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Approximation with harmonic and generalized harmonic polynomials in the partition of unity method

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The aim of the paper is twofold. In the first part, we present an analysis of the approximation properties of "complete systems", that is, systems of functions which satisfy a given differential equation and are dense in the set of all solutions. We quantify the approximation properties of these complete systems in terms of Sobolev norms. As a first step of the analysis, we consider the approximation of harmonic functions by harmonic polynomials. By means of the theory of Bergman and Vekua, the approximation results for harmonic polynomials are then extended to the case of general elliptic equations with analytic coefficients if the harmonic polynomials are replaced with their analogs, "generalized harmonic polynomials". In the second part of the paper, we present the Partition of Unity Method (PUM). This method has the feature that it allows for the inclusion of a priori knowledge about the local behavior of the solution in the ansatz space. Therefore, the PUM can lead to very effective and robust methods. We illustrate the PUM with an application to Laplace's equation and the Helmholtz equation.

1. Introduction

Many problems of practical importance are still beyond the capabilities of today's computers, if standard methods are used. For example, for the modeling of materials with microstructure, laminated materials, or the analysis of stiffeners, the classical h version finite element method requires that the mesh size h be so small that the smallest scale can be resolved—a requirement which may easily lead to prohibitively large linear systems. Further examples where the discretization has to be very fine are furnished by equations with highly oscillatory solutions such as the Helmholtz equation at high wave numbers; similar challenges arise in the numerical treatment of equations whose solutions exhibit boundary layers which is a typical situation in various plate and shell models.

However, in many of the examples above, the solution is not simply "bad" but still has some structure which is not exploited by the standard methods. If we are able to incorporate this extra knowledge about the structure into the method, then the computational cost may be lowered considerably. A number of methods have been proposed which do account for special features of the problem under consideration. For example, the "enriched spaces" of [10] mimic the singular behavior of a corner singularity. The Trefftz method of J. Jirousek [15, 16], A.P. Zielinski [46], O.C. Zienkiewicz [8], and the work of Bergman [7], Z.C. Li [20], E. Stein [40], and many others (see [45] for an overview) is based on the idea that, for differential equations with analytic coefficients, one can construct so-called "complete systems" with certain nice properties. The key point of these complete systems is that they are specific to the equation under consideration: they solve the differential equation and have very good approximation properties, viz., in comparison with the usual polynomials, fewer degrees of freedom are needed to achieve the same accuracy (cf. Section 2 for a detailed discussion of the approximation properties of complete systems).

The purpose of the present paper is twofold. Firstly, we present approximation results for certain types of complete systems (cf. Section 2). These results may contribute to the theory of various "knowledge-based" methods such as the Trefftz methods. Additionally, as complete systems are only one form of analytic information about the structure of the solution of a partial differential equation, we discuss other types of information and consider various questions related to the incorporation of analytic knowledge in a numerical method (cf. Sections 3, 4).

Secondly, we discuss the Partition of Unity Method (PUM), a new method presented first in [23, 38, 25, 24] (cf. Section 5). The aim of the PUM is to give the user the option to incorporate a priori knowledge about the structure of the equation into the numerical method. With the aid of a partition of unity, the PUM constructs a global, conforming space from user-provided local approximation spaces; these local approximation spaces can be chosen very specific to the particular problem under consideration, and the PUM may therefore be more efficient than the classical FEM. An additional feature of the PUM is that the ansatz spaces constructed by the PUM are conforming. In contrast to this, the Trefftz methods mentioned above are non-conforming. There, the solution is approximated locally (on "elements") by complete systems and the inter-element continuity is viewed as a constraint; typically this constraint is enforced by Lagrange multipliers or by minimizing (weighted) jumps of the function and/or derivatives across inter-element boundaries. This inter-element continuity constraint has to be treated carefully in order to ensure stability and convergence of the method. On the other hand, one of the advantages of conforming methods such as the PUM is that they inherit many of the features of the continuous problem. For example, if the original bilinear form is coercive, then the PUM produces in turn a coercive method and thus convergence is easily guaranteed.

The outline of the paper is as follows. In Section 2 we start with a detailed analysis of the approximation properties of harmonic polynomials and quantify the approximation properties in Sobolev norms. We then present the necessary details of the theory of Bergman and Vekua for general elliptic equations with analytic coefficients in order to construct the analogs of harmonic polynomials, the so-called "generalized harmonic polynomials". All the approximation results for Laplace's equation are seen to carry over to the case of analytic coefficients. We illustrate the theory of Bergman and Vekua and the notion of "generalized harmonic polynomials" by elaborating the case of Helmholtz's equation and the elasticity equations in two dimensions.

In Section 5 we present the details of the PUM. As we mentioned above, the main feature of the PUM is that it allows the user to include information about the local behavior of the solution in the ansatz spaces. The performance of the PUM therefore hinges on the availability of good descriptions of the local behavior of the solution. For that reason, we discuss in Section 3 a variety of situations in which knowledge about the local behavior of the solution is available and how we may use it in the context of the PUM. In Section 4 we give some mathematical guidelines for choosing local approximation spaces. Finally, in Section 6 we apply the PUM to two model problems, namely, Laplace's equation and the Helmholtz equation. In both cases, we use the local approximation spaces analyzed in Section 2.

2. Generalized harmonic polynomials for equations with analytic coefficients

In this section we want to present some results on the construction/definition of complete systems for elliptic equations. The main idea behind the notion of complete systems is to find systems of functions which solve a given differential equation and which are dense (in an appropriate norm) in the set of all solutions. The following questions arise naturally in this context.

1. How can we construct such systems?

2. Is a given system of functions dense (in some appropriate norm) in the set of all solutions?

3. What are the approximation properties of a given system?

4. As it turns out that there are many complete systems for a given differential equation, how do we justify a particular choice we may make?

There are many approaches to constructing complete systems and there are many sets of functions which are complete systems. We refer to the theory of Bergman [5], Vekua [43], and to Herrera [14] for a systematic way of constructing complete systems and proving various density assertions.

We will present in this section the theory of Bergman and Vekua in more detail. Their theory gives a way to construct complete systems for elliptic equations with analytic coefficients from complete systems for the simplest case, Laplace's equation. By means of their theory, any complete system for Laplace's equation gives rise to a complete system for any elliptic equation with analytic coefficients through an appropriate integral operator. The advantage of the approach of Vekua is that it allows us to quantify the approximation properties in terms of Sobolev norms. The theory of Bergman on the other hand leads to series expansions of these complete systems which is important from a practical point of view.

As their theory reduces the approximation problem (and the construction of complete systems) to the case of Laplace's equation, we will present old and new results for Laplace's equation. In particular, we will analyze the approximation properties of harmonic polynomials for the approximation of harmonic functions in terms of Sobolev norms. We then present the necessary details of the theory of Bergman and Vekua to see that the approximation results for Laplace's equation carry over the case of general elliptic equations with analytic coefficients. We illustrate this procedure with an application to Helmholtz's equation.

Additionally, we present the case of plane elasticity. By the theory of Muskhelishvili [27] the solutions of the elasticity equations can be expressed in terms of two holomorphic functions and thus the approximation results of Section 2.1 apply.

With respect to the last item of the above list ("which system should we choose?"), let us just mention that there are many issues to be considered, both implementational and approximation theoretical ones. We will discuss the approximation theoretical issues in Section 4.

2.1. Laplace's Equation

In this section we are interested in the approximation of harmonic functions by harmonic polynomials. By harmonic polynomials we mean polynomials which satisfy Laplace's equation, or, equivalently, polynomials which are the real part of a complex polynomial.

All harmonic functions are given as the real parts of holomorphic functions. In fact, the holomorphic function whose real part is a given harmonic function is unique up to a purely imaginary constant. A different way of putting this fact is to say that the operator Re (taking the real part) creates a one-to-one correspondence between

$$\{u \mid \Delta u = 0\} \text{ and } \{f \mid f \text{ is holomorphic and } f(z_0) \text{ is real}\}, \tag{1}$$

where z_0 is an arbitrary but fixed point of the domain of interest. Thus, harmonic polynomials are the images of complex polynomials under the operator Re. The approximation of harmonic function by harmonic polynomials is therefore reduced to the approximation of holomorphic functions by complex polynomials, and we address this issue in this subsection.

The question of approximating holomorphic functions by polynomials in a Hölder space setting (in particular, the question of uniform approximation, i.e., approximation in L^∞) was addressed a long time ago, [44, 26]. Let us quote one typical result.

Theorem 1 (Szegö) *Let $\Omega \subset \mathbb{C}$ be a simply connected, bounded Lipschitz domain. Let $\tilde{\Omega}$ be such that $\Omega \cup \partial\Omega \subset \tilde{\Omega}$ and assume that $f \in L^2(\tilde{\Omega})$ is holomorphic on $\tilde{\Omega}$. Then there are (complex)*

polynomials P_p of degree p such that

$$\|f - P_p\|_{L^\infty(\Omega)} \leq C_1 e^{-\gamma p} \|f\|_{L^2(\tilde{\Omega})}, \quad \|(f - P_p)'\|_{L^\infty(\Omega)} \leq C_2 e^{-\gamma p} \|f\|_{L^2(\tilde{\Omega})},$$

where the constants C_1, C_2, and $\gamma > 0$ depend only on Ω, $\tilde{\Omega}$.

Proof: See [41, 44]. □

For an application of approximation results in the context of variational formulations, it is more useful to have estimates in a Sobolev setting rather than in a Hölder setting. We have the following

Theorem 2 *Let $\Omega \subset \mathbb{C}$ be a bounded Lipschitz domain, star shaped with respect to a ball. Let the exterior angle of Ω be bounded from below at each boundary point by $\lambda\pi$ with $0 < \lambda < 2$ (i.e., let Ω satisfy an exterior cone condition with cone aperture $\lambda\pi$). Assume that $f \in H^k(\Omega)$, $k \geq 1$, is holomorphic on Ω. Then there are polynomials P_p of degree $p \geq k - 1$ such that*

$$\|f - P_p\|_{H^j(\Omega)} \leq C(\operatorname{diam}\Omega)^{k-j} \left(\frac{\ln p}{p}\right)^{\lambda(k-j)} \|f\|_{H^k(\Omega)}, \qquad j = 0, \ldots, [k],$$

where the constant $C > 0$ depends only on the shape of Ω and k.

Proof: [24]. □

Remark 1: We observe that for domains Ω with re-entrant corners $\lambda < 1$. Therefore, there is a loss of approximability in the approximation with polynomials which gets more pronounced as the angle of the re-entrant corner gets worse. This loss can actually be observed in numerical experiments (see the numerical example in Section 2.1.1) and is related to the question of the optimality of polynomials; cf. Section 4.

Remark 2: The restriction to star shaped domains is not a big restriction in the practical application of Theorem 2 within the framework of the PUM because we are interested in local approximation on "patches" only which are typically chosen to be star shaped.

2.1.1. Approximation of Singular Functions

In this subsection we want to consider the approximation of singular functions by polynomials. These singular functions arise naturally if one solves Laplace's equations in domains with corners, if interfaces are present, or if the type of boundary conditions changes at a point. Typically, these singular functions are of the form

$$z^\alpha \ln^\beta z$$

for some $\alpha, \beta \geq 0$. We will therefore formulate the ensuing theorem in terms of these functions for ease of exposition. We refer to [24] for the analysis of a more general situation.

Theorem 3 *Let Ω be a bounded domain with a corner of exterior angle $\lambda\pi$ at the origin. Let $f = z^\alpha \ln^\beta z$ for some $\alpha > 0$, $\beta \geq 0$. Then there are polynomials P_p of degree $p > \alpha$ such that*

$$\|f - P_p\|_{H^j(\Omega)} \leq C(\epsilon, f)(\operatorname{diam}\Omega)^{1+\alpha-j-\epsilon} p^{-\lambda(1+\alpha-j-\epsilon)}, \qquad j = 0, 1$$

for any $\epsilon > 0$, where $C(\epsilon, f)$ depends only on ϵ, f, and the shape of Ω.

Proof: [24]. □

Remark 3: The interesting case in Theorem 3 is the case $\lambda \geq 1$, i.e., the case that Ω has a convex corner at the origin. In this case, the convexity of the corner "helps" in the approximation with polynomials: The function f is in the Sobolev space $H^{1+\alpha-\epsilon}$, and an application of Theorem 2 would only yield approximation rates of the form $p^{-(\alpha-\epsilon)}$ for the approximation in the H^1 norm. In the case of a convex corner we have $\lambda > 1$, and thus Theorem 3 yields a better result for the approximation of singular functions than the general case of Theorem 2.

Remark 4: An alternative to approximating these singular functions by polynomials only is to include these singular functions in the approximation space (e.g., use polynomials *and* some singular functions).

In order to demonstrate that the rates of convergence of Theorem 3 are actually obtained, we conclude this section with a simple numerical experiment. We approximate the harmonic function $u := \operatorname{Im} z^{1/2}$ by harmonic polynomials on the sector $S(\omega)$ of aperture 2ω:

$$S(\omega) := \{z \in \mathbb{C} \,|\, |z| < 1 \text{ and } |\arg z| < \omega\}.$$

More precisely, we minimized the "energy"

$$E(\omega, p) := \min\{\|\nabla(u - u_p)\|^2_{L^2(S(\omega))} \,|\, u_p \text{ is a harmonic polynomial of degree } p\}.$$

For the values $\omega = \pi/16$, $\pi/8$, $\pi/4$, $\pi/3$, $\pi/2$, $3/4\pi$, and $5/6\pi$ we show the results in Fig. 1. We notice that indeed, just as Theorem 3 suggests, the rate of convergence improves as the angle where the singularity is located gets smaller. In fact, the slopes of the graphs in Fig. 1 are very close to the theoretical value of $2\alpha\lambda = 2\frac{1}{2}(2\pi - 2\omega)$.

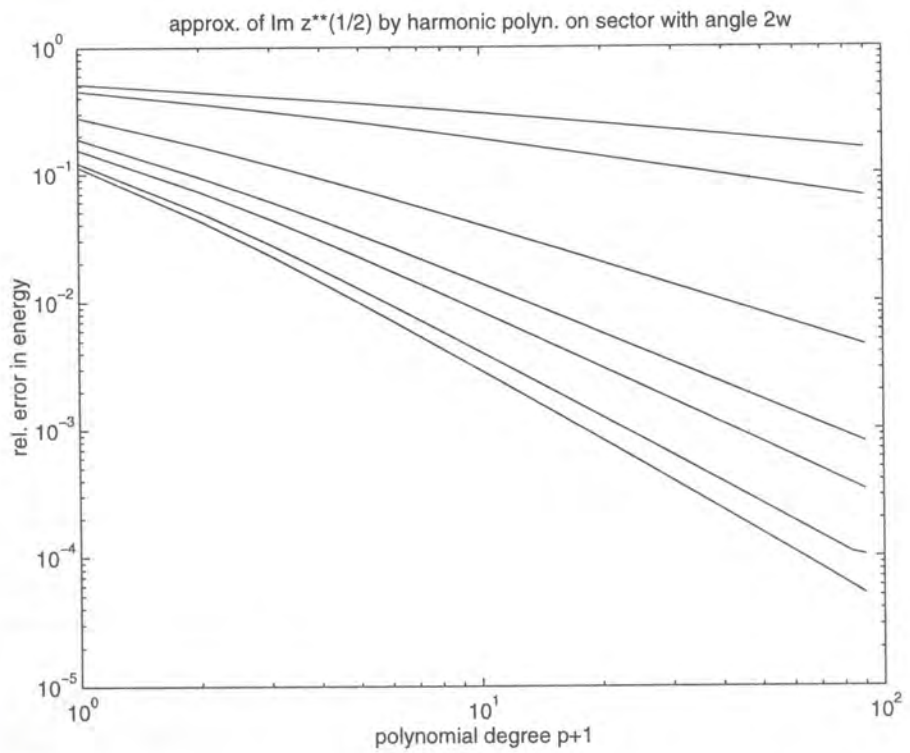

Fig. 1. Approximation of $\operatorname{Im} z^{1/2}$ on $S(\omega)$ for $\omega = \pi/16$, $\pi/8$, $\pi/4$, $\pi/3$, $\pi/2$, $3/4\pi$, and $5/6\pi$ (in ascending order)

2.2. Bergman and Vekua's Theory and Generalized Harmonic Polynomials

In this section we want to present very briefly the theory of I.N. Vekua and S. Bergman. We then show how that theory can be applied to motivate the definition of "generalized harmonic polynomials", and how we can derive the approximation properties of these "generalized harmonic polynomials".

I.N. Vekua [43] and S. Bergman [5] developed independently operators which map holomorphic functions onto solutions of (2) below. In fact, they showed that every solution of (2) can be represented as the image of a holomorphic function under this operator. We will only quote the results of their theory relevant to our notion of "generalized harmonic polynomials". For a more general survey and some applications of this theory we mention besides the above quoted references [33, 9, 13, 6, 7]. Let us finally remark that one of the differences between the theory of Vekua and Bergman is that Vekua proves the existence of the above mentioned operator by means of Picard iterations. The theory of Vekua lends itself more naturally to abstract error estimates and that is why we present it here. Bergman's theory on the other hand leads to series representations of the operator and hence may be of more practical interest for the explicit construction of the operator (and the "generalized harmonic polynomials"); we refer to [33] for an exposition on the relation between the theories of Bergman and Vekua and for an application of these series representations.

Let us consider

$$Lu \equiv \Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0 \qquad \text{on } \Omega \subset \mathbb{R}^2 \tag{2}$$

with real-valued, analytic coefficients a, b, and c. For notational convenience, we will identify in this section the plane $\mathbb{R}^2$ with $\mathbb{C}$ via $x+iy=z$. Since a, b, and c are assumed to be real analytic on Ω, they can be thought of as defined on a subset of $\mathbb{C}\times\mathbb{C}$.

A *fundamental domain* of the differential operator L is a simply connected set $\mathcal{D}\subset\mathbb{C}$ such that $\mathcal{A}(z,\zeta)$, $\mathcal{B}(z,\zeta)$, and $\mathcal{C}(z,\zeta)$ are holomorphic in both variables $z\in\mathcal{D}$, $\zeta\in\overline{\mathcal{D}}$, where

$$\begin{aligned}
4\mathcal{A}(z,\zeta) &= a\left(\frac{z+\zeta}{2},\frac{z-\zeta}{2i}\right) + ib\left(\frac{z+\zeta}{2},\frac{z-\zeta}{2i}\right),\\
4\mathcal{B}(z,\zeta) &= a\left(\frac{z+\zeta}{2},\frac{z-\zeta}{2i}\right) - ib\left(\frac{z+\zeta}{2},\frac{z-\zeta}{2i}\right),\\
4\mathcal{C}(z,\zeta) &= c\left(\frac{z+\zeta}{2},\frac{z-\zeta}{2i}\right).
\end{aligned}$$

Similarly, we can extend solutions u of (2) to functions $\mathcal{U}(z,\zeta)$ via

$$\mathcal{U}(z,\zeta) = u\left(\frac{z+\zeta}{2},\frac{z-\zeta}{2i}\right)$$

which solve the complex (formally hyperbolic) equation

$$\mathcal{L}\mathcal{U} \equiv \frac{\partial^2}{\partial z\partial\zeta}\mathcal{U} + \mathcal{A}\frac{\partial}{\partial z}\mathcal{U} + \frac{\partial}{\partial\zeta}\mathcal{U} + \mathcal{C}\mathcal{U} = 0. \tag{3}$$

Conversely, any complex solution $\mathcal{U}$ of (3) leads to two real solutions u of (2), namely, $u=\mathrm{Re}\,\mathcal{U}(z,\overline{z})$ and $u=\mathrm{Im}\,\mathcal{U}(z,\overline{z})$. Vekua constructed an operator which gives all solutions of (3) by means of the so-called *Riemann function* $G:\mathcal{D}\times\overline{\mathcal{D}}\times\mathcal{D}\times\overline{\mathcal{D}}\to\mathbb{C}$, which is continuous and holomorphic in each variable. On setting $\zeta=\overline{z}$ and taking the real part, Vekua then obtained a representation formula for all the solutions of (2):

Theorem 4 (Vekua) *Let Ω be simply connected and $\Omega\cup\partial\Omega\subset\mathcal{D}$. Fix $z_0\in\Omega$ and assume that u satisfies (2). Then there is a unique function ϕ, holomorphic on Ω with $\phi(z_0)$ real such that*

$$\begin{aligned}
u(x,y) &= \mathrm{Re}\left\{G(z,\overline{z}_0,z,\overline{z})\phi(z) + \int_{z_0}^{z}\phi(t)H(t,\overline{z}_0,z,\overline{z})dt\right\}\\
&\equiv \mathrm{Re}\,\mathrm{V}[\phi,z_0](z,\overline{z}),
\end{aligned}$$

where we wrote for brevity

$$H(t,\tau,z,\zeta) = B(t,\tau)G(t,\tau,z,\zeta) - \frac{\partial G}{\partial t}(t,\tau,z,\zeta)$$

and of course $z = x + iy$. *Conversely, any holomorphic function* ϕ *(without the constraint that* $\phi(z_0)$ *be real) is mapped under the operator* ReV *on a solution of (2).*

Thus the Riemann function G allows us to exhibit an integral operator (which we denoted ReV) which establishes a one-to-one correspondence between

$$\{u \mid u \text{ solves (2)}\} \text{ and } \{\phi \mid \phi \text{ holomorphic on } \Omega \text{ and } \phi(z_0) \text{ real}\} \tag{4}$$

Note the similarity to the simple case of Laplace's equation (1).

Example 1: Let $a = b = c = 0$, i.e., consider Laplace's equation. Then $G \equiv 1$ and the operator ReV reduces to taking the real part, that is, the operator Re, and we obtain the case (1).

Example 2: Let $a = b = 0$ and $c = -k^2$ with a constant $k > 0$. Then

$$G(t,\tau,z,\zeta) = I_0(k\sqrt{(z-t)(\zeta-\tau)}),$$

where I_0 is the modified Bessel function of order 0.

Example 3: Let $a = b = 0$ and $c = k^2$ with a constant $k > 0$. Then

$$G(t,\tau,z,\zeta) = J_0(k\sqrt{(z-t)(\zeta-\tau)}),$$

where J_0 is the Bessel function of the first kind of order 0.

We have seen above that the operator ReV is a bijection between the solutions of (2) and holomorphic functions. In fact, ReV and its inverse ReV^{-1} are continuous in Sobolev norms.

Theorem 5 *Let* $k \geq 1$ *and* $z_0 \in \Omega$. *Then there are* C_1, $C_2 > 0$ *such that for all holomorphic functions* ϕ

$$\|\mathrm{ReV}[\phi, z_0]\|_{H^j(\Omega)} \leq C_1 \|\phi\|_{H^j(\Omega)}, \qquad 0 \leq j \leq k,$$

$$\|\phi\|_{H^k(\Omega)} \leq C_2 \|\mathrm{ReV}[\phi, z_0]\|_{H^k(\Omega)} \qquad \text{if additionally } \phi(z_0) \text{ real.}$$

Proof: See [24]. A similar statement in Hölder spaces is proved in [9]. □

The point of Theorem 5 is that it allows us to quantify the approximation properties of complete systems in terms of Sobolev norms. Suppose u solves (2) and $u \in H^k(\Omega)$ for some $k > 1$, an assertion which one can usually make from the regularity of the data and the geometry of the domain. By Vekua's theorem, there is a holomorphic function ϕ such that $u = \mathrm{ReV}[\phi, z_0]$. Theorem 5 states that $\phi \in H^k(\Omega)$ as well. Letting ϕ_p be some approximation of ϕ such that $\|\phi - \phi_p\|_{H^1(\Omega)}$ is small (e.g., we could approximate ϕ by complex polynomials and invoke the theory of the preceding section), we can define an approximation of u by $u_p := \mathrm{ReV}[\phi_p, z_0]$. Again by Theorem 5 (with $j = 1$ in the first estimate) we have the estimate

$$\|u - u_p\|_{H^1(\Omega)} = \|\mathrm{ReV}[\phi - \phi_p, z_0]\|_{H^1(\Omega)} \leq C_2 \|\phi - \phi_p\|_{H^1(\Omega)}.$$

Therefore, if we have a complete system for the holomorphic functions, for which the approximation properties in terms of Sobolev norms are available, then we can construct a complete system for (2) for which the *same* properties in terms of Sobolev norms hold. As we have analyzed the case of polynomial approximation for holomorphic functions, we will use the images of (complex) polynomials under the operator ReV and will call them "generalized harmonic polynomials" (recall that for the simplest case, Laplace's equation, ReV reduces to taking the real part and thus the images of complex polynomials are precisely the harmonic polynomials). Let us formalize this in the following definition.

Definition 1 (Generalized Harmonic Polynomials) *As we are interested in a real formulation, define complex polynomials of degree p as the* real *span of*

$$\{1, z, iz, z^2, iz^2, \ldots, z^p, iz^p\}.$$

Then the "generalized harmonic polynomials" are given by

$$G(p) := \operatorname{span}\{\operatorname{ReV}[1, z_0], \operatorname{ReV}[z, z_0], \operatorname{ReV}[iz, z_0], \operatorname{ReV}[z^2, z_0], \operatorname{ReV}[iz^2, z_0], \ldots, \operatorname{ReV}[z^p, z_0], \operatorname{ReV}[iz^p, z_0]\}, \tag{5}$$

and $\dim g(P) = 2p + 1$.

We would like to condense the ideas discussed so far in the following local approximation theorem.

Theorem 6 *Let Ω be star shaped with respect to a ball and let the exterior angle of Ω be bounded from below by $\lambda\pi$ at each boundary point. Let $z_0 \in \Omega$ be given and let "generalized harmonic polynomials" be defined by (5). Assume that $u \in H^k(\Omega)$, $k > 1$, satisfies (2). Then there are "generalized harmonic polynomials" $P_p \in G(p)$ of degree $p \geq k - 1$ such that*

$$\|u - P_p\|_{H^j(\Omega)} \leq C(\operatorname{diam}\Omega)^{k-j} \left(\frac{\ln p}{p}\right)^{\lambda(k-j)} \|u\|_{H^k(\Omega)}, \qquad j = 0, 1,$$

where $C > 0$ depends only on the shape of Ω, the relative position of z_0 within Ω, and the coefficients of the differential operator of (2).

Proof: The proof follows essentially from the above observations and Theorem 2. The only statement which does not follow directly from the discussion above is the claim about how the constant C depends on the point z_0. We refer to [24] for the proof of this assertion. □

The fact that we can control the dependence of the constants on the location of the point z_0 in Theorem 6 is important for practical applications as it allows us to localize the definition of "generalized harmonic polynomials" and the approximation by them. One aspect is that the operator ReV is in general not known but has to be approximated. Both the Picard iterations of Vekua and the series expansions of Bergman can be expected to converge fast if the patch Ω is small and the point z_0 is in the patch Ω.

Remark 5: Obviously, an analogous theorem could be stated if we base the approximation of holomorphic functions not on Theorem 2 but on the estimate for singular functions (Theorem 3). In fact, in problems with corners, the function ϕ of Vekua's theorem which gives rise to the solution of (2) is exactly of the type discussed in Section 2.1.1. These "generalized harmonic polynomials" thus inherit the same improved rates of convergence.

2.3. Helmholtz's Equation

We want to formulate more explicitly the theory of Bergman and Vekua for the case of the Helmholtz equation $-\Delta u - k^2 u = 0$. We have stated above that the Riemann function G has the form

$$G(t, \tau, z, \zeta) = J_0(k\sqrt{(z - t)(\zeta - \tau)}).$$

For physical reasons, it is more natural to consider complex-valued solutions of the (homogeneous) Helmholtz equation. As the real and imaginary part of a complex valued solution satisfy Helmholtz's equation separately, we can use the representation theory of Section 2.2 for the real and imaginary part separately to arrive at the following form:

$$u(x, y) = \operatorname{ReV}[\phi_1, z_0](z, \overline{z}) + i\operatorname{ReV}[\phi_2, z_0](z, \overline{z}),$$

where ϕ_1 and ϕ_2 are holomorphic functions. The operator ReV here takes the form

$$\mathrm{ReV}[\phi, z_0](z, \overline{z}) = \mathrm{Re}\left\{\phi(z) - \int_{z_0}^{z} \phi(t) \frac{\partial}{\partial t} J_0\left(k\sqrt{(z-t)(\overline{z}-\overline{z}_0)}\right) \mathrm{d}t\right\}.$$

If the point z_0 is chosen as the origin and if we choose the path of integration to be the straight line going from the origin to the point z, this operator simplifies considerably, and we arrive at

$$u(x, y) = V[u_0](x, y) := u_0(x, y) - \int_0^1 u_0(tx, ty) \frac{\partial}{\partial t} J_0\left(kr\sqrt{1-t}\right) \mathrm{d}t,$$

where u_0 is a complex valued function that satisfies Laplace's equation and where $r = \sqrt{x^2 + y^2}$. In the spirit of the preceding section, we define the "generalized harmonic polynomials" for the Helmholtz equation as the images of a complete system for Laplace's equation under this operator V. One possible complete system for the set of all (complex valued) harmonic functions is given by the complex span (i.e., we have to take the span over $\mathbb{C}$)

$$\mathrm{span}\,\{1, z, \overline{z}, z^2, \overline{z}^2, \ldots,\},$$

and a lengthy calculation shows that, introducing the notation $x + iy = re^{i\theta}$,

$$\begin{aligned} V[z^n](x, y) &= n! \left(\frac{2}{k}\right)^n e^{in\theta} J_n(kr), \\ V[\overline{z}^n](x, y) &= n! \left(\frac{2}{k}\right)^n e^{-in\theta} J_n(kr), \end{aligned}$$

where the functions J_n are the Bessel functions of the first kind of order n. They are defined as

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{\nu=0}^{\infty} (-1)^\nu \left(\frac{x}{2}\right)^{2\nu} \frac{1}{\nu!(n+\nu)!}.$$

Thus, we may define the space of "generalized harmonic polynomials" for the Helmholtz equation by

$$G(p) = \mathrm{span}\,\{e^{\pm in\theta} J_n(kr) \,|\, n = 0, \ldots, p\}. \tag{6}$$

The approximation result Theorem 6 takes the following form.

Theorem 7 *Let Ω be a bounded Lipschitz domain, star-shaped with respect to a ball and assume that the origin is in Ω. Let the exterior angle of Ω be bounded from below by $\lambda\pi$ and assume that $u \in H^s(\Omega)$, $s \geq 1$, solves the homogeneous Helmholtz equation. Then there are functions $u_p \in G(p)$ for $p \geq s - 1$ such that*

$$\|u - u_p\|_{H^j(\Omega)} \leq C(\Omega, s, k)(\mathrm{diam}\,\Omega)^{s-j} \left(\frac{\ln p}{p}\right)^{\lambda(s-j)} \|u\|_{H^s(\Omega)}, \qquad j = 0, 1,$$

where $C(\Omega, s, k) > 0$ depends only on the shape of Ω, k, and s.

Proof: See [24]. □

As in the case of the approximation of solutions to Laplace's equation, there are many other alternative complete systems. For example, one can approximate the solutions of the homogeneous Helmholtz equation with systems of *plane waves*:

$$W(p) = \mathrm{span}\,\{\exp\,[ik(x\cos\theta_j + y\sin\theta_j)] \,|\, \theta_j = \frac{2\pi}{p} j, j = 0, \ldots, p-1\}. \tag{7}$$

One can show [24] that these systems of plane wave have approximation properties which are very similar to the approximation with "generalized harmonic polynomials":

Theorem 8 *Under the same assumptions as in Theorem 7 there are functions $u_p \in W(p)$ such that*

$$\|u - u_p\|_{H^j(\Omega)} \leq C(\Omega, s, k) \left(\frac{\ln^2 p}{p}\right)^{\lambda(s-j)} \|u\|_{H^s(\Omega)}, \qquad j = 0, 1,$$

where $C(\Omega, s, k) > 0$ depends only on Ω, k, and s.

Remark 6: We defined the "generalized harmonic polynomials" $G(p)$ for the Helmholtz equation as the images of (complex valued) harmonic polynomials under the operator V. As we discussed above, any complete system for the Helmholtz equation can be viewed as the image of a complete system for Laplace's equation under the operator V. It is actually possible to identify a complete system for Laplace's equation which generates systems of plane waves under the mapping V: On setting $a = \cos\theta + i \sin\theta$ with $\theta \in [0, 2\pi)$, we have

$$u_a(x, y) = -1 + e^{ik\bar{a}z/2} + e^{ika\bar{z}/2}, \tag{8}$$

$$V[u_a] = e^{ik(x\cos\theta + y\sin\theta)}. \tag{9}$$

Therefore, the approximation properties of systems of plane waves can be analyzed easily by studying the approximation properties of functions u_a for the approximation of harmonic functions, where a may range in a dense subset of $\{a \in \mathbb{C} \,|\, |a| = 1\}$.

What are the differences between the generalized harmonic polynomials and the systems of plane waves? In Section 4 we will see that harmonic polynomials and these "generalized harmonic polynomials" for the Helmholtz equation are optimal spaces in some sense. More precisely, they are optimal in the sense of n-width for the approximation on discs, that is, if we have to approximate a solution of the (homogeneous) Helmholtz equation on a disc and the only additional information we have is that it is in some $H^k(\Omega)$, then the best choice (in terms of error per degrees of freedom) of an approximation space we can make is precisely the choice of the spaces $G(p)$.

Of course, other factors besides the argument of optimality in the sense of n-width play a rôle in choosing a local approximation space in practice. An advantage of systems of plane waves is that they may be easier to use in practical applications. Plane waves can be written as products of functions of x and of y only and this may be exploited in the construction of the stiffness matrix (see the numerical example in Section 6).

Let us finish this section by mentioning that these "generalized harmonic polynomials" and the systems of plane waves lead to exponential rates of convergence if the function u is analytic up to boundary:

Theorem 9 *Let $\Omega \subset \mathbb{R}^2$ be a simply connected, bounded Lipschitz domain. Let $\tilde{\Omega} \supset\supset \Omega$ and assume that $u \in L^2(\tilde{\Omega})$ solves the homogeneous Helmholtz equation on $\tilde{\Omega}$. Then*

$$\inf_{u_p \in G(p)} \|u - u_p\|_{H^1(\Omega)} \leq Ce^{-\gamma p} \|u\|_{L^2(\tilde{\Omega})},$$

$$\inf_{w_p \in W(p)} \|u - w_p\|_{H^1(\Omega)} \leq \tilde{C}e^{-\tilde{\gamma}p/\ln p} \|u\|_{L^2(\tilde{\Omega})},$$

where C, $\tilde{C}$, γ, and $\tilde{\gamma}$ depend only on Ω, $\tilde{\Omega}$, and k.

2.4. Elasticity Equations

The solutions of the equations of linear elasticity (in the absence of body forces) in two dimensions can be expressed in terms of two holomorphic functions (see [27]). Let us consider the case of plain strain on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and let λ, μ be the Lamé constants of the material

(for the case of plain stress, replace in what follows λ by $\lambda^* = 2\lambda\mu/(\lambda + 2\mu)$). The displacement field (u, v) can be expressed by two holomorphic functions φ, ψ:

$$2\mu\,(u(x,y) + iv(x,y)) = \kappa\varphi(z) - z\overline{\varphi'}(z) - \overline{\psi}(z), \tag{10}$$

where $\kappa = (\lambda+3\mu)/(\lambda+\mu)$ and we set $z = x+iy$. For a given displacement state, the holomorphic functions φ, ψ are unique up to the normalization of $\varphi(z_0) = 0$ in a point $z_0 \in \Omega$. Thus, we may approximate the displacement field (u, v) by "generalized harmonic polynomials"

$$2\mu(u + iv) = \kappa\varphi_p(z) - z\overline{\varphi_p'}(z) - \overline{\psi_p}(z), \tag{11}$$

where the functions φ_p, ψ_p are complex polynomials of degree p

$$\varphi_p(z) = \sum_{n=1}^{p} a_n(z - z_0)^n,$$

$$\psi_p(z) = \sum_{n=0}^{p} b_n(z - z_0)^n$$

with complex coefficients a_n, b_n. In a real formulation, the displacements u and v are obtained by taking the real and imaginary parts of the elements of the space V (as a vector space over $\mathbb{R}$ of dimension $2 + 4p$)

$$\begin{aligned} V = \text{span}\,\{&1, i, \overline{(z - z_0)^n}, i\overline{(z - z_0)^n},\\ &\kappa(z - z_0)^n - n(z - z_0)\overline{(z - z_0)^{n-1}},\\ &i\kappa(z - z_0)^n - in(z - z_0)\overline{(z - z_0)^{n-1}} \,|\, n = 1, \ldots, p\}. \end{aligned}$$

The approximation properties of these "generalized harmonic polynomials" are very similar to the approximation properties of the harmonic polynomials for the approximation of solutions of Laplace's equation. Obviously, in the case that the displacement field satisfies the elasticity equations on a domain $\tilde{\Omega} \supset\supset \Omega$, the estimates of Theorem 1 produce similar estimates for the error in the displacement field and stress field for the approximation with "generalized harmonic polynomials". The analogous theorem to Theorem 2 takes the form

Theorem 10 *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, star-shaped with respect to a ball. Let the exterior angle of Ω be bounded from below by $\tilde{\lambda}\pi$. Assume that the displacement field $(u, v) \in H^k(\Omega)$, $k \geq 1$. Then (u, v) can be approximated by "generalized harmonic polynomials" of degree $p \geq k - 1$ such that*

$$\|2\mu(u + iv) - (\kappa\varphi_p - (z - z_0)\overline{\varphi_p'} - \overline{\psi_p})\|_{H^j(\Omega)} \leq C(\operatorname{diam}\Omega)^{k-j}\left(\frac{\ln p}{p}\right)^{\tilde{\lambda}(k-j)} \|(u, v)\|_{H^k(\Omega)}$$

for $j = 0, 1$. The constant C depends only on the shape of Ω and k.

Proof: The proof can be found [24]. □

Remark 7: The theory can be extended to problems with certain loads. In many practical applications the load is simple (constant, polynomial) and an explicit particular solution of the elasticity equations is known. Thus, augmenting the approximation space V by this particular solution allows us to deal with these problems successfully in the framework of approximating the sought solution by functions which solve the differential equation.

Remark 8: A statement completely analogous to Theorem 3 holds for the elasticity equations as well, that is, if the displacement field has a singularity at a corner but can be extended analytically across the boundary except for that corner, the statement of Theorem 3 holds true too for the approximation with "generalized harmonic polynomials" for the elasticity equations.

3. Describability of solution sets

The main feature of the PUM is that it allows us to separate the issue of conformity of the ansatz space from its approximation properties. The PUM permits us therefore to concentrate on finding local approximation spaces with good approximation properties. In order to do that, one has to be able to describe the local behavior of the exact solution well. Let us give some examples where knowledge about the local behavior of the solution is available and indicate how this knowledge can be used for the construction of local approximation spaces.

1. The simplest form of knowledge is that all we know is that the solution u is in some Sobolev space $H^k(\Omega)$ for some $k > 1$. As we will discuss in Section 4, this form of knowledge is not very useful as the classical piecewise polynomial spaces are practically the best possible choice.

2. As we discussed in the preceding section, we can construct complete systems for certain types of differential equations. E.g., if u is harmonic, or the solution of an elliptic equation with analytic coefficients, or if u solves the elasticity equations in two dimensions, we may approximate u locally with "generalized harmonic polynomials".

3. u is the solution of an elliptic partial differential equation with piecewise analytic data. For problems of this kind, the local behavior of u can be described very accurately in terms of so-called countably-normed spaces (see, e.g., [36]). This ability to describe the behavior of the solution so well allowed Babuška and Guo to construct *hp* methods for elliptic problems with piecewise analytic data which have exponential rates of convergence.

4. u can be decomposed in a smooth part (a function in H^k for large k, say) and in a singular part, which can be described very accurately. This happens typically in elliptic problems with corners, interfaces, or if points of sudden changes in boundary conditions are present. We refer to [12] for a good overview on results of this form.

5. u can be decomposed in a smooth part (again, a function in H^k for large k, say) and a boundary layer part. This kind of behavior happens typically in singularly perturbed problems, which appear naturally in various plate and shell models. Very good descriptions of the local behavior of the solutions of these singularly perturbed problems are available: see, e.g., [19] and [1] for the analysis of the Reissner–Mindlin plate model.

6. Analytic knowledge may be given in the form of regularity theory in anisotropic spaces. This was exploited in the construction of ansatz spaces for problems with uni-directionally rough coefficients [35]—see the discussion below.

7. Knowledge about the limiting behavior of the solution may be available. We have in mind problems with (periodic) microstructure, the analysis of laminated materials, or materials with stiffeners. For problems of this kind, asymptotic expansions are available which describe the limiting behavior of the solution.

   Various locking phenomena fall also in this category. For example, shear locking in the Timoshenko beam appears because in the limit, as the thickness of the beam tends to zero, the rotation of the fibers and the deflection field are not independent of each other (cf. the Euler–Bernoulli beam). We cite [39] for an application of the PUM to the Timoshenko beam, where the ansatz spaces are built in such a way that this limiting behavior is accounted for.

8. Another form of knowledge may be given by particular forms of essential boundary conditions. For example, if u solves Laplace's equation and homogeneous Dirichlet (or Neumann) boundary conditions are prescribed on a straight part of the boundary, we may approximate with antisymmetric (or symmetric) harmonic polynomials.

Of course, the different forms in which knowledge about the local behavior of the solution is available to us will naturally lead to different kinds of trial spaces. If no more knowledge is available than that u is in some H^k, then one cannot do much better than choosing a trial space which is motivated by some form of interpolation (see the discussion in Section 4). Let us mention three types of interpolation schemes which lead to slightly different types of methods. The first type of interpolation scheme is given by the classical piecewise polynomial spaces (or mapped polynomials) which lead to the classical FEM. Alternatives to the classical FEM have recently developed in the form of "meshless methods". In these meshless methods ([28, 3, 4, 2, 21, 22]) the ansatz spaces are motivated by interpolation techniques which are used in data fitting algorithms. Finally, a third type of interpolation scheme is given by the use of so-called "radial basis functions" (see [17, 18] for an application of these radial basis functions to the solution of partial differential equations; see [32] for an overview on radial basis functions). For a more detailed discussion of the these new meshless methods and their relationship with the PUM we refer to [38].

Let us elaborate in more detail the case of the presence of corner singularities and the case of the availability of regularity theory in anisotropic spaces. Let us first consider the case of corner singularities. If the exact nature of the singularity (i.e., the exponent) is known (as is the case for Laplace's equation), we could merely include these singular functions in the local approximation space. However, for other case of practical importance such as the elasticity equations, the exact nature of the singularity is not readily available. In this case, a mapping technique can be used which allows us to construct functions, which have at least approximation properties for these singular functions which are much better than the usual polynomials. This is the basic idea of the MAM ([29, 30]). The mapping is such that the solution u is mapped to a function $\tilde{u}$ which is smoother than the original function and thus may be approximated better by polynomials. The ansatz functions to be used then are the pull-back of polynomials under this mapping. Let us illustrate this procedure in a two dimensional setting. For that purpose, let us try to approximate a function u given on a sector of aperture 2ω

$$S := \{(r,\theta) \,|\, 0 < r < 1,\ |\theta| < \omega\},$$

where we assume that u has the form

$$u = u(r,\theta) = r^\alpha \Phi(r,\theta) \tag{12}$$

with $\alpha > 0$ and smooth Φ. The smoothness of u is governed by the size of α; in fact, $u \in H^{1+\alpha-\epsilon}(S)$ for all $\epsilon > 0$. Thus, polynomial approximation of u on S is poor whenever α is close to 0.

Consider now the conformal map $\tilde{z} \mapsto z = \tilde{z}^\beta$, or, in polar coordinates

$$(\tilde{r},\tilde{\theta}) \mapsto (r,\theta) = (\tilde{r}^\beta, \beta\theta)$$

for some (large) $\beta > 1$. This mapping maps the sector

$$\tilde{S} := \{(\tilde{r},\tilde{\theta}) \,|\, 0 < \tilde{r} < 1,\ |\tilde{\theta}| < \omega/\beta\}$$

onto the sector S. The function u transforms to

$$\tilde{u}(\tilde{r},\tilde{\theta}) = \tilde{r}^{\alpha\beta} \Phi(\tilde{r}^\beta, \tilde{\theta}\beta),$$

which is much smoother than the original function u, because $\tilde{u} \in H^{1+\alpha\beta-\epsilon}(\tilde{S})$. Therefore, $\tilde{u}$ can be approximated well by polynomials $\tilde{P}_p$ (in the variables $(\tilde{x},\tilde{y})$) of degree p. Denote P_p the pull-back of this polynomial, i.e., set $P_p(x,y) = \tilde{P}_p(\tilde{x},\tilde{y})$. A simple calculation shows that the H^1 semi norm is invariant under this transformation, that is, for all $v \in H^1(S)$ we have

$$\|\nabla v\|_{L^2(S)} = \|\nabla \tilde{v}\|_{L^2(\tilde{S})}.$$

In particular,

$$\|\nabla(u - P_p)\|_{L^2(S)} = \|\nabla(\tilde{u} - \tilde{P}_p)\|_{L^2(\tilde{S})},$$

and we see that indeed these "pull-back polynomials" (they are of course no longer polynomials in the variables x, y) have much better approximation properties for functions u of the type (12) than the usual polynomials.

A mapping idea is also used in [35] in their analysis of a problem with uni-directionally rough coefficients. In that particular example, a mapping leads to regularity theory in anisotropic spaces which could then in turn be used in the construction of good local approximation spaces. In [35] an elliptic equation with uni-directionally rough coefficients is considered:

$$\begin{aligned} &\frac{\partial}{\partial x}a(x,y)\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}a(x,y)\frac{\partial}{\partial y}u = f(x,y) \in L^2(\Omega) \qquad \text{in } \Omega, \\ &u = 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{13}$$

where the coefficient $a \in L^\infty$ may be rough in the x direction but is assumed to be smooth in the y direction. As usual, we assume that $0 < a_0 \le a(x,y) \le a_1 < \infty$ and that the boundary $\partial\Omega$ is smooth. The usual regularity theory may only give that the solution u is in some Sobolev space $H^{1+\epsilon}$ for some (potentially very small) $\epsilon > 0$. Thus, the usual piecewise polynomial ansatz functions fail for this problem. For the simplified model $a(x,y) = a(x)$ the mapping

$$x \mapsto \tilde{x} = \int_0^x \frac{\mathrm{d}t}{a(t)}, \qquad y \mapsto \tilde{y}$$

maps u onto a function $\tilde{u}$ which is in $H^2(\tilde{\Omega})$, where $\tilde{\Omega}$ is the image of Ω under the mapping. Thus, $\tilde{u}$ may be approximated locally well by linear functions. Transforming back to the original variables x, y gives that u can be approximated well (in the H^1 norm) by the linear combinations of the functions

$$\left\{1, y, \int^x \frac{\mathrm{d}t}{a(t)}\right\}.$$

Therefore, the local approximation spaces $V_i = \operatorname{span}\left\{1, y, \int^x \frac{\mathrm{d}t}{a(t)}\right\}$ have the same approximation properties for the approximation of the rough solution u of (13) as the usual linear functions for the approximation of smooth functions. The PUM based on these local approximation spaces is therefore robust in that the global error is independent of the roughness of the coefficient a.

4. Optimality of approximation spaces

We analyzed in Section 2.1 the approximation properties of harmonic polynomials for the approximation of solutions of Laplace's equation. Via the theory of Bergman and Vekua, we saw that these approximation properties are inherited by the "generalized harmonic polynomials", i.e., the images of polynomials under the operator ReV, which maps holomorphic functions onto solutions of (2). A natural question to ask is why we used harmonic polynomials for the approximation harmonic functions. There are many other systems of functions which are dense in the set of holomorphic functions, e.g., $\{e^{nz} \,|\, n \in \mathbb{N}_0\}$, $\{z^{-n} \,|\, n \in \mathbb{N}_0\}$ (if $0 \notin \Omega$) to mention but a few.

There are many factors which may determine the choice of a particular set of approximating functions. From an implementational point of view, one has to weigh the cost of evaluating or the cost of creating the approximating functions. Another practical consideration is, for example, the condition number of the resulting linear system. In this section, however, we want to justify the choice of harmonic polynomials from a mathematical point of view. The tool for these considerations is the theory of n-width [31] (see also the discussion of optimality in [38]). The theory of n-width answers the following question: Given n degrees of freedom, how well can we approximate elements of a given class of functions in a given norm? This is formalized in the following definition.

Definition 2 *Let $(X, \|\cdot\|)$ be a Banach space, $S \subset X$ be a class of functions we wish to approximate. For $n \in \mathbb{N}$ we define the n-width as*

$$d(n, \|\cdot\|, S) = \inf_{E_n} \sup_{f \in S} \inf_{g \in E_n} \|f - g\|,$$

where the first infimum is taken over all n dimensional subspaces E_n of X. A space E_n for which this infimum is attained is called an optimal space.

We see that the n-width depends on the number of degrees of freedom n, the norm $\|\cdot\|$ in which we measure approximability, and the class S of functions which we want to approximate. The class S encodes the information available to us.

Example 4: In this example, let $\Omega \subset \mathbb{R}^d$ and assume that we are trying to approximate a function u in the H^1 norm where we only know that it is in some space $H^k(\Omega)$, $1 < k \leq 2$. In this case, we would define $S = \{u \in H^k(\Omega) \,|\, \|u\|_{H^k} \leq 1\}$ and choose the norm $\|\cdot\| = \|\cdot\|_{H^1(\Omega)}$. If we approximate with piecewise linears on a (quasi-) uniform mesh with mesh size h, then we obtain the approximation rate Ch^{k-1}, which translates into $Cn^{-(k-1)/d}$ where n is the number of degrees of freedom. It turns out that this is already almost optimal: the rate (i.e., the exponent $(k-1)/d$) is optimal and choosing a different approximation space can only improve the constant C, [47].

The piecewise linears thus lead to optimal rates of convergence if we only know that the function u to be approximated is in some $H^k(\Omega)$. Note that they are uniformly optimal in the sense that for each k they lead to optimal asymptotic rates.

Example 5: Consider the same problem as in the preceding example but assume that the function to be approximated is in some $H^k(\Omega)$ for some k with $p < k \leq p+1$, $p \in \mathbb{N}$. Then piecewise polynomials of degree p lead to optimal rates just as in the preceding example.

Example 6: Assume now that we have additional information about the function u. Let us assume that we are trying to approximate a *harmonic* function u on the unit *disc* $B_1(0) \subset \mathbb{R}^2$. For $k \geq 1$ introduce the space

$$\mathcal{H}^k(B_1(0)) := \{u \in H^k(\Omega) \,|\, -\Delta u = 0 \text{ on } B_1(0)\}.$$

Then it can be shown ([38]) that harmonic polynomials of degree p *are optimal spaces* of dimension $2p+1$ for the approximation of elements of $\mathcal{H}^k$. More precisely, for every $k \geq 1$, we may set the class

$$S := \{u \in \mathcal{H}^k(B_1(0)) \,|\, \|u\|_{H^k} \leq 1\}$$

and then seek approximation in the H^1 norm. An optimal space E_n in the sense of Definition 2 for $n = 2p+1$ is given by the harmonic polynomials of degree p. Note that the harmonic polynomials are optimal for all $k \geq 1$. In this sense they are universally optimal.

Example 7: The proof of the statement in the preceding example relies not so much on properties of harmonic functions as it exploits a certain rotational invariance of the problem (the $\|\cdot\|_{H^1}$ norm, the space of harmonic polynomials, and the sets $\mathcal{H}^k$ are all rotationally invariant). This rotational invariance may be exploited to show that the "generalized harmonic polynomials" for the Helmholtz equation defined in (6) are also optimal. To be precise, if we introduce the set

$$\tilde{\mathcal{H}}^s(B_1(0)) := \{u \in H^s(\Omega) \,|\, u \text{ complex valued }, -\Delta u - k^2 u = 0 \text{ on } B_1(0)\},$$

then the spaces $G(p)$ defined in (6) are optimal spaces of dimension $2p+1$ (if we allow complex degrees of freedom) in the sense of n-width for every $s \geq 1$, if we measure approximation in the H^1 norm and define the class S by

$$S := \{u \in \tilde{\mathcal{H}}^s(B_1(0)) \,|\, \|u\|_{H^s} \leq 1\}.$$

Again, we note that these "generalized harmonic polynomials" are universally optimal in the sense that they are optimal for all $s > 1$.

We would like to point out that universally optimal spaces are very desirable. In practice, we may not know precisely in what space the exact solution may be. Thus, the approximation spaces should be large enough to cover a broad range of cases.

Let us give another example which illustrates the use of universally optimal approximation spaces. Suppose the function u which we wish to approximate is analytic on $\tilde{\Omega}$, a proper superset of the domain Ω of interest. In this case, $u \in H^k(\Omega)$ for all k and it is therefore not a priori clear how we should choose the class of functions S in Definition 2. Universally optimal spaces like the harmonic polynomials for the approximation of harmonic functions on discs avoid these difficulties. We refer to [34] for a more detailed discussion of the uncertainty issues we touched here.

We saw that harmonic polynomials are optimal for the approximation on discs. Of course, if we approximate on domains which are not discs, we cannot expect harmonic polynomials to be optimal any more. However, it is reasonable to expect that they are still near-optimal as long as the domain does not deviate too much from a disc. This ties in with Remark 1 where we observed that for domains with re-entrant corners the approximation properties of polynomials deteriorate as the angle of the re-entrant corner increases–or, put differently, as the domain deviates from a convex domain.

Let us finally mention that for the approximation of harmonic functions on sectors (i.e, we consider the case of a corner), expansions in terms of singular functions are optimal. For example, if we consider the sector

$$W = \{z \in \mathbb{C} \,|\, |z| < 1 \ |\arg z| < \omega\}$$

and approximate harmonic functions satisfying homogeneous Dirichlet boundary conditions on the two straight sides of W, then the functions $\operatorname{Im} z^{n\pi/\omega}$, $n = 1, \ldots, p$ form optimal spaces of dimension p.

5. Mathematical foundation of the PUM

In this section we want to describe in detail the PUM. As we have second order elliptic problems in mind, we will restrict the exposition in this paper to an H^1 setting, the extensions to other settings being straightforward.

The starting point of the PUM is, just as in the classical FEM, a variational formulation of the problem. Given such a variational formulation, the task is then to choose trial and test spaces judiciously. Let us therefore briefly review the main ingredients of the classical FEM which also serves to highlight the differences to the PUM. Once a variational formulation is chosen, the classical FEM rests on local approximation properties of the trial space, some inter-element continuity, and stability. For a large class of problems, stability is given by the coercivity of the bilinear form, and thus the performance of the classical FEM is completely determined by the approximation properties of the conforming ansatz spaces. In the classical FEM, these are given by the approximation properties of (piecewise) polynomials. Whenever polynomials can approximate the exact solution well locally, the classical FEM performs well. However, all that is really needed is that the trial space has good local approximation properties and is conforming.

With this observation in mind, the goal of the PUM is to create conforming ansatz spaces with good approximation properties. With the aid of a partition of unity the PUM constructs a global conforming space from user-provided local approximation spaces which, presumably, can approximate the exact solution well. The use of a partition of unity allows us to separate the issues of local approximability (which is encoded in the local approximation spaces) and the issue of conformity of the space. In fact, under very mild assumptions on the partition of unity (see below), the global space constructed by the PUM inherits the approximation properties of the local approximation space, i.e., the global approximation properties are as good as the local approximations permit;

additionally, the global space inherits the smoothness of the partition of unity. In particular, a Lipschitz partition of unity is sufficient to ensure H^1 conforming global spaces and smoother partitions of unity lead to smoother ansatz spaces (e.g., a C^1 partition of unity leads to H^2 conforming spaces important for some plate and shell models). Let us summarize these two main features of the PUM before we proceed with the description of the PUM.

1. The PUM allows for the inclusion of a-priori knowledge about the structure of the solution in the ansatz space. The PUM can be applied successfully whenever a good description of the local behavior of the solution is available and can therefore lead to efficient and robust methods.
2. The PUM allows for the creation of ansatz spaces of any desired regularity very easily. Therefore, trial spaces for variational formulations of various plate and shell models are available.

The main technical tool for the description of the PUM is that of a (M, C_∞, C_G) partition of unity.

Definition 3 *Let $\Omega \subset \mathbb{R}^n$ be an open set, $\{\Omega_i\}$ be an open cover of Ω satisfying a pointwise overlap condition*

$$\exists M \in \mathbb{N} \qquad \forall x \in \Omega \qquad \operatorname{card}\{i \,|\, x \in \Omega_i\} \leq M.$$

Let $\{\varphi_i\}$ be a Lipschitz partition of unity subordinate to the cover $\{\Omega_i\}$ satisfying

$$\operatorname{supp} \varphi_i \subset \operatorname{closure}(\Omega_i) \qquad \forall i, \tag{14}$$

$$\sum_i \varphi_i \equiv 1 \text{ on } \Omega, \tag{15}$$

$$\|\varphi_i\|_{L^\infty(\mathbb{R}^n)} \leq C_\infty, \tag{16}$$

$$\|\nabla \varphi_i\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C_G}{\operatorname{diam} \Omega_i}, \tag{17}$$

where C_∞, C_G are two constants. Then $\{\varphi_i\}$ is called a (M, C_∞, C_G) partition of unity subordinate to the cover $\{\Omega_i\}$. The partition of unity $\{\varphi_i\}$ is said to be of degree $m \in \mathbb{N}_0$ *if $\{\varphi_i\} \subset C^m(\mathbb{R}^n)$. The covering sets $\{\Omega_i\}$ are called* patches.

Definition 4 *Let $\{\Omega_i\}$ be an open cover of $\Omega \subset \mathbb{R}^n$ and let $\{\varphi_i\}$ be a (M, C_∞, C_G) partition of unity subordinate to $\{\Omega_i\}$. Let $V_i \subset H^1(\Omega_i \cap \Omega)$ be given. Then the space*

$$V := \sum_i \varphi_i V_i = \{\sum_i \varphi_i v_i \,|\, v_i \in V_i\} \subset H^1(\Omega)$$

is called the PUM space. *The PUM space V is said to be of degree $m \in \mathbb{N}$ if $V \subset C^m(\Omega)$. The spaces V_i are referred to as the* local approximation spaces.

Theorem 11 *Let $\Omega \subset \mathbb{R}^n$ be given. Let $\{\Omega_i\}$, $\{\varphi_i\}$, and $\{V_i\}$ be as in Definitions 3, 4. Let $u \in H^1(\Omega)$ be the function to be approximated. Assume that the local approximation spaces V_i have the following approximation properties: On each patch $\Omega_i \cap \Omega$, u can be approximated by a function $v_i \in V_i$ such that*

$$\|u - v_i\|_{L^2(\Omega_i \cap \Omega)} \leq \epsilon_1(i),$$

$$\|\nabla(u - v_i)\|_{L^2(\Omega_i \cap \Omega)} \leq \epsilon_2(i).$$

Then the function

$$u_{ap} = \sum_i \varphi_i v_i \in V \subset H^1(\Omega)$$

satisfies

$$\|u - u_{ap}\|_{L^2(\Omega)} \leq \sqrt{M} C_\infty \left(\sum_i \epsilon_1^2(i) \right)^{1/2}, \tag{18}$$

$$\|\nabla(u - u_{ap})\|_{L^2(\Omega)} \leq \sqrt{2M} \left(\sum_i \left(\frac{C_G}{\operatorname{diam} \Omega_i} \right)^2 \epsilon_1^2(i) + C_\infty^2 \epsilon_2^2(i) \right)^{1/2}. \tag{19}$$

Proof: See [25]. □

Remark 9: The constant M controls the overlap of the patches. In particular, not more than M patches overlap in any given point $x \in \Omega$ of the domain. The patches have to overlap because the functions φ_i are supposed to form a sufficiently regular (here: Lipschitz) partition of unity. Condition (17) expresses the fact that we need to control the gradient of the partition of unity functions φ_i if we are interested in H^1 estimates. Note that typically $\epsilon_1(i) \leq C(\operatorname{diam} \Omega_i)\epsilon_2(i)$ so that the terms in the sum of (19) are in a sense balanced.

Let us give a few examples of partitions of unity which satisfy all the requirements.

Example 8: The usual finite element hat functions on shape regular meshes satisfy all the requirements of Definitions 3. For example, the usual piecewise linear hat functions on a regular (triangular) mesh in two dimensions satisfy the above conditions of a (M, C_∞, C_G) partition of unity with $M = 3$, $C_\infty = 1$, and condition (17) is satisfied because of the regularity of the mesh, i.e., the minimum angle condition satisfied by the triangulation.

Similarly, the classical bilinear finite element functions on quadrilateral meshes form a (M, C_∞, C_G) partition of unity ($M = 4$, $C_\infty = 1$).

Therefore, if we want to construct a (M, C_∞, C_G) partition of unity for a domain Ω based on hat functions, we may use a regular mesh (based on triangles, quadrilateral, or both) as outlined above on any $\tilde{\Omega} \supset \Omega$. The hat functions given by the mesh on $\tilde{\Omega}$ then form a partition of unity for $\tilde{\Omega}$ and therefore for Ω as well.

Example 9: Smooth partitions of unity may also be constructed easily. Let $\{\Omega_j\}$ be a collection of overlapping patches which cover Ω and let ψ_j be a collection of smooth (at least Lipschitz) functions which are supported by the patches. Then the functions

$$\varphi_i := \frac{\psi_i}{\sum_j \psi_j}$$

form a partition of unity subordinate to the covering $\{\Omega_j\}$. Note that the functions φ_i are as smooth as the functions ψ_i. Thus, this "normalization" process allows us to create partitions of unity of any desired regularity.

In order to make sure that the functions φ_i generated in this way do indeed satisfy the conditions of a (M, C_∞, C_G) partition of unity, a few minor technical conditions need to be placed on the patches and the functions ψ_i. For example, we can ensure that the φ_i satisfy all the necessary requirements if we assume that

1. the patches Ω_i satisfy the overlap condition (i.e., no more than M patches overlap in any given point $x \in \Omega$);

2. the patches are locally comparable in size, i.e., there are C_1, $C_2 > 0$ such that $C_1 \operatorname{diam} \Omega_i \leq \operatorname{diam} \Omega_j \leq C_2 \operatorname{diam} \Omega_i$ whenever $\Omega_i \cap \Omega_j \neq \emptyset$;

3. the functions ψ_i satisfy

$$\begin{aligned} \|\psi_i\|_{L^\infty} &\leq C_3, \\ \|\nabla\psi_i\|_{L^\infty} &\leq \frac{C_4}{\operatorname{diam}\Omega_i}, \\ \sum_i \psi_i(x) &\geq C_5 \qquad \text{on } \Omega \end{aligned}$$

for some constants $C_1, \ldots, C_5$.

These conditions are rather easily met in practice.

As a very simple example, let us consider the PUM with *polynomial* local approximation spaces. We then expect the PUM to perform similarly to the classical h, p, and hp finite element methods. Let $\{\Omega_i\}$ be a covering of $\Omega \subset \mathbb{R}^2$ and $\{\varphi_i\}$ be a (M, C_∞, C_G) partition of unity subordinate to the covering. Choose as the local approximation spaces the space of all polynomials of degree p, i.e., $V_i(p) := \operatorname{span}\{1, x, y, x^2, \ldots, y^p\}$. Then the global PUM space is given by $V = \sum_i \varphi_i(x) V_i$. For the approximation properties of V, let us assume that the function u to be approximated is in $H^k(\Omega)$, $k > 1$. In the notation of Theorem 11, we have the standard local estimates

$$\epsilon_1(i) \leq C(\operatorname{diam}\Omega_i)^{\mu+1} p^{-k} \|u\|_{H^k(\Omega_i \cap \Omega)},$$

$$\epsilon_2(i) \leq C(\operatorname{diam}\Omega_i)^{\mu} p^{-(k-1)} \|u\|_{H^k(\Omega_i \cap \Omega)},$$

where $\mu = \min(k-1, p)$. Theorem 11 gives the existence of a global approximant $v \in V$ (the PUM space) such that

$$\|u - v\|_{L^2(\Omega)} \leq CMC_\infty h^{\mu+1} p^{-k} \|u\|_{H^k(\Omega)},$$

$$\|\nabla(u - v)\|_{L^2(\Omega)} \leq CM\sqrt{2(C_\infty^2 + C_G^2)}\, h^{\mu} p^{-(k-1)} \|u\|_{H^k(\Omega)},$$

where we set $h = \max_i \operatorname{diam}\Omega_i$. We see that the PUM with polynomial local approximation spaces produces an approximation space with approximation properties very similar to the usual finite element method: If the local approximation spaces consist of polynomials of fixed degree p and if the approximation in V_i is achieved through the smallness of the patch Ω_i, then the method behaves like the h version. If the patches are kept fixed and the local approximation is achieved by increasing the degree p of the polynomials, which form the local spaces V_i, the method behaves like the p version. In this sense, the PUM is a generalization of the h and p version.

In the numerical examples in Section 6 we will choose local approximation spaces custom tailored to the problem under consideration. As we consider Laplace's equation and the Helmholtz equation, we will use harmonic polynomials and "generalized harmonic polynomials" which have approximation properties very similar to the classical polynomials except that the same accuracy can be achieved with considerably fewer degrees of freedom.

6. Numerical examples

In this section we want to illustrate the performance of the PUM for two examples, Laplace's equation and the Helmholtz equation. In both examples, we will be interested in the p version of the PUM, that is, we will keep the number of patches small and consider the effect of enriching the local approximation spaces. We refer to [25] for an example of the h version of the PUM for a model problem with rough coefficients. In that example, the PUM leads to a robust method, that is, a method which perform well independently of the roughness of the coefficients.

As we will use the same partition of unity for both examples, we will describe it at this point. Let $\Omega = (0,1)^2 \subset \mathbb{R}^2$. Subdivide the square Ω into $n \times n$ ($n \in \mathbb{N}$) squares. There are $(n+1) \times (n+1)$

nodes and the classical, piecewise bilinear hat functions associated with these $(n+1)\times(n+1)$ nodes form a (M, C_∞, C_G) partition of unity (cf. Section 5). For the patches Ω_i, we choose the supports of the $(n+1)\times(n+1)$ piecewise bilinear functions.

6.1. Laplace's Equation

We consider the following problem:

$$\begin{aligned} \Delta u &= 0 \qquad \text{on } \Omega := (0,1)\times(0,1), \\ u &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{20}$$

with

$$g = \partial_n \mathrm{Re}\left((a^2 - z^2)^{-1} + (a^2 + z^2)^{-1}\right)$$

and the parameter $a = 1.05$. Problem (20) has a unique solution if we fix u in one point (e.g., $(0,0)$). The weak form of this differential equation is to find $u \in H^1(\Omega)$ such that

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = F(v) = \oint_{\partial\Omega} gv \mathrm{d}s.$$

The PUM may be applied to this problem, and we compare the performance of the PUM for this problem with the classical p versions of the finite element method in Fig. 2. We consider a "p version" of the PUM, that is, we fix a covering and a partition of unity and then increase the

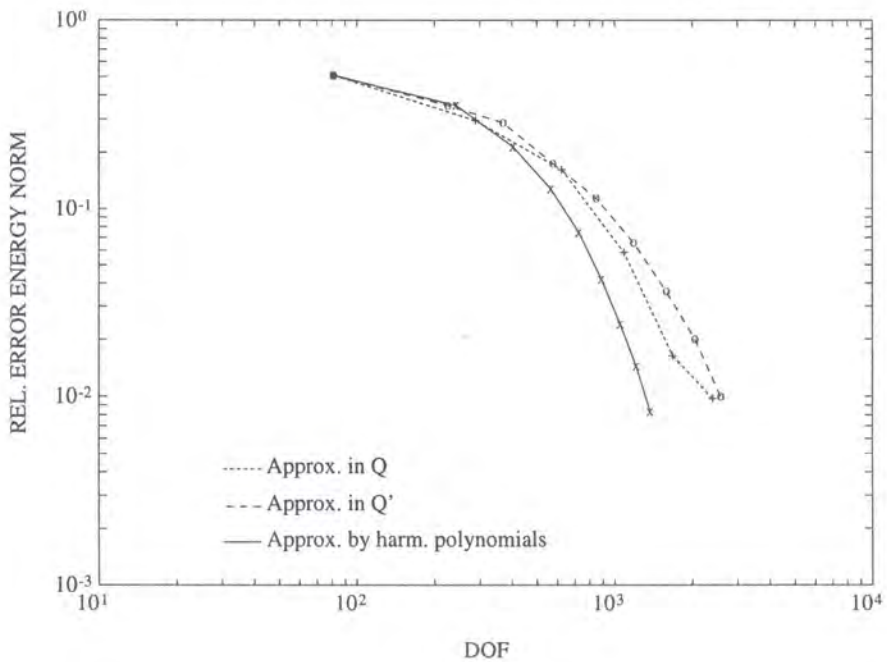

Fig. 2. PUM, classical p version for Laplace's equations; $a = 1.05$, 8×8 elements

local approximation spaces. Here, the partition of unity is chosen as described above with $n = 8$, i.e., there are 81 patches and the partition of unity consists of 81 piecewise bilinear functions. The local approximation spaces are chosen as *harmonic* polynomials of degree p (p ranges from 0 to 8). These approximation spaces are compared with the usual p version elements (both the full spaces Q_p and the trunk spaces Q_p') on the same mesh, i.e., on an 8×8 mesh. The approximation

error is measured in the "energy norm", i.e., the H^1 semi norm. We see that the PUM performs better than the classical p versions: to achieve an accuracy of 1%, the PUM based on harmonic polynomials needs only half as many degrees of freedom as the classical p version. We also note that the discrepancy between the PUM and the classical p versions becomes larger as the accuracy requirement is increased.

These observations are in agreement with the approximation theory we developed for the PUM. The global error of the PUM is controlled by the approximation properties of the local approximation spaces which we analyzed in Section 2.1. In this particular case, the solution u is analytic up to the boundary and thus Theorem 1 gives error bounds of the form $e^{-\gamma p}$ for some $\gamma > 0$ for the approximation with harmonic polynomials. Note that there are $2p+1$ harmonic polynomials of degree p, and therefore, we get error bounds of the form $e^{-\gamma_1 DOF}$ for the "p version" of the PUM. The classical p versions, on the other hand, also lead to error bounds of the form $e^{-\gamma_2 p}$ for some $\gamma_2 > 0$. But, as $DOF = O(p^2)$ in the p version, we only get bounds of the form $e^{-\gamma_3\sqrt{DOF}}$ for the classical p versions. Therefore, asymptotically the PUM is better than the classical p version, and the discrepancy between the performance of the PUM based on harmonic polynomials and the classical p version has to become bigger as the accuracy requirement is increased.

6.2. Helmholtz's Equation

In this section we want to apply the PUM to Helmholtz's equation. We consider

$$\begin{aligned} -\Delta u - k^2 u &= 0 \qquad \text{on } \Omega = (0,1)\times(0,1) \subset \mathbb{R}^2, \\ \partial_n u + iku &= g \qquad \text{on } \partial\Omega, \end{aligned} \tag{21}$$

where g is chosen such that the exact solution u is a plane wave of the form

$$u(x,y) = \exp\{ik(x\cos\theta + y\sin\theta)\} \qquad \text{with } \theta = \frac{\pi}{16}, k = 32 \text{ and } k = 100.$$

The weak formulation of this problem is to find $u \in H^1(\Omega)$ such that

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla \overline{v}\,\mathrm{d}x\mathrm{d}y - k^2 \int_\Omega u\overline{v}\,\mathrm{d}x\mathrm{d}y + ik \oint_{\partial\Omega} u\overline{v}\mathrm{d}s = F(v) = \oint_{\partial\Omega} g\overline{v}\mathrm{d}s \tag{22}$$

for all $v \in H^1(\Omega)$. In Section 2.3 we analyzed two types of local approximation spaces, viz., "generalized harmonic polynomials" and systems of plane waves. We present the PUM with these two choices as local approximation spaces. The partition of unity consists again of piecewise bilinear functions of an $n \times n$ mesh as outlined above.

As the exact solution is a plane wave, the Theorem 9 suggests that the PUM leads to exponential rates of convergence for both choices of local approximation spaces. This exponential rate can indeed be observed in Fig. 3 for the case of "generalized harmonic polynomials" as local approximation spaces and in Fig. 4, 5 (in both cases, the wave number $k = 32$). We note that the approximation properties of both the "generalized harmonic polynomials" and the systems of plane waves are very similar.

We mentioned in Section 2.3 that implementational aspects also affect the choice of the local approximation spaces. In this particular example, the choice of systems of plane waves seems preferable to "generalized harmonic polynomials". The plane waves with which local approximation is done can be written as products of functions in x and in y. As the partition of unity functions can also be written as products of functions depending only on x and y, the construction of the stiffness matrix can be done cheaply in this case.

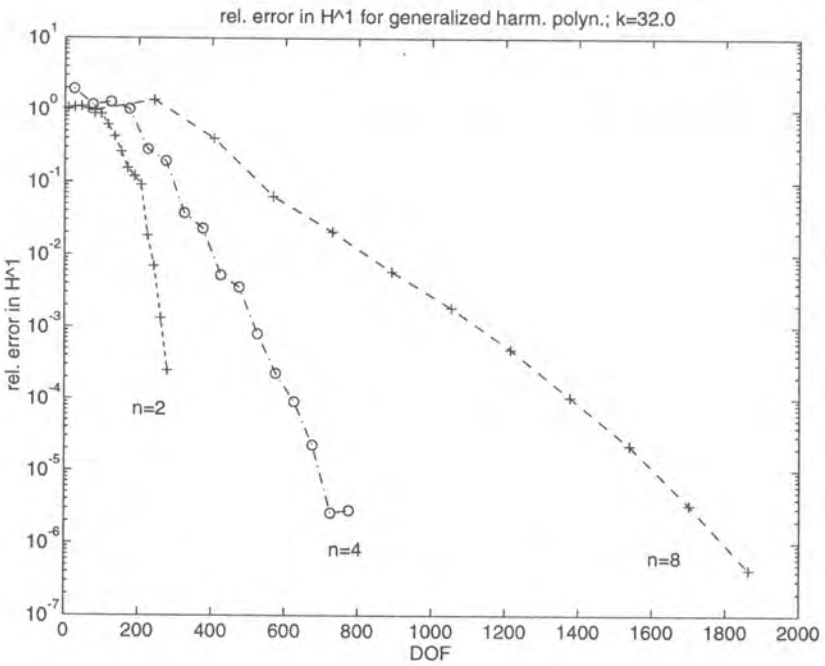

Fig. 3. "p version" of PUM for Helmholtz's equation ($k = 32$); Generalized Harmonic Polynomials as local approximations

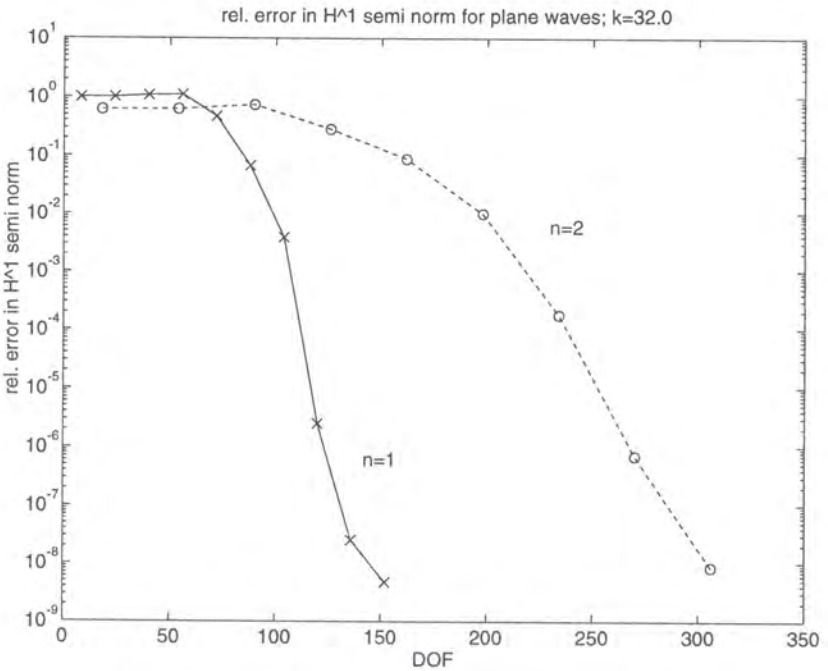

Fig. 4. "p version" of PUM for Helmholtz's equation ($k = 32$); plane waves as local approximations

In Tables 1–4 we compare the PUM based on systems of plane waves with several methods which are based on piecewise linear functions, namely, the classical FEM, the generalized least squares finite element method (GLSFEM) [42] and the QSFEM [37]. In Table 1 we compare the number of degrees of freedom needed to achieve a given accuracy in L^2 for the PUM and these h version type methods and see that the PUM needs even fewer degrees of freedom than the piecewise linear best approximant would need. This reduction in the degrees of freedom needed to achieve a certain

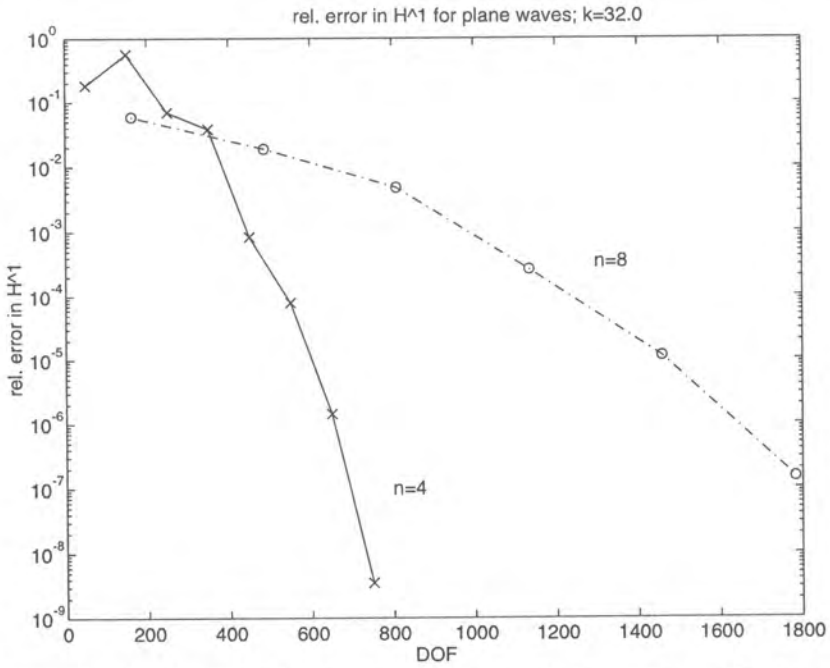

Fig. 5. "p version" of PUM for Helmholtz's equation ($k = 32$); plane waves as local approximations

accuracy translates directly into an improvement in the operation count for the solution of the resulting linear system (Table 2). Finally, in Tables 3, 4, we compare the performance of the PUM with the piecewise linears based methods, if the linear systems are solved using the iterative solver proposed in [11]. Again, the PUM outperforms these h version methods.

Table 1. DOF necessary to achieve various accuracies in L^2 for PUM with $n = 4$ and piecewise linear based methods; $k = 100$

p	L^2 error	PUM	best approx.	QSFEM	FEM
26	10.8%	6.50D+2	3.80D+3	7.95D+3	2.08D+5
30	0.69%	7.50D+2	5.89D+4	1.23D+5	3.23D+6
34	0.11%	8.50D+2	3.45D+5	7.23D+5	1.90D+7

Table 2. Number of operations using band elimination – the p version of the PUM; $n=4$; $k=100$; error in L^2

p	L^2 error	PUM	QSFEM	FEM
26	10.8%	1.76D+7	6.3D+7	4.3D+11
30	0.69%	2.71D+7	1.5D+10	1.01D+13
34	0.11%	3.94D+7	5.2D+11	3.6D+14

Table 3. Operation count for solving linear system; error in H^1 norm;$k = 32$

	Galerkin			QSFEM		
$\sqrt{DOF}$	H^1 error	No. iter	NOP	H^1 error	No. iter	NOP
32	65%	232	4.51D+6	30.5%	272	5.29D+6
64	21.7%	434	3.37D+7	14.3%	492	3.82D+7
128	8.16%	831	2.68D+8	7.02%	953	2.96D+8
256	3.64%	1665	2.07D+9	3.48%	1863	2.31D+9
512	1.72%	3263	1.62D+10	1.69%	3752	1.86D+10

Table 4. Operation count for band elimination for PUM; $k = 32$, error in H^1; $n = 1$

p	H^1 error	NOP PUM
18	46%	1.3D+5
22	6.7%	2.3D+5
26	0.38%	3.8D+5
30	0.00025%	5.9D+5

7. Conclusions

In the first part of the paper, we presented approximation results for certain types of complete systems. We started with the simple case of the approximation of harmonic functions by harmonic polynomials and quantified this approximation in terms of Sobolev norms. These approximation results were then extended to the case of general elliptic equations with analytic coefficients via the theory of Bergman and Vekua.

In the second part of the paper, we presented a new method, the PUM. This method has the special feature that it allows the user to include knowledge about the structure of the solution in the finite element space. This feature can be very useful in the analysis of problems where the classical methods are very expensive, e.g., the numerical solution of equations with rough coefficients or the analysis of problems with highly oscillatory solutions. In these cases, feeding knowledge about the solution into the ansatz space can lead to highly efficient and robust methods.

The PUM permits to custom tailor the ansatz space to the particular problem under consideration. In order to make use of this ability, one has to be able to describe the local behavior of the solution. One form of descriptions is provided by complete systems. In Section 3, we discussed other forms of analytic knowledge about the local behavior of the solution such as the presence of corner singularities, boundary layers, and limiting behavior described by asymptotic expansions.

Finally, we demonstrated the viability and the capabilities of the PUM by an application to the Helmholtz equation at high wave numbers.

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