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Trefftz spectral method for initial-boundary problems

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We describe a new Quasi Trefftz-type Spectral Method (QTSM) for solving boundary value and initial value problems. QTSM combines the properties of the Trefftz method with the spectral approach. The special feature of QTSM is that we use trial functions which satisfy the corresponding homogeneous equation only approximately. These trial functions are represented in the terms of a truncated series of eigenfunctions of some eigenvalue problem associated with the problem considered. The method has been found to work well for different elliptic problems with the Laplace, the Helmholtz and the biharmonic operators. We also consider some nonstationary parabolic problems including the problem in the domain with moving boundaries. The possibilities of further development of QTSM are also discussed.

1. Introduction

In this paper we describe a new numerical method for solving boundary value and initial value problems. The elliptic problem

$$Lu\,(\bar{x}) = f\,(\bar{x})\,, \quad \bar{x} \in \Omega \subset \mathcal{R}^i, \quad i = 1, 2, 3, \tag{1}$$

$$Bu\,(\bar{x}) = g\,(\bar{x})\,, \quad \bar{x} \in \partial\Omega, \tag{2}$$

where L is a linear differential elliptic operator and B is a boundary operator, will be our initial concern. Nonstationary parabolic problems in the domains with fixed and moving boundary will be considered in Sec. 4, 5.

Boundary methods based on the use of the complete system of solutions are applied widely to the problems like (1), (2). They are usually called Trefftz methods [1,2].

The Trefftz idea is that the approximate solution of (1), (2) is represented as

$$u_K\,(\bar{x}) = u_p(\bar{x}) + \sum_{k=1}^{K} \Phi(\bar{x}\,|\,\bar{a}_k), \tag{3}$$

where $u_p(\bar{x})$ is any particular solution of (1), $\Phi(\bar{x}\,|\,\bar{a})$ is the solution of the homogeneous equation

$$L\Phi(\bar{x}\,|\,\bar{a}) = 0, \tag{4}$$

and $\bar{a}_k$ are some parameters which are chosen to satisfy approximately boundary condition (2):

$$\min_{\bar{a}} \|Bu_K - g\|\,, \tag{5}$$

where the norm $\|\cdot\|$ is defined on $\partial\Omega$.

Various algorithms can be derived from specific choices of norm in (5) and trial functions in (3). For example, the fundamental solutions of homogeneous equation (4) with singularities located outside the domain Ω can be used to obtain the functions Φ. This approach, known as the Method of Fundamental Solutions (MFS), has been intensively developed in recent years [3–11].

MFS for the homogeneous elliptic equation

$$Lu(\bar{x}) = 0, \quad \bar{x} \in \Omega \tag{6}$$

with boundary condition (2) assumes, as an approximation to the solution, a function

$$u_K(\bar{x}) = \sum_{k=1}^{K} q_k \cdot \Psi(\bar{x}\,|\,\bar{\xi}_k), \tag{7}$$

where $\Psi(\bar{x}\,|\,\bar{\xi}_k)$ is any solution of the problem

$$L\Psi(\bar{x}\,|\,\bar{\xi}_k) = \delta(\bar{x} - \bar{\xi}_k). \tag{8}$$

Here δ denotes the Dirac delta-function and the source points $\bar{\xi}_k$ are located outside Ω. When MFS is applied, equation (8) is not subjected to any boundary condition and, therefore, the free space Green functions in representation (7) are used.

The parameters $\{q_k, \bar{\xi}_k\}_{k=1}^{K}$ are selected in order to approximate the boundary data, e.g.,

$$\min_{q_k, \bar{\xi}_k} \int_{\partial\Omega} \left(\sum_{k=1}^{K} q_k \cdot \Psi(\bar{x}\,|\,\bar{\xi}_k) - g(\bar{x}) \right)^2 \mathrm{d}S \tag{9}$$

for the Dirichlet boundary condition $u(\bar{x}) = g(\bar{x}), \bar{x} \in \partial\Omega$. A nonlinear minimization process should be used because not only q_k in (9) but also the source points $\bar{\xi}_k$ are unknown. Nonetheless, the linear version of such technique, when only the coefficients q_k are determined, is also applicable [7].

MFS was applied effectively to two- and three-dimensional linear potential problems [8], to biharmonic problems [5,9], to problems involving the boundary singularities [6] and free boundaries [10]. The nonlinear version of MFS is particularly effective in application to problems which are governed by a linear equation but are subjected to nonlinear boundary conditions [11].

At the same time MFS is applicable only to problems for which the fundamental solutions of the governing equation are known. Furthermore, it loses its advantages when the governing equation of the problem under consideration is inhomogeneous. These restrictions considerably narrow the field of the MFS application.

In this paper we propose a new technique which is close to MFS but is based on the spectral representation of the solution. It should be remarked that spectral methods have been long and succesfully used for numerical solving the partial differential equations. They demonstrate the superior approximation properties for problems possessing smooth solutions and which are defined in simple regions (e.g., rectangular or circular ones) [12, 13]. The method proposed in this paper seeks to combine the flexibility of the Trefftz methods with the superior approximation properties of the spectral methods. We call this method QTSM (Quasi Trefftz-type Spectral Method) [14–16]. QTSM, in our opinion, allows to overcome some of the MFS limitations and extends a field of application of the spectral methods.

The detailed description of QTSM is given in Sec. 2. Here we only remark that in accordance with the Trefftz approach we seek the solution of (1), (2) in the form

$$u_K(\bar{x}) = u_p(\bar{x}) + \sum_{k=1}^{K} q_k \cdot \Psi(\bar{x}\,|\,\bar{\xi}_k). \tag{10}$$

But contrary to other Trefftz methods we use the trial functions $\Psi(\bar{x}\,|\,\bar{\xi}_k)$ which satisfy the corresponding homogeneous equation

$$L\Psi(\bar{x}\,|\,\bar{\xi}_k) = 0, \quad \bar{x} \in \Omega \tag{11}$$

only approximately and, according to a spectral approach, are represented in the terms of a truncated series of smooth global functions from the orthogonal set $\{\varphi_n(\bar{x})\}$:

$$\Psi(\bar{x}\,|\,\bar{\xi}_k) = \sum_{n=1}^{M} d_n(\bar{\xi}_k) \cdot \varphi_n(\bar{x}), \tag{12}$$

where $\varphi_n(\bar{x})$ are the eigenfunctions of some eigenvalue problem associated with the problem considered. The fact that we use the trial functions which do not exactly satisfy (11) on one hand, needs additional efforts to control the calculation accuracy but, on the other hand, it considerably extends the set of trial functions. The spectral representation makes such trial functions especially effective when dealing with nonstationary problems (see [15, 16]).

In order to test QTSM we consider mainly two-dimensional problems with an exact analytic solution (see Sec. 3, 5). But the method suggested can be applied to more general problems in the domains with a complex geometry, including three-dimensional ones.

2. General description of QTSM

As mentioned above, QTSM falls into spectral methods. Unfortunately, eigenvalue problems can be solved analytically only for simple L, B and Ω. Therefore, we consider the solution domain Ω as a part of some simple domain Ω_0 (e.g., square or rectangle) extending in a proper way the operator L on the whole Ω_0 and choosing the appropriate boundary conditions on $\partial\Omega_0$ so that corresponding eigenvalue problem in Ω_0 can be easily solved. Namely, let us denote by B_0 and L_0 the boundary operator on $\partial\Omega_0$ and the extension of the operator L on Ω_0 respectively. It should be noted that if $\partial\Omega$ and $\partial\Omega_0$ have a common part, then B_0 must correspond to boundary condition (2) on this common part. We assume that the eigenvalue problem

$$\begin{cases} L_0\varphi = -\lambda\varphi & \text{in } \Omega_0, \\ B_0\varphi = 0 & \text{on } \partial\Omega_0 \end{cases} \tag{13}$$

can be easily solved analytically, the eigenfunctions $\{\varphi_n\}_{n=1}^{\infty}$ form a complete orthonormal system in $L^2(\Omega_0)$, the corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ are positive and $\lambda_n \to \infty$ as $n \to \infty$. Some examples of the choice of the domain Ω_0 and the boundary operator B_0 are presented in Sec. 3, 5.

The basic idea of QTSM is that we look for the approximate solution of (1) in the form (10), where $u_p(\bar{x})$ is the particular solution of (1), q_k are the unknown coefficients, the source points $\bar{\xi}_k \in \Omega_0 \setminus \Omega$ and the function $\Psi(\bar{x}\,|\,\bar{\xi})$ satisfies the inhomogeneous problem

$$\begin{cases} L_0\Psi(\bar{x}\,|\,\bar{\xi}) = I(\bar{x}\,|\,\bar{\xi}) & \text{in } \Omega_0, \\ B_0\Psi(\bar{x}\,|\,\bar{\xi}) = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{14}$$

Here $I(\bar{x}\,|\,\bar{\xi})$ is a δ-shaped source function which satisfies the following conditions:

(i) The function $I(\bar{x}|\,\bar{\xi})$ considered as function of $\bar{x}$ is a finite linear combination of the functions $\{\varphi_n\}$:

$$I(\bar{x}\,|\,\bar{\xi}) = \sum_{n=1}^{M} c_n(\bar{\xi}) \cdot \varphi_n(\bar{x}). \tag{15}$$

(ii) For any small $\varepsilon > 0$ there exist M and ε-neighbourhood $C_\varepsilon(\bar{\xi})$ of the source point $\bar{\xi}$ such that

$$|I(\bar{x}\,|\,\bar{\xi})| < \varepsilon \tag{16}$$

for any $\bar{x} \notin C_\varepsilon(\bar{\xi})$.

It should be stressed that, in fact, we replace equation (1) by the equation

$$L_0 u(\bar{x}) = f_0(\bar{x}) + \sum_{k=1}^{K} q_k \cdot I(\bar{x} \,|\, \bar{\xi}_k), \quad \bar{x} \in \Omega_0, \tag{17}$$

where $f_0(\bar{x})$ is the extension of the function $f(\bar{x})$ on Ω_0.

If the source functions $I(\bar{x} \,|\, \bar{\xi}_k)$ are sufficiently compact, i.e. for any small $\varepsilon \ll 1$

$$\bigcup_{k=1}^{K} C_\varepsilon(\bar{\xi}_k) \subset \Omega_0 \setminus \Omega, \tag{18}$$

then we have the following bound of the error caused by replacing equation (1) with equation (17):

$$|Lu_K(\bar{x}) - f(\bar{x})| = \left| \sum_{k=1}^{K} q_k \cdot I(\bar{x} \,|\, \bar{\xi}_k) \right| < \varepsilon \cdot \sum_{k=1}^{K} |q_k|, \quad \bar{x} \in \Omega.$$

Substituting (15) in (14), we get the function $\Psi(\bar{x} \,|\, \bar{\xi})$ in the form of expansion (12) with

$$d_n(\bar{\xi}) = -\frac{c_n(\bar{\xi})}{\lambda_n}. \tag{19}$$

If the right-hand side of inhomogeneous equation (1) is represented as a finite expansion with respect to the functions $\{\varphi_n\}$:

$$f(\bar{x}) = \sum_{n=1}^{M} F_n \cdot \varphi_n(\bar{x}), \tag{20}$$

then the particular solution $u_p(\bar{x})$ can be easily found in the same form:

$$u_p(\bar{x}) = \sum_{n=1}^{M} V_n \cdot \varphi_n(\bar{x}), \quad V_n = -\frac{F_n}{\lambda_n}. \tag{21}$$

From (10), (12), and (21) we obtain that the solution $u_K(\bar{x})$ is represented as a finite expansion with respect to the functions $\{\varphi_n\}$:

$$u_K(\bar{x}) = \sum_{n=1}^{M} U_n \cdot \varphi_n(\bar{x}), \quad U_n = V_n + \sum_{k=1}^{K} q_k \cdot d_n(\bar{\xi}_k). \tag{22}$$

The unknown coefficients q_k have been determined by a collocation procedure from boundary condition (2).

Here some comments are necessary:

1) The central point of the QTSM realization is the construction of the source functions $I(\bar{x} \,|\, \bar{\xi})$ with the properties described above (see (15), (16)) for different systems of the functions $\{\varphi_n\}$. This question have been discussed in [14, 17]. In this paper we use the following complete orthonormal in $L^2([0,1])$ systems of the one-dimensional functions:

$$\varphi_n^{(1)}(x) = \sqrt{2}\,\sin n\pi x, \quad \lambda_n^{(1)} = n^2\pi^2, \quad n = 1, \ldots, \infty; \tag{23}$$

$$\varphi_n^{(2)}(x) = \sqrt{2}\,\cos(n - 1/2)\pi x, \quad \lambda_n^{(2)} = (n - 1/2)^2\pi^2, \quad n = 1, \ldots, \infty; \tag{24}$$

$$\varphi_n^{(3)}(x) = \frac{\sqrt{2}\,J_0(\mu_n x)}{J_1(\mu_n)}, \quad \lambda_n^{(3)} = \mu_n^2, \quad n = 1, \ldots, \infty, \tag{25}$$

where $J_0(x)$ and $J_1(x)$ are the Bessel functions of order 0 and 1 respectively, μ_n is n^{th} zero of the function $J_0(x)$. Functions (25) form a complete orthonormal system in $L^2([0,1]; x)$.

We define for each of the systems $\{\varphi_n^{(i)}\}$ the source functions

$$I^{(i)}(x\,|\,\xi) = \sum_{n=1}^{M} r_n^{(i)} \cdot \varphi_n^{(i)}(\xi) \cdot \varphi_n^{(i)}(x), \quad i = 1,2,3, \tag{26}$$

where

$$r_n^{(i)} \equiv r_n^{(i)}(M,l) = \left(\sin \frac{\sqrt{\lambda_n^{(i)}}}{M+1} \Big/ \frac{\sqrt{\lambda_n^{(i)}}}{M+1}\right)^l . \tag{27}$$

Here l is a free parameter which we use to optimize the source function $I^{(i)}(x \mid \xi)$. It should be noted that in the case $i = 1$ formulae (26), (27) correspond to a well-known method of the Lanczos σ-multipliers [18].

To judge the properties of source functions (26) we present in Table 1 the values of

$$G_{M,l}(x\,|\,\xi) = \max_{|z-\xi|\ge|x-\xi|} \left| I^{(1)}(z\,|\,\xi) \big/ I^{(1)}(\xi\,|\,\xi) \right| \tag{28}$$

for $\{M = 30,\ l = 6\}$, $\{M = 50,\ l = 10\}$, $\{M = 70,\ l = 10\}$, and $\{M = 100,\ l = 12\}$. We introduce the function $G_{M,l}(x\,|\,\xi)$ for the following reasons: (1) to scale the sources with different M and l; (2) to smooth the oscillation of $\left|I^{(1)}(x\,|\,\xi)\right|$. So the function $G_{M,l}(x \mid \xi)$ is a majorant which monotonically decreases as $|x-\xi|$ increases. The calculations carried out show that if the source point ξ is sufficiently far from the boundaries of the interval, then the function $I^{(1)}(x \mid \xi)$ is dependent only on the difference $x - \xi$. So we set $\xi = 0.5$ in all the calculations presented in Table 1.

Table 1. The majorant $G_{M,l}(x\,|\,0.5)$.

x	$G_{30,6}$	$G_{50,10}$	$G_{70,10}$	$G_{100,12}$
0.50	1.0	1.0	1.0	1.0
0.55	0.58	0.39	0.16	$0.41\cdot10^{-1}$
0.60	$0.95\cdot10^{-1}$	$0.19\cdot10^{-1}$	$0.18\cdot10^{-3}$	$0.36\cdot10^{-7}$
0.65	$0.21\cdot10^{-2}$	$0.27\cdot10^{-4}$	$0.59\cdot10^{-9}$	$0.50\cdot10^{-12}$
0.70	$0.44\cdot10^{-5}$	$0.76\cdot10^{-9}$	$0.18\cdot10^{-10}$	$0.12\cdot10^{-13}$
0.80	$0.12\cdot10^{-6}$	$0.82\cdot10^{-11}$	$0.12\cdot10^{-12}$	$0.18\cdot10^{-14}$
0.90	$0.18\cdot10^{-7}$	$0.20\cdot10^{-12}$	$0.53\cdot10^{-14}$	$0.18\cdot10^{-14}$
1.00	$0.53\cdot10^{-15}$	$0.89\cdot10^{-15}$	$0.15\cdot10^{-14}$	$0.18\cdot10^{-14}$

Let us remark that the behaviour of the other source functions $I^{(i)}(x\,|\,\xi)$, $i = 2,3$ is similar to the one demonstrated in Table 1 [17]. Thus, formulae (26), (27) allow to obtain the appropriate source functions $I^{(i)}(x\,|\,\xi)$ for each system of the functions $\varphi^{(i)}(x)$, $i = 1,2,3$.

It should be stressed that the source functions $I^{(i)}(x \mid \xi)$ can be applied directly to the one-dimensional problems only. For two- and three-dimensional cases the source functions can be obtained by multiplying $I^{(i)}$ by each other (see Sec. 3, 5).

2) As shown above, QTSM is applicable to the inhomogeneous equations with the right-hand side expanded with respect to the functions $\{\varphi_n\}$. A variety of techniques may be used to obtain expansion (20). For example, it is possible to use first some smooth extension of the function $f(\bar{x})$ to the whole Ω_0 and then the standard Fourier series expansion procedure. However, the calculations carried out show that this approach provides a relatively low accuracy. Another way of looking at

it is as follows: let $\{\bar{z}_i\}_{i=1}^{J}$ $(J \geq M)$ be some points distributed in the domain Ω. We write the collocation conditions at these points:

$$\sum_{n=1}^{M} F_n \cdot \varphi_n(\bar{z}_i) = f(\bar{z}_i), \quad i = 1, \ldots, J. \tag{29}$$

So we obtain the system of J linear equations and solve it by least squares procedure. This method provides a very high accuracy (see subsec. 5.2 where it is applied to expand the initial function h).

We note that if the right-hand side of the governing equation can be represented as a product of functions of one variable or as a linear combination of such products, then the simpler alternate version of this algorithm can be used (see (46) – (48)).

3) Evidently that the accuracy provided by QTSM (as well as the stability of the method in the case of nonstationary problems) is dependent on the location of the source points $\bar{\xi}_k$. In this paper we don't include the coordinates of $\bar{\xi}_k$ in the number of the unknowns. But the nonlinear version of QTSM, when we seek not only q_k but also $\bar{\xi}_k$, is also possible.

Some remarks about the location of the source points in the one-dimensional case are given in [16]. The examples of the distribution of the source points in the two-dimensional domains are presented in Sec. 3, 5.

3. Boundary Value Problems: Numerical Results

3.1. Example 1

First, we apply QTSM to the two-dimensional inhomogeneous problem

$$\Delta u\,(x, y) = f\,(x, y)\,, \qquad (x, y) \in \Omega \subset \mathcal{R}^2, \tag{30}$$

$$u\,(x, y) = 0, \qquad (x, y) \in \partial\Omega. \tag{31}$$

Here Δ is the Laplace operator. We restrict our consideration to the case when the solution domain Ω is located inside the square $\Omega_0 = \{(x, y) : 0 \leq x, y \leq 1\}$ (see Fig. 1).

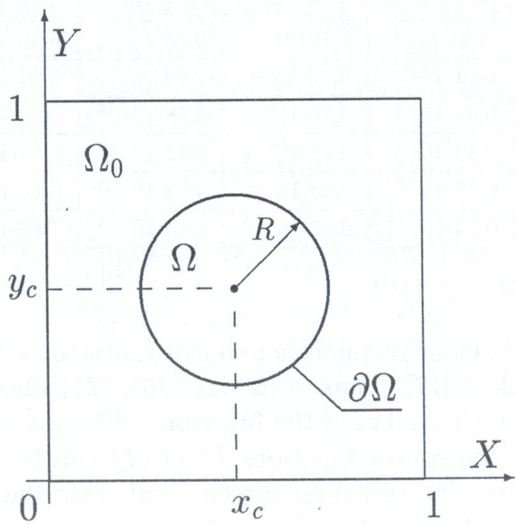

Fig. 1. Solution domain for Examples 1 and 2.

We choose the complete orthonormal in $L^2(\Omega_0)$ system of the functions

$$\varphi_{nm}(x, y) = \varphi_n^{(1)}(x) \cdot \varphi_m^{(1)}(y) \tag{32}$$

and denote by λ_{nm} the eigenvalues of the Laplace operator corresponding to functions (32):

$$\lambda_{nm} = \pi^2 \left(n^2 + m^2\right) \tag{33}$$

(see (23)). Then we introduce the source function

$$I(x, y \,|\, \xi, \eta) = I^{(1)}(x \,|\, \xi) \cdot I^{(1)}(y \,|\, \eta) = \sum_{n,m=1}^{M} c_{nm}(\xi, \eta) \cdot \varphi_{nm}(x, y), \tag{34}$$

$$c_{nm}(\xi, \eta) = r_n^{(1)} \cdot r_m^{(1)} \cdot \varphi_{nm}(\xi, \eta)$$

(see (26), (27)) and replace equation (30) by

$$\Delta u\,(x, y) = f\,(x, y) + \sum_{k=1}^{K} q_k \cdot I(x, y \,|\, \xi_k, \eta_k), \quad (\xi_k, \eta_k) \in \Omega_0 \setminus \Omega \tag{35}$$

(cf. (17)), where q_k are the unknown coefficients. The source points (ξ_k, η_k) are located on some curve which is equidistant from the boundary $\partial\Omega$. We consider this problem for each specific domain Ω separately.

We assume that

$$f\,(x, y) = \sum_{n,m=1}^{M} F_{nm} \cdot \varphi_{nm}(x, y) \tag{36}$$

and seek the solution of (34) as

$$u_K\,(x, y) = u_p(x, y) + \sum_{k=1}^{K} q_k \cdot \Psi(x, y \,|\, \xi_k, \eta_k), \tag{37}$$

where (see (21), (19))

$$u_p(x, y) = \sum_{n,m=1}^{M} V_{nm} \cdot \varphi_{nm}(x, y), \quad V_{nm} = -\frac{F_{nm}}{\lambda_{nm}}\,, \tag{38}$$

$$\Psi(x, y \,|\, \xi, \eta) = \sum_{n,m=1}^{M} d_{nm}(\xi, \eta) \cdot \varphi_{nm}(x, y), \quad d_{nm}(\xi, \eta) = -\frac{c_{nm}(\xi, \eta)}{\lambda_{nm}}\,. \tag{39}$$

Then we choose N $(N \geq K)$ collocation points (x_i, y_i) on the boundary $\partial\Omega$ and, using boundary condition (31), obtain the linear algebraic system for the coefficients q_k:

$$\sum_{k=1}^{K} \Psi(x_i, y_i \,|\, \xi_k, \eta_k) \cdot q_k = -u_p(x_i, y_i), \quad i = 1, \ldots, N. \tag{40}$$

We solve this system by least squares procedure.

Thus, we represent the approximate solution of (30), (31) in the form

$$u_K\,(x, y) = \sum_{n,m=1}^{M} U_{nm} \cdot \varphi_{nm}(x, y), \quad U_{nm} = V_{nm} + \sum_{k=1}^{K} q_k \cdot d_{nm}(\xi_k, \eta_k). \tag{41}$$

The algorithm described above can be applied to problem (30), (31) in any good enough bounded domain Ω. But to test QTSM we consider a simple example when Ω is the disk of radius R centered at the point (x_c, y_c):

$$\Omega = \{(x, y) : (x - x_c)^2 + (y - y_c)^2 \leq R^2\} \tag{42}$$

and the right-hand side of (30) takes the form

$$f(x,y) = F \cdot (x - x_c) \cdot (y - y_c), \quad F = \text{const}. \tag{43}$$

In this case problem (30), (31) has the exact analytic solution

$$u_e(x,y) = f(x,y) \cdot \frac{r^2 - R^2}{12}, \quad r^2 = (x - x_c)^2 + (y - y_c)^2. \tag{44}$$

To estimate the accuracy we introduce the maximum absolute error of the calculations

$$e_A = \max_{\Omega} |u_K(x,y) - u_e(x,y)|. \tag{45}$$

In practice, we take N_1 checking points which are uniformly distributed inside Ω.

Some results are presented in Table 2. In all the variants we use $x_c = y_c = 0.5$, $R = 0.1$, $N = 40$, $N_1 = 79$. The source points are located on the circle of radius $R + H$ ($H = 0.2$) centered at (x_c, y_c). We take $F = 10^6$ to obtain the maximum value of the solution of order 1. So the data in Table 2 characterize not only the absolute error but also the relative one.

Table 2. Errors e_A for Example 1 with $R = 0.1$, $F = 10^6$, $N = 40$, $H = 0.2$.

	$K = 20$	$K = 30$	$K = 40$
$M = 30$, $l = 6$	$0.18 \cdot 10^{-3}$	$0.37 \cdot 10^{-4}$	$0.37 \cdot 10^{-4}$
$M = 50$, $l = 10$	$0.17 \cdot 10^{-4}$	$0.45 \cdot 10^{-7}$	$0.13 \cdot 10^{-7}$
$M = 70$, $l = 10$	$0.47 \cdot 10^{-5}$	$0.16 \cdot 10^{-8}$	$0.56 \cdot 10^{-11}$

It should be noted that the right-hand side of (30) is a product of functions of one variable. Therefore, for obtaining expansion (36) we can use the following algorithm. First, using the collocation procedure together with least squares method, we obtain two one-dimensional expansions:

$$x - x_c \approx \sum_{n=1}^{M_F} f_n^{(x)} \cdot \varphi_n^{(1)}(x), \quad x \in [x_c - R, x_c + R], \tag{46}$$

$$y - y_c \approx \sum_{m=1}^{M_F} f_m^{(y)} \cdot \varphi_m^{(1)}(y), \quad y \in [y_c - R, y_c + R]. \tag{47}$$

If $x_c = y_c$, then, of course, $f_n^{(x)} = f_n^{(y)}$. We take $J = 30$ collocation points on each segment and $M_F = 20 \leq M$ terms in (46), (47). Next, we obtain the necessary representation by multiplying (46) and (47). Finally, we get

$$F_{nm} = \begin{cases} F \cdot f_n^{(x)} \cdot f_m^{(y)}, & n \leq M_F, \quad m \leq M_F, \\ 0, & n > M_F \text{ or } m > M_F. \end{cases} \tag{48}$$

We compute the function $f(x,y)$ from formulae (36), (48) at the same points in which we calculate the error e_A. The deviation from (43) is less than $0.18 \cdot 10^{-10}$.

3.2. Example 2

Here we consider the equation akin to the previously described in subsection 3.1 but with the Helmholtz operator:

$$\left(\Delta - p^2\right) u(x,y) = f(x,y), \quad (x,y) \in \Omega \subset \mathcal{R}^2. \tag{49}$$

We subject it to boundary condition (31).

A special feature of QTSM is that the algorithm of solving (49), (31) is exactly the same as for (30), (31). We must only use in (38), (39) the eigenvalues

$$\lambda_{nm} = \pi^2 \left(n^2 + m^2\right) + p^2 \tag{50}$$

of the Helmholtz operator instead of (33).

To test QTSM we consider the problem in the circular domain Ω (see (42)) and set $f(x,y) = F = \text{const}$. Then problem (49), (31) has the exact analytic solution

$$u_e(x,y) = \frac{F}{p^2}\left(\frac{I_0(pr)}{I_0(pR)} - 1\right), \quad r^2 = (x - x_c)^2 + (y - y_c)^2, \tag{51}$$

where I_0 is a modified Bessel function of order 0.

The values of the error e_A (it is defined similarly to (45)) are shown in Table 3. We use $p = 10$, $F = -500$, $x_c = y_c = 0.5$, $R = 0.1$, $N = 40$, $N_1 = 79$, $H = 0.2$.

Table 3. Errors e_A for Example 2 with $R = 0.1$, $p = 10$, $F = -500$, $N = 40$, $H = 0.2$.

	$K=20$	$K=30$	$K=40$
$M=30$, $l=6$	$0.16 \cdot 10^{-5}$	$0.83 \cdot 10^{-6}$	$0.84 \cdot 10^{-6}$
$M=50$, $l=10$	$0.42 \cdot 10^{-6}$	$0.34 \cdot 10^{-8}$	$0.34 \cdot 10^{-8}$
$M=70$, $l=10$	$0.32 \cdot 10^{-6}$	$0.32 \cdot 10^{-8}$	$0.32 \cdot 10^{-8}$

3.3. Example 3

In this subsection we deal with the inhomogeneous biharmonic equation in a closed bounded domain in the (x,y)-plane. We assume that the problem in hand has two axes of symmetry $\{x = 0\}$ and $\{y = 0\}$ and restrict our consideration to the quarter of the plane $\{(x,y) : x \geq 0, y \geq 0\}$. Namely, we consider the problem

$$\Delta^2 u(x,y) = f(x,y), \qquad (x,y) \in \Omega, \tag{52}$$

$$u(x,y) = \frac{\partial u}{\partial n}(x,y) = 0, \qquad (x,y) \in \Gamma, \tag{53}$$

$$\frac{\partial u}{\partial x}(0,y) = \frac{\partial^3 u}{\partial x^3}(0,y) = 0, \quad (0,y) \in \Omega, \tag{54}$$

$$\frac{\partial u}{\partial y}(x,0) = \frac{\partial^3 u}{\partial y^3}(x,0) = 0, \quad (x,0) \in \Omega. \tag{55}$$

Here Γ is a part of the domain boundary lying in the quadrant $\{(x,y) : x \geq 0, y \geq 0\}$. As before, we imbed the solution domain into the square $\Omega_0 = \{(x,y) : 0 \leq x, y \leq 1\}$ (see Fig. 2).

This time we introduce the functions

$$\varphi_{nm}(x,y) = \varphi_n^{(2)}(x) \cdot \varphi_m^{(2)}(y) \tag{56}$$

(see (24)). Due to the choice of the functions φ_{nm}, conditions (54), (55) are always fulfilled. Boundary condition (53) is used to obtain the unknown coefficients q_k.

It is easy to verify that

$$\Delta^2 \varphi_{nm}(x,y) = -\lambda_{nm}\,\varphi_{nm}(x,y), \tag{57}$$

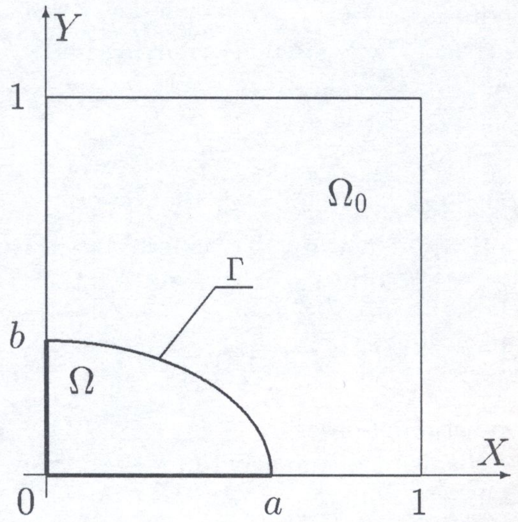

Fig. 2. Solution domain for Example 3

where

$$\lambda_{nm} = -\left((n-1/2)^2 + (m-1/2)^2\right)^2 \cdot \pi^4 . \tag{58}$$

We use here the source function

$$I(x, y \,|\, \xi, \eta) = I^{(2)}(x \,|\, \xi) \cdot I^{(2)}(y \,|\, \eta) = \sum_{n,m=1}^{M} c_{nm}(\xi, \eta) \cdot \varphi_{nm}(x, y), \tag{59}$$

$$c_{nm}(\xi, \eta) = r_n^{(2)} \cdot r_m^{(2)} \cdot \varphi_{nm}(\xi, \eta).$$

Further, the algorithm is identical to one described in subsection 3.1. It should be stressed that from boundary condition (53) we obtain not only equations (40) but also

$$\sum_{k=1}^{K} \frac{\partial \Psi}{\partial n}(x_i, y_i \,|\, \xi_k, \eta_k) \cdot q_k = -\frac{\partial u_p}{\partial n}(x_i, y_i), \quad i = 1, \ldots, N. \tag{60}$$

Thus, we have $2N$ linear algebraic equations (40), (60) for the coefficients q_k.

We carry out calculations for the domain Ω which is the quarter of the ellipse

$$\Omega = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, \quad x, y \geq 0 \right\}$$

and $f\,(x, y) = F = \text{const}$. Then problem (52) – (55) has the analytic solution [19]

$$u_e(x, y) = u_0 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2, \quad u_0 = F\left(\frac{24}{a^4} + \frac{24}{b^4} + \frac{16}{a^2 b^2}\right)^{-1}. \tag{61}$$

The results of the calculations are presented in Table 4. The error e_A is defined similarly to (45). The source points are located on the arcs of two ellipses

$$\left\{ (x, y) : \frac{x^2}{(a+H_j)^2} + \frac{y^2}{(b+H_j)^2} = 1, \quad x, y \geq 0 \right\}, \quad j = 1, 2.$$

We take $K/2$ source points on each arc. In all the variants we use $a = 0.25$, $b = 0.15$, $F = 0.75 \cdot 10^5$, $N = 40$, $H_1 = 0.2$, $H_2 = 0.3$, $N_1 = 79$.

Table 4. Errors e_A for Example 3 with $a = 0.25$, $b = 0.15$, $F = 0.75 \cdot 10^5$, $N = 40$, $H_1 = 0.2$, $H_2 = 0.3$.

	$K = 20$	$K = 30$	$K = 40$
$M = 30,\ l = 6$	$0.54 \cdot 10^{-7}$	$0.54 \cdot 10^{-7}$	$0.54 \cdot 10^{-7}$
$M = 50,\ l = 10$	$0.44 \cdot 10^{-9}$	$0.11 \cdot 10^{-10}$	$0.95 \cdot 10^{-11}$
$M = 70,\ l = 10$	$0.58 \cdot 10^{-9}$	$0.63 \cdot 10^{-11}$	$0.31 \cdot 10^{-11}$

As evident from Examples 1–3, we can apply the algorithm described above to any two-dimensional equation of the type (1) with the operator

$$L = P\left(-\Delta\right), \tag{62}$$

where P is some polynomial with constant coefficients. We need only that P has no zeros on the positive semi-axis. We must replace only the eigenvalues λ_{nm} of the Laplace operator in (38), (39) by $-P\left(\lambda_{nm}\right)$.

It should be also noted that in [14] we consider examples of the QTSM application to some other elliptic problems including axisymmetric and periodic ones, the problem with infinitely long boundaries, the problem in the doubly connected domain.

4. Initial value problems: general description

In this section we investigate the peculiarities of the QTSM application to nonstationary problems [15, 16]. Let us consider the parabolic equation

$$\frac{\partial u}{\partial t}\left(\bar{x},t\right) = Lu\left(\bar{x},t\right) + f\left(\bar{x},t\right), \quad \bar{x} \in \Omega \subset \mathcal{R}^i,\ i = 1,2,3, \tag{63}$$

with the boundary condition

$$Bu\left(\bar{x},t\right) = g\left(\bar{x},t\right), \quad \bar{x} \in \partial\Omega \tag{64}$$

and the initial condition

$$u\left(\bar{x},0\right) = h\left(\bar{x}\right), \quad \bar{x} \in \Omega. \tag{65}$$

For solving problem (63) – (65) we leave the spatial variable $\bar{x}$ continuous and discretize (63) in time by using any implicit time-stepping method. In particular, we use the Crank-Nicholson scheme

$$\frac{u^{(j+1)}(\bar{x}) - u^{(j)}(\bar{x})}{\Delta t} = \frac{Lu^{(j+1)}(\bar{x}) + Lu^{(j)}(\bar{x})}{2} + f^{(j+1/2)}(\bar{x}), \tag{66}$$

but it is possible to apply any other, including multistep ones. Here $u^{(j)}(\bar{x}) \equiv u\left(\bar{x}, t^{(j)}\right)$, $f^{(j+1/2)}(\bar{x}) \equiv f\left(\bar{x}, \left(t^{(j)} + t^{(j+1)}\right)/2\right)$, $t^{(j)} = j \cdot \Delta t$, and Δt is a time discretization step.

Denoting $p^2 = 2/\Delta t > 0$, we obtain the sequence of the coupled stationary problems:

$$\left(L - p^2\right) u^{(j+1)}(\bar{x}) = S^{(j+1)}(\bar{x}) \equiv -\left(L + p^2\right) u^{(j)}(\bar{x}) - 2f^{(j+1/2)}(\bar{x}). \tag{67}$$

If $f^{(j+1/2)}(\bar{x})$ and $u^{(j)}(\bar{x})$ have the form of the spectral expansions with respect to the eigenfunctions $\{\varphi_n(\bar{x})\}$:

$$f^{(j+1/2)}(\bar{x}) = \sum_{n=1}^{M} F_n^{(j+1/2)} \cdot \varphi_n(\bar{x}), \tag{68}$$

$$u^{(j)}(\bar{x}) = \sum_{n=1}^{M} U_n^{(j)} \cdot \varphi_n(\bar{x}), \tag{69}$$

then the right-hand side of (67) can be easily represented in the same form:

$$S^{(j+1)}(\bar{x}) = \sum_{n=1}^{M} \left(\left(\lambda_n - p^2 \right) \cdot U_n^{(j)} - 2\, F_n^{(j+1/2)} \right) \cdot \varphi_n(\bar{x}), \tag{70}$$

So, having originally expanded the initial function $u^{(0)}(\bar{x}) = h(\bar{x})$, we can solve stationary problems (67) obtaining every time the right-hand side $S^{(j+1)}(\bar{x})$ in a convenient form of spectral representation.

Let us remark that if the function $f(\bar{x}, t)$ takes the form

$$f(\bar{x}, t) = \sum_{i=1}^{N_f} \alpha_i(t) \cdot f_i(\bar{x}), \tag{71}$$

then we can, using one of the procedures described above, expand each of the stationary functions $f_i(\bar{x})$:

$$f_i(\bar{x}) = \sum_{n=1}^{M} F_{i,n} \cdot \varphi_n(\bar{x}). \tag{72}$$

It is evident that at some instant $t = t^{(j+1/2)}$ we get

$$F_n^{(j+1/2)} = \sum_{i=1}^{N_f} \alpha_i \left(t^{(j+1/2)} \right) \cdot F_{i,n}. \tag{73}$$

Finally, let us note that if the time discretization step Δt and the source points are fixed, then in solving each of problems (67) the same set of the trial functions $\{\Psi(\bar{x}\,|\,\bar{\xi}_k)\}$ is used.

5. Initial value problems: numerical results

5.1. Example 4

In this subsection we apply QTSM to the axisymmetric initial value problem with the Laplace space operator and Dirichlet boundary condition. We assume that the problem in hand has the plane of symmetry $\{z = 0\}$ and restrict our consideration to the half-space $\{z \geq 0\}$. Then the problem may be written in cylindrical coordinates as

$$\frac{\partial u}{\partial t}(r, z, t) = Lu(r, z, t) + f(r, z, t), \quad (r, z) \in \Omega, \tag{74}$$

$$u(r, z, t) = g(r, z, t), \quad (r, z) \in \Gamma, \tag{75}$$

$$\frac{\partial u}{\partial z}(r, 0) = 0, \qquad (r, 0) \in \Omega, \tag{76}$$

$$\frac{\partial u}{\partial r}(0, z) = 0, \qquad (0, z) \in \Omega. \tag{77}$$

$$u(r, z, 0) = h(r, z), \quad (r, z) \in \Omega. \tag{78}$$

Here

$$L = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} \tag{79}$$

and Γ is the boundary of the cross-section of the three-dimensional solution domain considered in (r, z)-plane (see Fig. 3). We also assume that it is possible to put Ω into cylinder

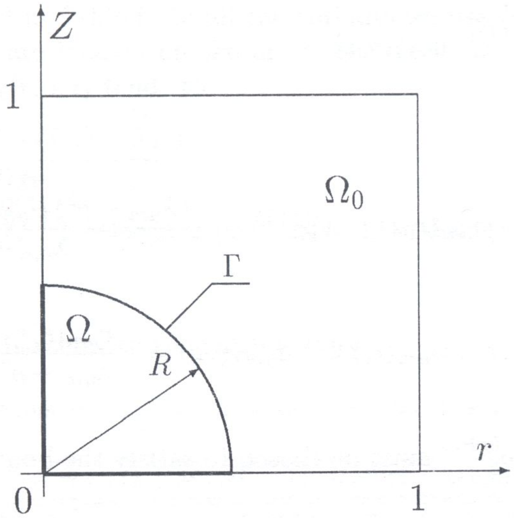

Fig. 3. Solution domain for Example 4

$$\Omega_0 = \{(r, z) : 0 \leq r, z \leq 1\}. \tag{80}$$

In accordance with the general QTSM approach described in Sec. 4 we discretize (74) in time by using the Crank-Nicholson difference scheme and obtain the sequence of the stationary equations (cf. (67)):

$$\left(L - p^2\right) u^{(j+1)}(r, z) = -\left(L + p^2\right) u^{(j)}(r, z) - 2f^{(j+1/2)}(r, z), \quad p^2 = 2/\Delta t. \tag{81}$$

In this case we take the system of eigenfunctions

$$\varphi_{nm}(r, z) = \varphi_n^{(3)}(r) \cdot \varphi_m^{(2)}(z), \tag{82}$$

which possess the needed symmetry (76), (77). The corresponding eigenvalues of the axisymmetric Laplace operator are

$$\lambda_{nm} = \mu_n^2 + (m - 1/2)^2 \pi^2 \tag{83}$$

(see (24), (25)).

We also introduce the source function

$$I(r, z \,|\, \xi, \eta) = I^{(3)}(r \,|\, \xi) \cdot I^{(2)}(z \,|\, \eta) = \sum_{n,m=1}^{M} c_{nm}(\xi, \eta) \cdot \varphi_{nm}(r, z), \tag{84}$$

$$c_{nm}(\xi, \eta) = r_n^{(3)} \cdot r_m^{(2)} \cdot \varphi_{nm}(\xi, \eta)$$

and seek the approximate solution of (81) in the form

$$u_K^{(j+1)}(r, z) = u_p^{(j+1)}(r, z) + \sum_{k=1}^{K} q_k^{(j+1)} \cdot \Psi(r, z \,|\, \xi_k, \eta_k). \tag{85}$$

We represent the solution $u_K^{(j)}(r, z)$ and the function $f^{(j+1/2)}(r, z)$ as finite expansions with respect to the functions $\{\varphi_{nm}\}$:

$$u_K^{(j)}(r, z) = \sum_{n,m=1}^{M} U_{nm}^{(j)} \cdot \varphi_{nm}(r, z), \tag{86}$$

$$f^{(j+1/2)}(r,z) = \sum_{n,m=1}^{M} F_{nm}^{(j+1/2)} \cdot \varphi_{nm}(r,z). \tag{87}$$

Then we get

$$u_p^{(j+1)}(r,z) = \sum_{n,m=1}^{M} V_{nm}^{(j+1)} \cdot \varphi_{nm}(r,z), \quad V_{nm}^{(j+1)} = -\frac{(\lambda_{nm} - p^2)\, U_{nm}^{(j)} - 2\, F_{nm}^{(j+1/2)}}{\lambda_{nm} + p^2}, \tag{88}$$

$$\Psi\,(r, z\,|\,\xi, \eta) = \sum_{n,m=1}^{M} d_{nm}(\xi,\eta) \cdot \varphi_{nm}(r,z), \quad d_{nm}(\xi,\eta) = -\frac{c_{nm}(\xi,\eta)}{\lambda_{nm} + p^2}. \tag{89}$$

The unknown coefficients $q_k^{(j+1)}$ must be chosen to satisfy the boundary condition

$$u_K^{(j+1)}\,(r,z) = g^{(j+1)}\,(r,z)\,, \quad (r,z) \in \Gamma. \tag{90}$$

We take the collocation points $\{(r_i, z_i)\}_{i=1}^{N}$ on the boundary Γ ($N \geq K$) and, using (85), obtain the linear algebraic system

$$\sum_{k=1}^{K} \Psi(r_i, z_i\,|\,\xi_k, \eta_k) \cdot q_k^{(j+1)} = -u_p^{(j+1)}(r_i, z_i) + g^{(j+1)}\,(r_i, z_i)\,, \quad i = 1, \ldots, N. \tag{91}$$

Let us remark that, as it is mentioned above, if the time discretization step and the source points are fixed, then the trial functions $\{\Psi(r, z\mid \xi_k, \eta_k)\}$ are independent of j. Therefore, if the collocation points are also fixed, then systems (91) have the same matrix (but, of course, the different right-hand sides) at various j. This allows to construct an effective numerical algorithm.

After determining $q_k^{(j+1)}$ we can write $u_K^{(j+1)}\,(r,z)$ in the same form as (86):

$$u_K^{(j+1)}(r,z) = \sum_{n,m=1}^{M} U_{nm}^{(j+1)} \cdot \varphi_{nm}(r,z), \quad U_{nm}^{(j+1)} = V_{nm}^{(j+1)} + \sum_{k=1}^{K} q_k^{(j+1)} \cdot d_{nm}(\xi_k, \eta_k) \tag{92}$$

and so the algorithm is closed.

As a checking example, we consider the particular case when the solution domain Ω is the quarter of the disk of radius $R < 1$,

$$g\,(r,z,t) \equiv 0, \quad h\,(r,z) \equiv 0$$

and

$$f\,(r,z,t) = F \cdot \left(1 - \frac{r^2 + z^2}{R^2}\right), \quad F = \text{const.}$$

In this case the problem considered has the exact analytic solution [20]

$$u_e(r,z,t) = \frac{F \cdot (R^2 - \rho^2) \cdot (7R^2 - 3\rho^2)}{60\, R^2} - \frac{12\, F\, R^3}{\rho\, \pi^5} \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin \alpha_n}{n^5} \cdot e^{-\alpha_n^2\, t}, \tag{93}$$

where we denote $\rho = \sqrt{r^2 + z^2}$ and $\alpha_n = n\pi/R$.

Similarly to (45) we define the maximum absolute error at the time $t = t^{(j)}$:

$$e_A^{(j)} = \max_{\Omega} \left| u_K^{(j)}\,(r,z) - u_e\left(r, z, t^{(j)}\right) \right|. \tag{94}$$

As before, we take N_1 checking points which are uniformly distributed inside Ω.

Some results are presented in Table 5. In all the variants we use $R = 0.25$, $N = 40$, $\Delta t = 10^{-4}$, $N_1 = 79$. The source points are located on the arc of the circle of radius $R + H$ centered at the origin. Let us remark that $u_e(r, z, t)$ tends to

$$u_e(r, z, \infty) = \frac{F \cdot (R^2 - \rho^2) \cdot (7R^2 - 3\rho^2)}{60\, R^2}$$

as $t \to \infty$. We take $F = 60\big/7R^2$ to obtain $u_e(0, 0, \infty) = 1$. In doing so we obtain that

$$|u_e(0, 0, 0.1) - 1| < 10^{-7}.$$

So the data in Table 5 also characterize the relative error.

Table 5. Errors $e_A^{(j)}$ for Example 4 with $R = 0.25$, $N = 40$, $\Delta t = 10^{-4}$.

t	M=30, l=6, H=0.18	M=50, l=10, H=0.15	M=70, l=10, H=0.12
0.00	$0.52{\cdot}10^{-8}$	$0.52{\cdot}10^{-8}$	$0.52{\cdot}10^{-8}$
0.02	$0.10{\cdot}10^{-4}$	$0.43{\cdot}10^{-5}$	$0.30{\cdot}10^{-5}$
0.04	$0.13{\cdot}10^{-4}$	$0.59{\cdot}10^{-5}$	$0.35{\cdot}10^{-6}$
0.06	$0.15{\cdot}10^{-4}$	$0.67{\cdot}10^{-5}$	$0.22{\cdot}10^{-6}$
0.08	$0.16{\cdot}10^{-4}$	$0.72{\cdot}10^{-5}$	$0.23{\cdot}10^{-6}$
0.10	$0.16{\cdot}10^{-4}$	$0.76{\cdot}10^{-5}$	$0.25{\cdot}10^{-6}$

5.2. Example 5

Here we consider the problem which has only one essential difference from the problem described in subsection 5.1: the solution domain varies with time. Namely, we consider the axisymmetric problem with governing equation (74), (79) with $f(r, z, t) \equiv 0$ in the sphere of variable radius $R(t)$. We use cylindrical coordinates and denote

$$\Omega(t) = \Big\{(r, z) : 0 \le r^2 + z^2 \le R(t), \quad r, z \ge 0\Big\}. \tag{95}$$

We assume that the boundary motion law is known and take it in the form:

$$R(t) = \beta\,(1 + t)^{1/2}, \quad \beta = R(0) = \text{const} < 1. \tag{96}$$

We restrict our consideration to such t that the solution domain $\Omega(t)$ is in cylinder (80).

We subject equation (74) to conditions (75) – (78) with

$$g(r, z, t) = \frac{1}{R(t)} \tag{97}$$

and

$$h(r, z) = \frac{1}{\rho} \cdot \frac{\mathrm{Erf}(\rho/2)}{\mathrm{Erf}(\beta/2)}, \quad \rho = \sqrt{r^2 + z^2}, \tag{98}$$

where

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy.$$

It is easy to verify that the function

$$u_e(r, z, t) = \frac{1}{\rho} \cdot \frac{\mathrm{Erf}(w(\rho, t))}{\mathrm{Erf}(\beta/2)}, \quad w(\rho, t) = \frac{\rho}{2\sqrt{1 + t}} \tag{99}$$

is the exact solution of the problem in hand.

The solving algorithm is almost identical to the one described above. It should be only stressed that the collocation points (r_i, z_i) are now located on the moving boundary

$$\Gamma(t) = \left\{(r,z) : 0 \leq r^2 + z^2 = R(t), \quad r, z \geq 0\right\} \tag{100}$$

and the source points (ξ_k, η_k) are also moved. In particular, we place the source points on the arc of the circle of variable radius $R(t) + H$, $H = \text{const}$. So, in this case to determine the coefficients $q_k^{(j+1)}$ we obtain the sequence of linear algebraic systems like (91) with different matrices.

To expand the initial function $h(r,z)$ with respect to the functions φ_{nm} we use the procedure described in Sec. 2 (see (29)).

Table 6. Errors $e_A^{(j)}$ for Example 5 with $M = 30$, $l = 8$, $R(0) = 0.25$, $N = 40$, $K = 30$, $\Delta t = 10^{-3}$.

t	$e_A^{(j)}$
0.0	$0.16 \cdot 10^{-7}$
0.5	$0.54 \cdot 10^{-5}$
1.0	$0.44 \cdot 10^{-5}$
1.5	$0.38 \cdot 10^{-5}$
2.0	$0.34 \cdot 10^{-5}$
2.5	$0.32 \cdot 10^{-5}$
3.0	$0.30 \cdot 10^{-5}$

We present some results of the calculations in Table 6. The value $e_A^{(j)}$ has the same meaning as in the previous subsection. We use $M = 30$, $l = 8$, $R(0) = 0.25$, $N = 40$, $K = 30$, $H = 0.2$, $\Delta t = 10^{-3}$. We note that $t = 3$ corresponds to the doubling of the initial disk radius.

Let us remark that in [16] we consider some another problems with moving boundaries including the problems in which the boundary motion law must be determined in the solving process (so-called Stefan problems).

6. Conclusions

The main aim of this paper is to give a general description of the Quasi Trefftz-type Spectral Method (QTSM) and to test it on simple model problems. QTSM combines the properties of the Trefftz and spectral methods. It is applicable to stationary as well as time-dependent problems. QTSM preserves all the appealing features of Trefftz methods such as flexibility and adaptiveness. Its implementation on a computer leads to simple codes and is not expensive on CPU time or memory.

The following lines of the present work generalization seem to be of special interest. The first is the extension to the three-dimensional problems. The second is the treatment of problems with a more general governing equation. The third direction concerns the application of the nonlinear versions of QTSM. These will be the subject of future investigations.

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