

## Self-adaptive Trefftz procedure for harmonic problems

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The paper propose two adaptive algorithms based on a Trefftz method for two-dimensional Laplace equation satisfying the maximal principle. First one for given the error tolerance and an initial number of terms in the solution expansion, the algorithm computes expansion coefficient by location of boundary conditions and evaluates the maximum absolute error on the boundary. If error exceeds the error the tolerance, additional expansion terms and boundary collocation points are added and process repeated until the tolerance is satisfied. The second one is based on Galerkin formulation of Trefftz method and utilizes the exact potential error norm for predict a new mesh and new solution expansion until the tolerance is satisfied.

### 1. INTRODUCTION

A successful application of various self-adaptive techniques in FEM and BEM prompted attempts to use them also in the Trefftz method [1–5], with the hope of overcoming some numerical difficulties inherent in this method.

One of the problems encountered when using the Trefftz method is the selection of a number of trial functions used in the approximate solution. If the number of trial functions is small, the accuracy of the solution may be unsatisfactory; on the other hand too large a number of trial functions may lead to ill-conditioned system of the linear equations from which function multipliers are determined. This phenomenon is common for all versions of the Trefftz method. In the case of using the boundary collocation method, one may be confronted with an additional difficulty, which is the so called Runge phenomenon. It manifests itself in the occurrence of great errors in satisfaction of the boundary conditions between the collocation points, despite the fact that at the collocation points themselves, the conditions are fulfilled exactly.

The numerical difficulties mentioned above can be avoided by using some adaptive versions of the Trefftz method. Two such versions are proposed in this paper. They are intended for solving the two-dimensional Laplace equation satisfying the maximal error principle. In the first version, for a given error tolerance and for initial number of terms in the solution expansion, the algorithm computes the expansion coefficients by the collocation of boundary conditions and evaluates the maximum absolute error on the boundary. If the error exceeds the error tolerance, additional expansion terms and boundary collocation points are added and the process repeats until the tolerance is satisfied.

The second version is based on the Galerkin formulation of the Trefftz method and utilises the exact potential error norm for predicting a new mesh and new solution expansion until the tolerance is satisfied.

## 2. VERSIONS OF TREFFTZ METHOD USED IN ADAPTATION

We consider the two-dimensional potential problem. The governing equation and the boundary conditions, expressed in the polar co-ordinate system  $(R, \theta)$  are

$$\frac{\partial^2 \Phi}{\partial R^2} + \frac{1}{R} \frac{\partial \Phi}{\partial R} + \frac{1}{R^2} \frac{\partial^2 \Phi}{\partial \theta^2} = 0 \quad \text{in } \Omega \quad (1)$$

and

$$\Phi = \bar{\varphi} \quad \text{on } \Gamma_1, \quad (2)$$

$$\frac{\partial \Phi}{\partial n} = \bar{q} \quad \text{on } \Gamma_2, \quad (3)$$

where  $\Phi$  is the potential function.  $\bar{\varphi}$  and  $\bar{q}$  are the specified values of the potential and of its normal derivations, respectively.  $\Omega$  is the domain occupied by the object, while  $\Gamma_1$  and  $\Gamma_2$  are the parts of boundary ( $\Gamma = \Gamma_1 + \Gamma_2$ ).  $n$  denotes the outward unit vector, normal to the boundary.

For the Trefftz method the potential  $\Phi$  is approximated by superposition of a finite number of the T-complete functions  $N_i(R, \theta)$  ( $i = 1, \dots, n$ ), each multiplied by a coefficient  $a_i$

$$\bar{\Phi} = \sum_{i=1}^n a_i N_i(R, \theta). \quad (4)$$

The set functions  $N_i(R, \theta)$  are adopted in a manner that they satisfy the harmonic equation

$$\frac{\partial^2 N_i}{\partial R^2} + \frac{1}{R} \frac{\partial N_i}{\partial R} + \frac{1}{R^2} \frac{\partial^2 N_i}{\partial \theta^2} = 0. \quad (5)$$

The undetermined coefficients  $a_i$ , in turn, are calculated in a way that guarantees approximate satisfaction of the boundary conditions. Depending on how the trial functions are constructed and how the boundary conditions are satisfied, several versions of the Trefftz method can be distinguished. In presentation of our adaptation procedure we will consider only two versions, namely: a) boundary collocation method, b) boundary Galerkin method.

Before proceeding to a detailed discussions of the two methods let us specify the residuals on the boundaries. They are of the form:

$$\frac{\partial \tilde{\Phi}}{\partial n} = \sum_{i=1}^n a_i \frac{\partial N_i}{\partial n}, \quad (6)$$

$$R_i \equiv \sum_{i=1}^n a_i N_i - \bar{\varphi} \neq 0 \quad \text{on } \Gamma_1. \quad (7)$$

The first one is obtained by substituting Eq. (4) into Eq. (2). The second one results from substituting the approximate normal derivative

$$R_2 \equiv \sum_{i=1}^n a_i \frac{\partial N_i}{\partial n} - \bar{q} \neq 0 \quad \text{on } \Gamma_2 \quad (8)$$

into Eq. (3).

### 2.1. Boundary collocation method

In the boundary collocation method we require vanishing of the residuals  $R_1$  and  $R_2$  at some selected points belonging to boundaries  $\Gamma_1$  and  $\Gamma_2$ .



For  $M_1$  collocation points selected on boundary  $\Gamma_1$  this requirement has the form

$$\sum_{i=1}^n a_i N_i(P_j) = \bar{\varphi}(P_j), \quad P_j \in \Gamma_1, \quad j = 1, \dots, M_1, \quad (9)$$

while for  $M_2$  collocation points on boundary  $\Gamma_2$  it is

$$\sum_{i=1}^n a_i \frac{\partial N_i(Q_j)}{\partial n} = \bar{q}(Q_j), \quad Q_j \in \Gamma_2, \quad j = 1, \dots, M_2. \quad (10)$$

If  $n = M_1 + M_2$  the equations (9)–(10) for a sufficient basis for determining the searched coefficients  $a_i$ .

## 2.2. Boundary Galerkin method

Introducing the weighting functions  $W_1$  and  $W_2$  of residuals (6) and (7), respectively, we can form the weighted residual equation

$$\int_{\Gamma_1} W_1 R_1 d\Gamma + \int_{\Gamma_2} W_2 R_2 d\Gamma = 0. \quad (11)$$

If we choose  $W_1$  and  $W_2$  as follows:  $W_1 = \frac{\partial N_j}{\partial n}$  and  $W_2 = -N_j$  then Eq. (11) becomes

$$\int_{\Gamma_1} \frac{\partial N_j}{\partial n} \left( \sum_{i=1}^n a_i N_i - \bar{\varphi} \right) d\Gamma - \int_{\Gamma_2} N_j \left( \sum_{i=1}^n a_i \frac{\partial N_i}{\partial n} - \bar{q} \right) d\Gamma = 0, \quad j = 1, \dots, n \quad (12)$$

or

$$\sum_{i=1}^n B_{ji} a_i = f_j, \quad (13)$$

where

$$B_{ji} = \int_{\Gamma_1} \frac{\partial N_j}{\partial n} N_i d\Gamma - \int_{\Gamma_2} N_j \frac{\partial N_i}{\partial n} d\Gamma, \quad (14)$$

$$f_i = \int_{\Gamma_1} \frac{\partial N_j}{\partial n} \bar{\varphi} d\Gamma - \int_{\Gamma_2} N_j \bar{q} d\Gamma. \quad (15)$$

It is easy to recognise that Eq. (12) reflects the essence of the Galerkin method.

## 3. ADAPTIVE ALGORITHMS

To measure the quality of the approximate solution (4) one must utilize an appropriate norm of error which will provide a scalar measure of the distance between the exact and approximate solutions. Here, as a norm, we adopt the maximal absolute distance between the exact solution  $\Phi$  and the solution  $\tilde{\Phi}$

$$ERR_1 = \max |\Phi - \tilde{\Phi}|. \quad (16)$$

According to the maximal error principle [6], the maximal error for boundary method applied to the Laplace equation, occurs at the boundary.

For the examples considered below the maximal error  $ERR_1$  is found by incremental search in 200 equally distributed control points on this part of boundary where boundary condition is fulfilled approximately.

### 3.1. Boundary collocation method

The maximal error in the boundary collocation method is a function of a number of trial functions  $n$  (number of collocation points) and co-ordinates of collocation points. Consequently, the adaptation procedure should minimise the error by searching for the optimal position of collocation points and by optimisation of their number. In general, the optimisation of the co-ordinates of the collocation points when their number  $n$  is fixed, is a relatively complicated nonlinear problem. On the other hand, by increasing a number of collocation points with the fixed co-ordinates, in the end, we get ill-conditioned set of linear equations (9)–(10).

Some simplifications of the above nonlinear problem can be achieved by using the following adaptive collocation algorithm:

- Step 1.* Choose an initial number of expansion terms (trial function)  $n$  in approximate solution (4).
- Step 2.* Input the error tolerance  $TOL_1$  and  $TOL_2$  of satisfaction of boundary conditions (2) and (3), and choose  $M_1$  and  $M_2$  ( $n = M_1 + M_2$ ) which are the initial numbers of equidistant collocation points on boundary  $\Gamma_1$  and  $\Gamma_2$ , respectively.
- Step 3.* Find an approximate solution (4) to the problem by collocation on points chosen in *Step 2*, by solving the equations (9)–(10).
- Step 4.* Evaluate the approximate solution at test points along the boundary. Find the maximum absolute error  $ERR_1$  and its location  $LOC_1$  on  $\Gamma_1$  and maximum absolute error  $ERR_2$  and its location  $LOC_2$  on  $\Gamma_2$ .
- Step 5.* If  $ERR_1 > TOL_1$  and  $ERR_2 > TOL_2$ , then introduce additional collocation points at the locations  $LOC_1$  and  $LOC_2$ , update the number of equations  $n = n + 2$ , and find the approximate solution by collocation of the boundary conditions. Otherwise terminate the calculations and output the results.
- Step 6.* Repeat *Step 4*.

### 3.2. Boundary Galerkin method

The maximal error in the Galerkin method is a function of the number of trial functions and of the integration procedure applied at the boundary. In boundary integration we choose some number of boundary elements and some number of integral points using the Gauss quadrature on element. The purpose of the proposed adaptive algorithm is the optimisation of the number of trial functions and of the number of boundary elements. The number of Gauss quadrature points is prescribed and is not subjected to optimisation.

The adaptive algorithm for Galerkin method consists of the following steps:

- Step 1.* Input:  $INN$  – initial number of trial functions,  $INNI$  – initial number of boundary elements,  $NIMA$  – maximal number of boundary elements,  $MMA$  – maximal number of trial functions,  $TOL$  – error tolerance of satisfaction of boundary conditions,  $TOL1$  – maximal difference between two following values of integrals.
- Step 2.* Calculate  $K_{ij}$ ,  $f_i$  and solve linear set of equations (13)
- Step 3.* Find the maximal error of satisfaction of boundary condition  $ERR$ .
- Step 4.* If  $ERR < TOL$  then terminate the calculations and output the results, else
- Step 5.* If  $INN > MMA$  or  $INNI > NIMA$  then stop, else



Step 6. If difference between the last two errors  $ERR$  is greater than  $TOL1$ , then  $INNI = INNI + 1$  and go to Step 2 adding one more boundary element else

Step 7. Substitute initial value for the number of element  $INNI$ , i.e.  $INNI = INNI + 1$  and go to Step 2 adding one more trial function.

#### 4. TEST PROBLEMS

In this section, the proposed adaptation algorithms are used for solving two test problems. In all these problems, special trial functions which fulfil not only governing equation but also boundary condition on some part of boundary are applied [7].

In order to evaluate the conditioning of the linear set of equations (13) the following measure is used:

$$\beta = \frac{1}{n} \|\mathbf{B}\| \|\mathbf{B}^{-1}\|, \tag{17}$$

where

$$\|\mathbf{B}\| = \sqrt{\sum_{i=1}^n \sum_{j=1}^n b_{ij}^2}. \tag{18}$$

#### Problem I

#### Saint-Venant torsion problem of a square rod

The cross-section of the rod under consideration is shown in Fig. 1a. Because of multiple symmetry of the cross section of the bar, the torsion problem can be considered for a selected repeated element of the section. The element itself, as well as its boundary value problem, are presented in Fig. 1b.

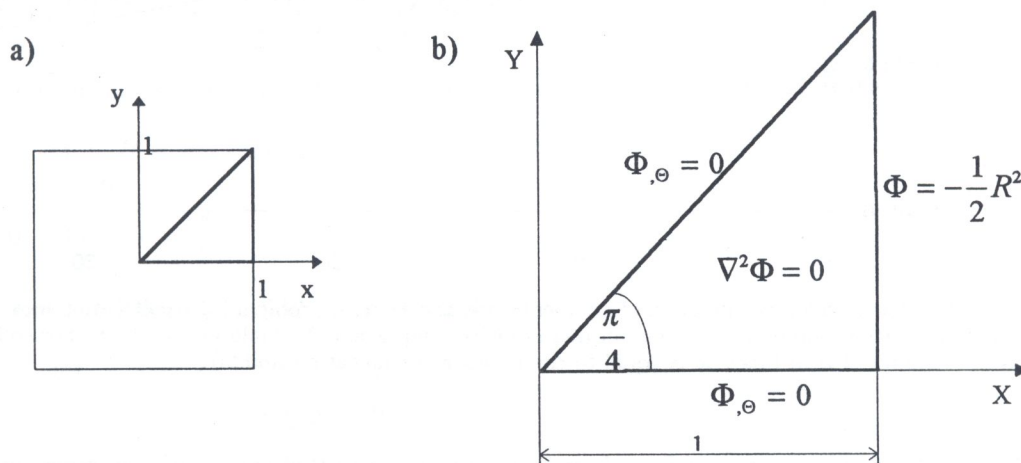


Fig. 1.

Let us assume the approximate solution for the repeated element in the following form:

$$\tilde{\Phi} = \sum_{k=1}^n a_k R^{4(k-1)} \cos(4(k-1)\theta). \tag{19}$$

This solution fulfils exactly not only the differential equation, but also boundary conditions:

$$\frac{\partial \Phi}{\partial \theta} = 0 \quad \text{for} \quad \theta = 0 \quad \text{and for} \quad \theta = \frac{\pi}{4}. \tag{20}$$

The only condition that is satisfied approximately is that related to segment AB:

$$\Phi = -\frac{1}{2}R^2 = 0 \quad \text{for} \quad X = 1, \quad 0 \leq Y \leq 1. \quad (21)$$

The behaviours of maximal local errors as a function of number of trial functions for three versions of the method, namely: (1) collocation method with equidistant collocation points, (2) collocation method with adaptation and (3) Galerkin method with adaptation are shown in Fig. 2. One can observe that for the first method, the maximal local error decreases with the increase of the number of collocation points monotonically. However, this takes place only below a critical certain number of collocation point. With the increase of the number collocation points the condition number increases, what is shown in Table 1, and set of linear equations is ill-conditioned. At

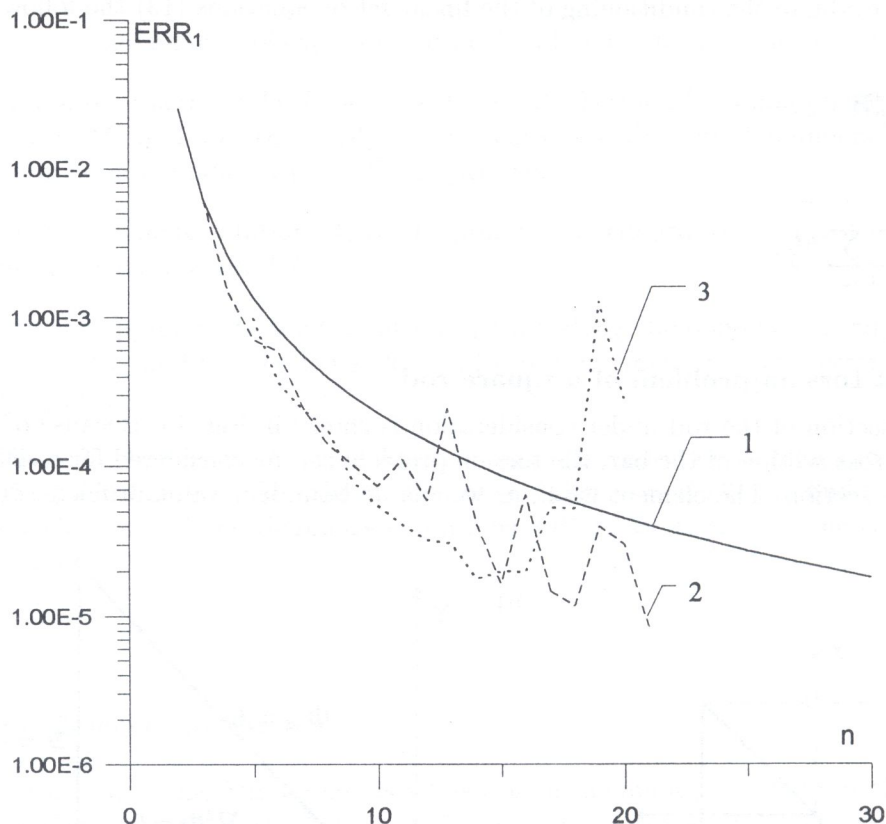


Fig. 2. Maximal local error as a function of number of trial function for problem I. 1 – collocation method with equidistant collocation points, 2 – collocation method with adaptation, 3 – Galerkin method; (For collocation method the number of trial function is equal to the number of collocation points)

Table 1. Maximal local error  $ERR_1$  and condition number  $\beta$  versus of number of collocation points  $n$

$n$	COLLOCATION ERROR $ERR_1$	CONDITION NUMBER $\beta$	$n$	COLLOCATION ERROR $ERR_1$	CONDITION NUMBER $\beta$
3	0.62752E - 02	0.53319E + 01	30	0.18000E - 04	0.12639E + 22
6	0.83438E - 03	0.15193E + 03	33	0.13975E - 04	0.73676E + 24
9	0.29524E - 03	0.60685E + 04	36	0.69075E - 05	0.44576E + 27
12	0.14687E - 03	0.31583E + 06	39	0.92770E - 05	0.27825E + 30
15	0.86792E - 04	0.51692E + 08	42	0.90482E - 05	0.17836E + 33
18	0.57066E - 04	0.19657E + 11	45	0.21096E - 05	0.11697E + 36
21	0.39744E - 04	0.87759E + 13	48	0.18280E - 04	0.78251E + 38
24	0.29588E - 04	0.43058E + 16	49	0.22470E - 04	0.68708E + 39
27	0.22938E - 04	0.22685E + 19	50	0.55810E - 03	0.60451E + 40

large number of the collocation points errors increase and oscillate. When we use the adaptation techniques, the maximal local error, in general, is less than in the version without adaptation, but it oscillates.

### Problem II

#### Potential flow past circular cylinder placed between parallel walls

In the analysis presented, the infinite region of the flow is replaced by a finite region indicated in Fig. 3a by the broken line. The formulation of the boundary problem for the repeated element the selected finite region is shown in Fig. 3b.

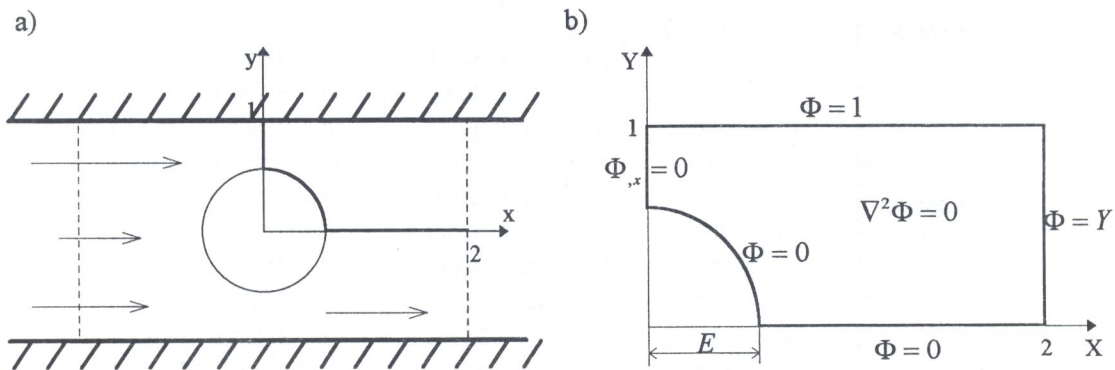


Fig. 3.

The approximate solution for the considered element has been assumed in the form:

$$\tilde{\Phi} = \sum_{k=1}^n a_k \left( R^{2k-1} - E^{2(2k-1)} R^{1-2k} \right) \sin((2k-1)\theta). \quad (22)$$

This solution fulfils exactly the differential equation and the boundary conditions:

$$\Phi = 0 \quad \text{for} \quad R = E \quad \text{and for} \quad \dot{\theta} = 0, \quad (23)$$

$$\frac{\partial \Phi}{\partial \theta} = 0 \quad \text{for} \quad \theta = \frac{\pi}{2}. \quad (24)$$

The only two conditions satisfied approximately are:

$$\Phi = Y \quad \text{for} \quad X = 2, \quad 0 \leq Y \leq 1, \quad (25)$$

$$\Phi = 1 \quad \text{for} \quad Y = 1, \quad 0 \leq X \leq 2. \quad (26)$$

Results of the maximal local error and the conditioning index for this problem are shown in Fig. 4, Fig. 5 and Table 2.

## 5. CONCLUSIONS

From the present study the following conclusions can be drawn.

(1) In contrary to the adaptive boundary element method or adaptive finite element method, in the adaptive Trefftz method (both in the collocation and Galerkin's versions), the error estimator does not need to be used. Firstly, it is known from the maximum error principle that the maximal error occurs at the boundary, secondly, the solution is given in the explicit form that enables one to determine the maximal error in a simply way by incremental searches.



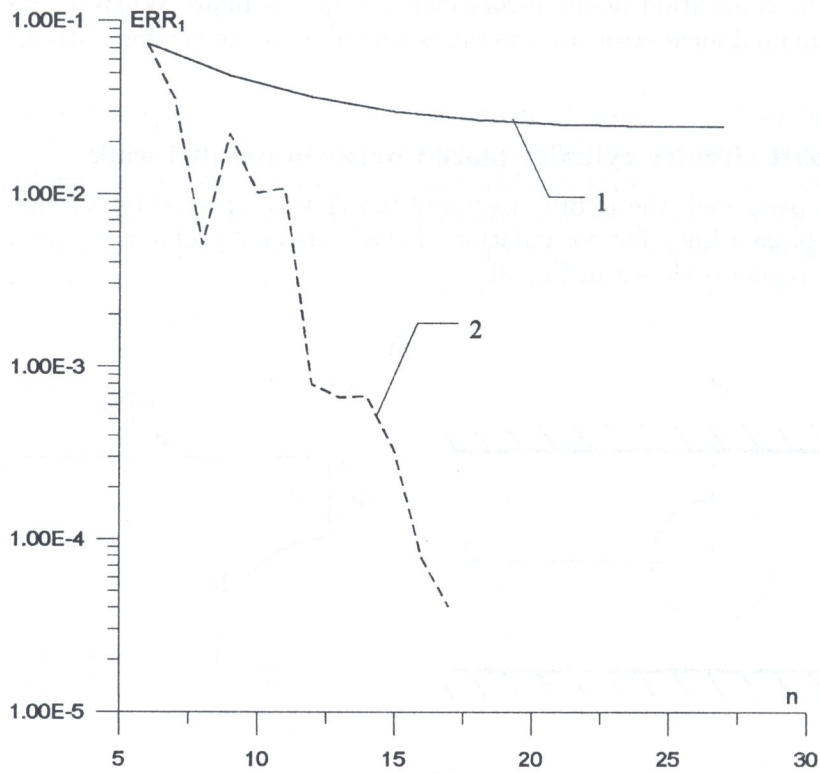


Fig. 4. Maximal local error as a function of number of collocation points for problem II for boundary  $Y = 1$ . 1 – collocation method with equidistant collocation points, 2 – collocation method with adaptation

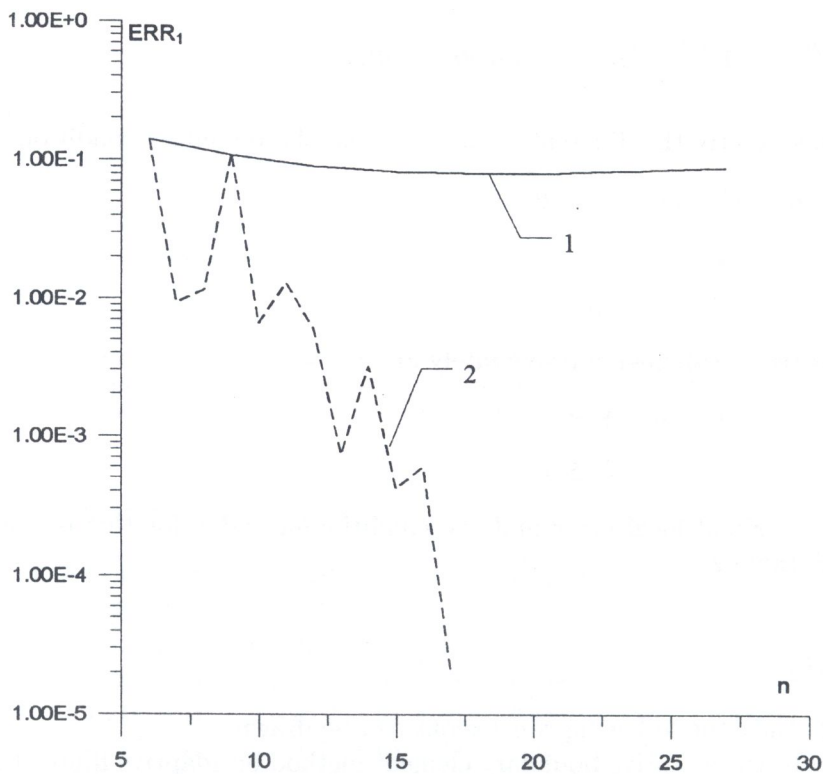


Fig. 5. Maximal local error as a function of number of collocation points for problem II for boundary  $X = 2$ . 1 – collocation method with equidistant collocation points, 2 – collocation method with adaptation



**Table 2.** Maximal local error  $ERR_1$  and condition number  $\beta$  versus of number of collocation points  $n$ 

FOR EDGE $X = 2$			FOR EDGE $Y = 1$		
$n$	COLLOCATION ERROR $ERR_1$	CONDITION NUMBER $\beta$	$n$	COLLOCATION ERROR $ERR_1$	CONDITION NUMBER $\beta$
6	0.1404E + 00	0.1297E + 04	6	0.7397E - 01	0.1297E + 04
9	0.1072E + 00	0.1066E + 06	9	0.4838E - 01	0.1066E + 06
12	0.9025E - 01	0.8972E + 07	12	0.3643E - 01	0.8972E + 07
15	0.8242E - 01	0.9900E + 09	15	0.3033E - 01	0.9900E + 09
18	0.7995E - 01	0.3249E + 12	18	0.2718E - 01	0.3249E + 12
21	0.8070E - 01	0.1431E + 15	21	0.2581E - 01	0.1431E + 15
24	0.8379E - 01	0.3343E + 16	24	0.2516E - 01	0.3343E + 16
27	0.8833E - 01	0.2260E + 18	27	0.2516E - 01	0.2260E + 18
30	0.9393E - 01	0.3978E + 20	30	0.2504E - 01	0.3978E + 20
33	0.1002E + 00	0.4950E + 22	33	0.2578E - 01	0.4950E + 22
36	0.9128E - 01	0.2028E + 25	36	0.2251E - 01	0.2028E + 25
39	0.4962E - 01	0.1404E + 28	39	0.1261E - 01	0.1404E + 28
42	0.8124E + 00	0.6491E + 29	42	0.1944E + 00	0.6491E + 29
45	0.2153E + 00	0.3572E + 33	45	0.5315E - 01	0.3572E + 33
48	0.5243E - 01	0.7097E + 33	48	0.1867E - 01	0.7097E + 33
51	0.1254E - 01	0.1721E + 37	51	0.2378E - 02	0.1721E + 37
54	0.1148E + 00	0.2936E + 39	54	0.8251E - 01	0.2936E + 39
57	0.7868E - 01	0.2480E + 43	57	0.7868E - 01	0.2480E + 43

(2) For the boundary collocation method with equidistant collocation points, the maximal local errors initially decrease monotonically with the increase of the number of collocation points. After passing a critical number of the collocation points, such behaviour fails to continue and the errors increase and oscillate. The reason for that behaviour is the worsening of the equation set conditioning with increase  $n$ .

(3) In using the adaptive collocation method, the maximal local error, in general, decreases with the increase of the number of collocation points but not in a monotonic way. One can observe oscillations, but in principle the error in the adaptive version of the boundary collocation method is lesser than in the version with equidistant collocation points. Moreover, in the adaptive version, error is controlled in the calculation process and calculations can be stopped if the error in several consecutive adaptation cycles increases.

(4) Similarly as in the boundary collocation with equidistant collocation points, for the boundary Galerkin method the maximal local errors initially decrease monotonically with the increase of the number of trial functions. After passing a critical number of the trial functions, such a behaviour fails to continue and the errors increase and oscillate. In the adaptive boundary Galerkin method, the maximal error depends on the number of the trial functions significantly more than on the number of the boundary elements. Thus, in the adaptation process the basic problem resides in finding the optimal number of trial functions using a reasonable number of the boundary elements and reasonable number of points in the Gauss quadrature.

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