-\Pi_{HR}$ + constant

In consequence of the above statements and of the identifications given in Table 12, and as the constant terms do not affect the optimal values of the objective functions, it may be stated that:

(S21) *For the hybrid-Trefftz displacement and stress models, the dual program (157) {primal program (156)} encodes the theorem on the minimum potential energy {co-energy}.*

Under the same sufficiency conditions [15, 16] the alternative pair of quadratic programs (158) and (159) is also equivalent to system (153):

$$\min z_* = -\frac{1}{2}\mathbf{x}^t\mathbf{H}\mathbf{x} + \frac{1}{2}\mathbf{y}^t\mathbf{G}\mathbf{y} + \mathbf{y}^t\mathbf{M}\mathbf{x} - \mathbf{x}^t\mathbf{y}_0 - \mathbf{y}^t\mathbf{x}_0, \tag{158}$$

$$\min w_* = -\frac{1}{2}\mathbf{x}^t\mathbf{H}\mathbf{x} + \frac{1}{2}\mathbf{y}^t\mathbf{G}\mathbf{y} + \mathbf{y}^t\mathbf{M}\mathbf{x} \quad \text{subject to system (153)}. \tag{159}$$

Program (158) enjoys the feature of being unconstrained, meaning that a *stationary* value of functional z_* recovers the solution set of the governing system (153). The identifications for this functional given in Table 12 are obtained using a process similar to that described above for functionals z and w . Again, as the constant terms do not affect the objective function at optimality, it can be stated that:

(S22) *For the hybrid-Trefftz displacement {stress} model, as the compatibility {equilibrium} condition is locally satisfied, program (158) encodes the stationary condition on the Hu-Washizu {Hellinger-Reissner} functional.*

13. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Cottle's duality theorem [17] when specialised to programs (156) and (157) justifies the following statements on the existence of solutions to system (153):

(S23) *If $(\mathbf{q}_V^s, \mathbf{X}_\Gamma^s)$ and $(\mathbf{q}_V^k, \mathbf{X}_\Gamma^k)$ $\{(\mathbf{X}_V^s, \mathbf{q}_\Gamma^s)$ and $(\mathbf{X}_V^k, \mathbf{q}_\Gamma^k)\}$ are respectively finite element (weak) statically and kinematically admissible solutions of the hybrid-Trefftz displacement {stress} model, then $\Pi + \Pi_* > 0$.*

(S24) *If $(\mathbf{q}_V^s, \mathbf{X}_\Gamma^s)$ and $(\mathbf{q}_V^k, \mathbf{X}_\Gamma^k)$ $\{(\mathbf{X}_V^s, \mathbf{q}_\Gamma^s)$ and $(\mathbf{X}_V^k, \mathbf{q}_\Gamma^k)\}$ are optimal solutions of the primal and dual programs, respectively, there exists for the hybrid-Trefftz displacement {stress} model a solution $(\mathbf{q}_V, \mathbf{p}_\Gamma)$ $\{(\mathbf{X}_V, \mathbf{q}_\Gamma)\}$ satisfying $\mathbf{D}_\Gamma\mathbf{q}_V = \mathbf{D}_\Gamma\mathbf{q}_V^s$ $\{\mathbf{D}_\Gamma\mathbf{X}_V = \mathbf{D}_\Gamma\mathbf{X}_V^k\}$ such that $(\mathbf{q}_V, \mathbf{p}_\Gamma)$ and $(\mathbf{q}_V^k, \mathbf{p}_\Gamma)$ $\{(\mathbf{X}_V^s, \mathbf{q}_\Gamma)$ and $(\mathbf{X}_V, \mathbf{q}_\Gamma^k)\}$ are respectively optimal solutions of the dual and primal programs, and $\min \Pi + \min \Pi_* = 0$.*

(S25) *If (weak) finite element kinematically {statically} admissible solutions do not exist but statically {kinematically} solutions do exist, then Π $\{\Pi_*\}$ is unbounded.*

(S26) *If either the primal or the dual program has an optimal solution, then there is a solution which is optimal for both primal and dual programs.*

(S27) *If both (weak) statically and kinematically admissible solutions exist, then both primal and dual programs have optimal solutions.*

The sufficient condition stated below for the existence of unique optimal solutions results from the application of the uniqueness theorem [18] to the quadratic programs (156) and (157):

(S28) *If, for the hybrid-Trefftz displacement model, the primal system (156b) {dual system (157b)} is non-empty and $(\mathbf{q}_V^s, \mathbf{X}_\Gamma^s)$ $\{(\mathbf{q}_V^k, \mathbf{X}_\Gamma^k)\}$ is a (weak) statically {kinematically} admissible solution of program (156) {(157)}, then if the elasticity matrix \mathbf{D}_Γ is positive definite, the kinematic solution $\mathbf{q}_V^s = \mathbf{q}_V = \mathbf{q}_V^k$ is the unique optimal solution; the static solution may be multiple at optimality, $\mathbf{X}_\Gamma^s \neq \mathbf{X}_\Gamma^k$, unless the equilibrium matrix \mathbf{B}_Γ is full-rank.*

(S29) *If, for the hybrid-Trefftz stress model, the primal system (156b) {dual system (157b)} is non-empty and $(\mathbf{X}_V^s, \mathbf{q}_\Gamma^s)$ $\{(\mathbf{X}_V^k, \mathbf{q}_\Gamma^k)\}$ is a (weak) statically {kinematically} admissible solution of program (156) {(157)}, then if the elasticity matrix \mathbf{D}_Γ is positive definite, the static solution $\mathbf{X}_V^s = \mathbf{X}_V = \mathbf{X}_V^k$ is the unique optimal solution; the kinematic solution may be multiple at optimality, $\mathbf{q}_\Gamma^s \neq \mathbf{q}_\Gamma^k$, unless the compatibility matrix \mathbf{A}_Γ is full-rank.*

If, because of poor judgement in setting-up the approximation bases, rank deficiency does occur but the solution set is non-empty, the quality of the static {kinematic} solution obtained with the displacement stress model can be strongly affected by the emerging spurious modes and thus be rendered useless for practical applications. However, under these circumstances the kinematic {static} solution thus obtained still has practical relevance because it derives from the consistent implementation of the finite element state conditions and can be used to compute both the stress and displacement distributions from the corresponding domain approximation conditions.

14. CLOSURE

Two alternative models for the hybrid-Trefftz finite element formulation are derived from the fundamental conditions of linear elastodynamics. It is shown that the resulting finite element systems are consistent with the relevant energy statements. These systems embody the essential features of the governing systems obtained with the conventional finite element and boundary element formulations; symmetry, sparsity and boundary integral expressions for all the intervening arrays. Because hierarchical bases are used, the hybrid-Trefftz models can be implemented on coarse meshes of few but highly rich and naturally hierarchical elements with arbitrary geometry and low-level inter-element connectivity. These properties lead to important numerical implementation advantages, in particular when parallel processing is used. The hybrid-Trefftz elements produce, in general, weak statically and kinematically admissible solutions. However, the stress and the displacement elements are so designed as to be capable of generating finite element solutions that satisfy locally the static and the kinematic admissibility conditions, respectively. The practical implementation of the sufficient conditions that ensure the existence and uniqueness of the finite element solutions offers no particular difficulty and is necessary to support the development of reliable computer codes.

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