Computer Assisted Mechanics and Engineering Sciences, 4: 157–165, 1997.

Steady flow in a linearly diverging asymmetrical channel

Oluwole Daniel Makinde
School of Mathematics, University of Bristol, BS8 lTW, Bristol, UK

(Received February 28, 1996)

In this paper the steady flow of a viscous incompressible fluid in a slightly asymmetrical channel is considered. The flow is considered for channel with a small aspect ratio ε. The solution is expanded into a Taylor series with respect to the Reynolds number. Using the D–T method (Drazin and Tourigny, [7]), a bifurcation study is performed. Parameter ranges for the Reynolds number, where no, one or two solutions of the given type exist, are computed.

1. Introduction

The study of flow through channel of varying width has received much attention owing to their applications in mathematical modelling of biological and engineering systems. The idea of mathematical treatment of this type of problem was first introduced into the literature by Blasius [3]. He perturbed the steady Poiseuille flow between parallel plates to approximate the separated flow in a slowly varying exponentially diverging plane channel. A similar problem, popularly known as J–H flow, were investigated by Jeffery [13] and Hamel [11]. They obtained a similarity solution that describes the steady two-dimensional radial flow between two inclined rigid planes driven by a line source at the intersection of the planes.

However, for over four decades, workers in statistical mechanics have effectively employed the technique of extending a regular perturbation series to high order by computer, and then analysing the coefficients to reveal the structure of the solution, Gaunt and Gutmann, [10]. That procedure was later adapted to a variety of problems in fluid mechanics, Van Dyke [19]. Most recently, Drazin and Tourigny [7] presented a novel computational approach to the investigation of bifurcations that relies on the use of power series in the bifurcation parameter for a particular solution branch. Their initial motivation was to solve boundary-value problems for nonlinear systems of ordinary and partial differential equations. The procedure leads to a special type of Hermite–Padé approximant. Let us suppose that the partial sum

$$U_N(\lambda) = \sum_{n=1}^{N} a_n \lambda^n = U(\lambda) + O(\lambda^{N+1}) \quad \text{as} \quad \lambda \to 0\,, \tag{1}$$

is given. We shall make the simplest hypothesis in the context of nonlinear problems by assuming that $U(\lambda)$ is the local representation of an algebraic function u of λ. Therefore, we seek a polynomial $F_d = F_d(\lambda, u)$ of degree $d \geq 2$,

$$F_d(\lambda, u) = \sum_{m=1}^{d} \sum_{k=0}^{m} f_{m-k,k} \lambda^{m-k} u^k \tag{2}$$

such that

$$\frac{\partial F_d(0,0)}{\partial u} = 1\,, \tag{3}$$

and

$$F_d(\lambda, U_N(\lambda)) = O(\lambda^{N+1}) \quad \text{as } \lambda \to 0\,. \tag{4}$$

Condition (3), which yields $f_{0,1} = 1$, ensures that the polynomial has only one root which vanishes at $\lambda = 0$ and also normalises F_d. There are thus

$$1 + \sum_{m=2}^{d} (m+1) = \frac{1}{2}(d^2 + 3d - 2)\,, \tag{5}$$

undetermined coefficients in the polynomial (2). The requirement (4) reduces the problem to a system of N linear equations for the unknown coefficients of F_d. The entries of the underlying matrix depend only on the N given coefficients a_n. Henceforth, we shall take

$$N = \frac{1}{2}(d^2 + 3d - 2), \tag{6}$$

so that the number of equations equals the number of unknowns. A bifurcation occurs where the solutions of a nonlinear system change their qualitative character as a parameter changes. In particular, bifurcation theory is about how the number of steady solutions of a system depends on a parameter. The bifurcations for $F_d(\lambda, u)$ can then be analysed locally by means of Newton's diagram (Vainberg and Trenogin, [20]).

In the present paper the steady flow in a linearly diverging asymmetrical channel is considered. Our objective is to demonstrate the applicability of the D–T method in solving nonlinear systems of partial differential equations as well as understanding the bifurcation that takes place in the flow field as the Reynolds number increases. In Sections 2 and 3, we establish the mathematical formulation for the problem. Computer extension of the resulting perturbation series solution and the bifurcation study are examined in Section 4. In Section 5, we discuss the entire findings.

2. Mathematical formulation

Consider the steady flow of an incompressible viscous fluid through a channel of varying width. It is assumed that the channel is long enough to neglect both the entrance and the end effects. Take a Cartesian co-ordinate system (x, y) where y is the distance measured in the normal section of the channel and x is in the streamwise direction. Let u and v be the velocity components in the directions of x and y increasing respectively and $b(x)$ is the variation in the channel's width. Then,

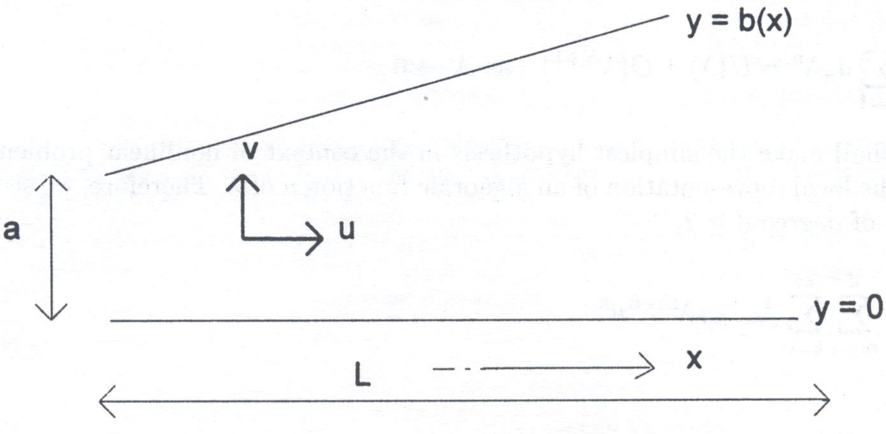

Fig. 1. Schematic diagram of the problem.

for two-dimensional flow the Navier–Stokes equations in terms of stream-function ψ and vorticity ω are

$$\frac{\partial(\omega,\Psi)}{\partial(x,y)}=\nu\nabla^2\omega\,, \tag{7}$$

$$\omega=-\nabla^2\Psi\,, \tag{8}$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$ and ν the kinematic viscosity of the fluid. The channel's width ($y=b(x)=aS(x/L)$) is assumed to vary slowly with axial distance such that $0<\varepsilon=a/L\ll 1$, where a is the channel's characteristic width and L is the characteristic length. In the limit $\varepsilon\to 0$, the channel is of constant width. The following dimensionless variables are introduced

$$\omega'=\frac{a^2\omega}{Q}\,,\quad x'=\frac{\varepsilon x}{a}\,,\quad y'=\frac{y}{a}\,,\quad \Psi'=\frac{\Psi}{Q}, \tag{9}$$

where Q is the constant flux across any cross-section of the channel. On the assumption that viscous forces are dominant or of the order of magnitude of the inertial forces, the dimensionless governing equations together with the appropriate boundary conditions, (neglecting primes for clarity) can be written as

$$\frac{\partial^2\omega}{y^2}=\mathrm{Re}\frac{\partial(\omega,\Psi)}{\partial(x,y)}\,, \tag{10}$$

$$\omega=-\frac{\partial^2\Psi}{y^2}\,, \tag{11}$$

$$\Psi=0\,,\quad \frac{\partial\Psi}{\partial y}=0\,,\qquad \text{on } y=0, \tag{12}$$

$$\frac{\partial\Psi}{\partial y}=0\,,\quad \Psi=1\,,\qquad \text{on } y=S, \tag{13}$$

where $\mathrm{Re}=\varepsilon Q/\nu$ is the effective flow Reynolds number and we emphasis that $\mathrm{Re}\geq O(1)$ as $\varepsilon\to 0$.

3. Method of Solution

Since the nonlinear character of the differential equation (10) precludes its solution exactly, we seek the solution as an asymptotic power series in terms of the perturbation parameter Re i.e.

$$\Psi=\sum_{i=0}^{\infty}\mathrm{Re}^i\Psi_i\,,\quad \omega=\sum_{i=0}^{\infty}\mathrm{Re}^i\omega_i\,. \tag{14}$$

Substituting the above expressions (14) into (10)–(13) and collecting the coefficients of like powers of Re, we obtained and solved the equations governing Ψ and ω up to order forty-four along with their corresponding boundary conditions. We have written a MAPLE program that calculates successively the coefficients of the solution series. In outline, it consists of the following segments:

(1) Declare arrays for the solution series coefficients e.g. $\Psi=$ `array(0..43)`, $\omega=$ `array(0..43)`, etc.

(2) Input the leading order term and their derivatives i.e. ω_0, Ψ_0, etc.

(3) Input the slope i.e. $\mathrm{d}S/\mathrm{d}x=m$ (where $S=1+mx$)

(4) Using a MAPLE loop procedure, solve equations (10)–(13) for the higher order terms i.e. $\omega_i, \Psi_i, i = 1, 2, 3, \ldots$, etc.

(5) Compute the wall shear stress and axial pressure gradient coefficients.

Details of the MAPLE program can be found in the appendix. Some of the solution for stream-function and vorticity are then given as follows;

$$\Psi = -\left[2\eta^3 - 3\eta^2\right] - \frac{3\,\mathrm{Re}\,S_x}{35}\left[2\eta^7 - 7\eta^6 + 7\eta^5 - 3\eta^3 + \eta^2\right] + \ldots, \tag{15}$$

$$\omega = \frac{6}{S^2}(2\eta - 1) + \frac{6\,\mathrm{Re}\,S_x}{35S^2}\left[42\eta^5 - 105\eta^4 + 70\eta^3 - 9\eta + 1\right] + \ldots, \tag{16}$$

where S_x represents the derivative of S with respect to x and $\eta = y/S$. The shear stress at the wall of the channel is given by

$$t_{\mathrm{w}} = -\frac{1}{(1 + b_x^2)}\left[(\sigma_{yy} - \sigma_{xx})b_x + (1 - b_x^2)\sigma_{xy}\right] \quad \text{on } y = b(x), \tag{17}$$

where $\sigma_{yy}, \sigma_{xx}, \sigma_{xy}$ are the usual stress components, i.e.,

$$\sigma_{xy} = \mu[\Psi_{yy} - \Psi_{xx}]\sigma_{yy} - \sigma_{xx} = -4\mu\Psi_{xy}, \tag{18}$$

and μ is the dynamic coefficient of viscosity. The subscripts (x, y) denote partial differentiation with respect to (x, y), respectively. The dimensionless form of wall shear stress can be written as

$$\beta = \frac{a^2S^2}{\mu Q}t_{\mathrm{w}} = -\frac{S^2}{(1 + \varepsilon^2S_x^2)}\left[(\Psi_{yy} - \varepsilon^2\Psi_{xx})(1 - \varepsilon^2S_x^2) - 4\varepsilon^2S_x\Psi_{xy}\right], \tag{19}$$

and we obtain

$$\beta = 6 - \frac{6S_x\,\mathrm{Re}}{35} + \ldots \quad \text{on } y = S. \tag{20}$$

From axial component of the Navier–Stokes equation, we can determine the fluid pressure distribution. The solution for the pressure field is taken as

$$P = \frac{\mu Q}{a^2}q, \tag{21a}$$

where q is given as i.e.,

$$q = \frac{1}{\varepsilon}P_0 + P_1 + \varepsilon P_2 + O(\varepsilon^2) + \ldots. \tag{21b}$$

The fluid pressure gradient in the longitudinal direction is given as

$$\frac{\partial q}{\partial x} = \frac{1}{\varepsilon}\Delta P = \frac{1}{\varepsilon S^3}\left\{-12 + \frac{54S_x}{35}\mathrm{Re} + \ldots\right\}, \tag{21a}$$

and solution for the pressure field is

$$q = \frac{1}{\varepsilon}\int_x \Delta P\,\mathrm{d}x. \tag{21b}$$

4. Computer Extension and Bifurcation Study

Here, we examine the flow of a viscous incompressible fluid driven steadily through a slowly varying linearly diverging asymmetrical channel defines by $S = 1 + mx$, where $0 < m \ll 1$ is the geometry slope. The state variables β and H (i.e. the wall shear stress and the axial pressure gradient respectively) are expanded in powers of Rd (Rd = Rem) to obtain

$$\beta = 6 - \frac{6}{35}\mathrm{Rd} - \frac{79}{26950}\mathrm{Rd}^2 - \frac{79}{943250}\mathrm{Rd}^3 - \frac{476031}{174271097000}\mathrm{Rd}^4 - \dots , \tag{21}$$

$$H = S^3\varepsilon\frac{\partial q}{\partial x} = -12 + \frac{54}{35}\mathrm{Rd} + \frac{156}{13475}\mathrm{Rd}^2 + \frac{933}{2452450}\mathrm{Rd}^3 + \frac{5920197}{435677742500}\mathrm{Rd}^4 + \dots ,$$

$$\text{as } \mathrm{Rd} \to 0. \tag{22}$$

The first 44 coefficients of the above series were obtained. A graphical ratio test together with Neville's extrapolation techniques at $1/n = 0$ (i.e., $n \to \infty$) reveal the radius of convergence Rd = 21.8324, (Domb and Sykes, [6]). Using D–T method, we compute a turning point $\mathrm{Rd_c}$ together with β_{-1} and H_{-1} on the secondary solution branch as Rd → 0, since $\beta \sim \beta_{-1}/\mathrm{Rd}$ and $H \sim H_{-1}/\mathrm{Rd}$. From Table 1 below, our results show that $\mathrm{Rd_c} \approx 21.832434744444767$, $\beta_{-1} \approx -1139.586$ and $H_{-1} \approx 10311.612$. We also noticed that $\beta \to 0$ as $\mathrm{Rd} \to 18.849555921538759430$, i.e., separation and reversal of the flow at the wall occur.

Table 1. Computations for bifurcation diagram using the D–T method.

d	N	$\mathrm{Rd_c}$	$\beta_{-1}^{(d)}$	$H_{-1}^{(d)}$
2	4	19.973272956800874	-873.9049438738	—
3	8	20.892553028665588	-1131.464294610	—
4	13	21.834727876353050	-1118.157759518	8849.093
5	19	21.832414257048973	-1139.536465765	10252.652
6	26	21.832434728390886	-1139.602488871	10310.922
7	34	21.832434744444767	-1139.586311605	10311.603
8	43	21.832434744444767	-1139.586282243	10311.612

5. Graphical Results and Discussion

From the equations (21)–(22) and the numerical results in Table 1, one can easily sketch the bifurcation diagrams as shown in figure 2. The bifurcation diagrams show how the flows evolve and bifurcate, as the flow Reynolds number increases. From these diagrams, we noticed three important solution regions, namely; (i) Rd < 0, (ii) $0 \le \mathrm{Rd} < \mathrm{Rd_c}$, (iii) $\mathrm{Rd} > \mathrm{Rd_c}$. In region (i), the solution is single valued, this corresponds to the case of negative flux and it represents the flow in a slowly varying linearly converging asymmetrical channel. The wall shear stress and axial pressure gradient increase with an increase in flow Reynolds number in this region. In region (ii), dual solution exists, namely; I (i.e. primary branch) and II (i.e. secondary branch). The two solution branches (I and II) show an increase or decrease in wall shear stress and axial pressure gradient as the flow Reynolds number increases. In the secondary branch of the solution, we estimated the asymptotic behaviour of wall shear stress parameter (β) and axial pressure gradient (H) as Rd → 0. A turning point exists at $\mathrm{Rd_c}$ ($\mathrm{Rd_c} = m\mathrm{Re_c}$). We also noticed that $\beta \to 0$ as Rd → 18.849, i.e., separation and reversal of the flow at the wall occur. Fraenkel [9] used the Jacobian elliptic functions to classify the solutions of J–H flow (i.e. the symmetrical case of the present problem), as types I, II_n, III_n, IV_n, or V_n, for $n = 1, 2, \dots$ He showed that plane Poiseuille flow arises for solutions of types I and III_1 as $\alpha \to 0$, (α is the semi-angle between the plane walls), for a fixed Re and obtained the turning points, one of which is $\mathrm{Re_c} = 5.461/\alpha$. Here, we observe that our result at the turning point is four

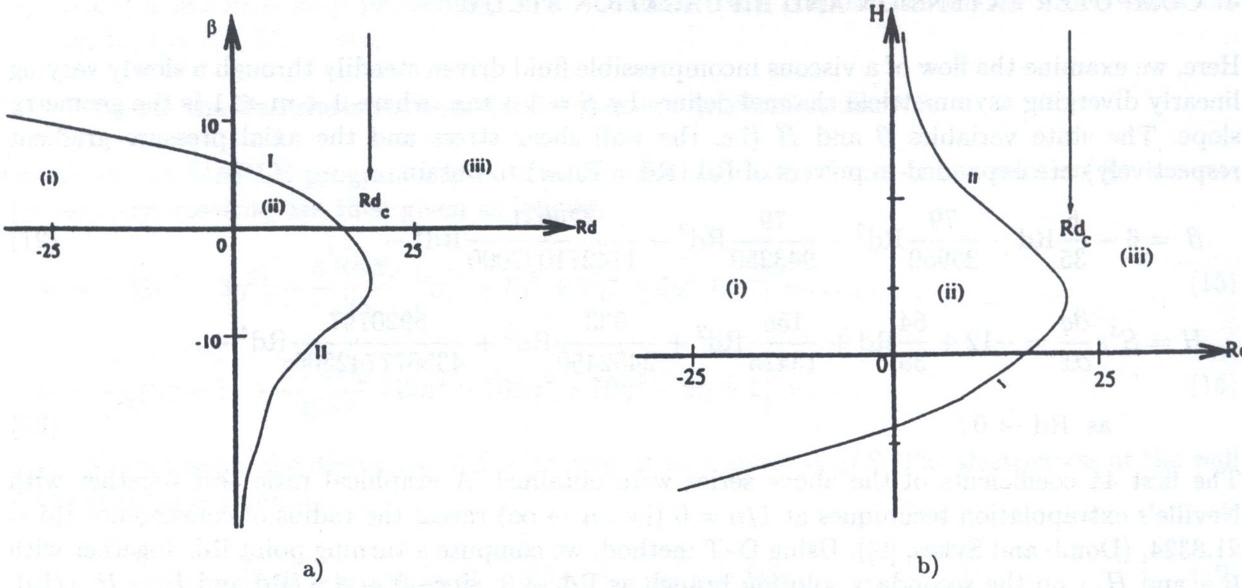

Fig. 2. A sketch of bifurcation diagram for the problem; (a) wall shear stress, and (b) axial pressure gradient with respect to Rd.

times the result obtained by Fraenkel [9] at the turning point (i.e. $4 \cdot 5.461 = 21.84$). This implies that the results obtained for flow in asymmetrical channel can be easily transformed to that in a symmetrical channel. In region (iii) no real solution of the assumed form is found. Figure 3 shows the Domb–Sykes plot for the coefficients of (β). The radius of convergence that corresponds to the nearest singularity lies on the positive real axis of Rd and is given as Rd = 21.8324.

In general, we emphasize that the D–T method is, in essence, a numerical form of analytic continuation and, as such, can only be expected to reveal those branches that are analytic continuations of the original power series. Furthermore, it is noteworthy that the applicability of the D–T method as well as other series summation and improvement techniques depend on the availability

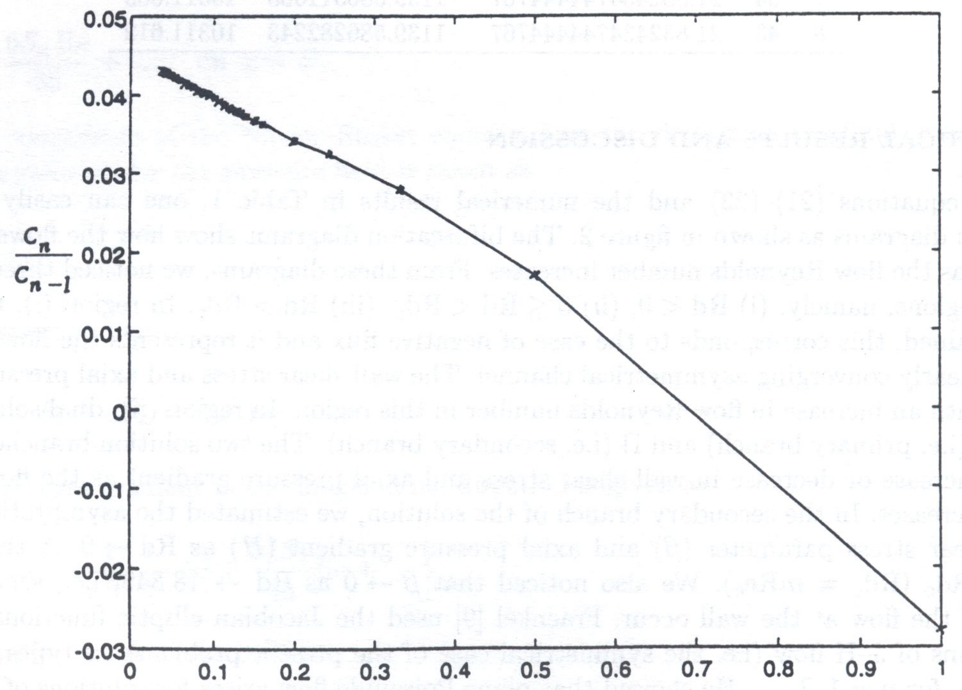

Fig. 3. Domb–Sykes plot for coefficients of (Radius of convergence = 21.8324).

of partial sums of the solution series for the problem under consideration. In many situations, it is possible to obtain the Taylor coefficients exactly (for instance by perturbation methods). Reliance upon the exactness of the Taylor coefficients may, however, limit the usefulness of the procedure to a rather small portion of the global bifurcation diagram. Hence, one may compute the approximate Taylor coefficients using the standard numerical part-following techniques (Lyness, [14]), thus greatly enhancing the scope and range of applicability of the D–T method. The D–T method has a wide range of application and may fail to reveal the bifurcation point for some problems (for instance when the pattern of signs of the coefficients of solution series is irregular or cannot be easily established), but whenever it works, the error decays faster than exponentially with the number of terms of the series used as illustrated by our example. Therefore, it is not a panacea for all applied mathematics problems. Moreover, the D–T method utilised in this paper is obviously amenable to various types of generalizations. For instance, it is quite straightforward to devise extensions to cater for problems with more than one bifurcation parameter or with more than one scalar state variable. Such generalizations may, however, be very demanding of computing time and memory.

ACKNOWLEDGEMENTS

The author wishes to thank Prof. P. G. Drazin, School of Mathematics, University of Bristol, United Kingdom, for his encouragement during the preparation of this paper, the referees for their useful suggestions and the Commonwealth Scholarship Commission in the United Kingdom for the financial support.

APPENDIX

A1: The table showing the coefficients of wall shear stress (i.e. β)

i	$C[i]$	i	$C[i]$
0	-6.0000000000000000000	24	0.395401587489238920613
1	0.171428571428571428571	25	0.170704549876543797104
2	0.002931354359925788497	26	0.738628809286882392960
3	0.000837529817121653856	27	0.320265863097518068460
4	0.273155450441675936658	28	0.139134440782352661493
5	0.950535206939780716154	29	0.605537526578935814141
6	0.344924460276584897823	30	0.263984466116357809944
7	0.129053975416975692812	31	0.115265669219927882979
8	0.494400631643117337441	32	0.504038364033901083137
9	0.192997763894894649590	33	0.220714437618860320244
10	0.765007209450438534208	34	0.967756955575966099962
11	0.307095974103714306801	35	0.424852448547783853064
12	0.124594950761946000334	36	0.186731128659573413171
13	0.510104236879815536088	37	0.821627498659435430599
14	0.210477146970403835081	38	0.361899391047135941476
15	0.874377825418023712963	39	0.159563032630296609025
16	0.365411442884592643568	40	0.704185165367339402582
17	0.153516979843578649754	41	0.311051134813595496108
18	0.647996773394044893598	42	0.137514536360579978899
19	0.274676048523360630675	43	0.608443386752716127668
20	0.116874722018139144606		
21	0.499019773243511912939		
22	0.213736396475714584380		
23	0.918091853313549007244		

A2: The Maple procedure to solve the system of equations (10)–(13) and the values of the coeff cients of wall shear stress.

```
#       Declaration of arrays for the solution series coefficients
        Digits:=50:
        ψ:=array(0..44): ω:=array(0..44): ψy:=array(0..44):
        ψx:=srray(0..44): ωy:=array(0..44): ωx:=array(0..44):
#       Input the leading order term and their derivatives
        ψ[0]:=3*(y/S)^2-2*(y/S)^3:
        ω[0]:=12*y/S^3-6/S^2:
        ψy[0]:=diff(ψ[0],y):
        ψx[0]:=Sx*diff(ψ[0],S):
        ωy[0]:=diff(ω[0],y):
        ωx[0]:=Sx*diff(ω[0],S):
#       Input the slope
        Sx:=m:
#       Solving equations (10)-(13) for the higher order terms
        for n from 1 by 1 to 44 do
        A:=R*sum(yry[i]*ωx[n-i-I]-ψrx[i]*ωy[n-i-1],i=0..n-1):
        g11:=int(A,y)+K1:
        g1:=int(g11,y)+K2:
        fl1:=int(-g1,y):
        fl:=int(fl1,y):
        y:=S:
        K1:=solve(fl1=0,K1):
        K2:=solve(fl=0,K2):
        y:='y':
        ψ[n]:=normal(fl):
        ω[n]:=normal(gl ):
        ψy[n]:=diff(ψ[n],y):
        ψx[n]:=Sx*diff(ψ[n],S):
        ωy[n]:=diff(ω[n],y):
        ωx[n]:=Sx*diff(ω[n],S):
        K1:='K1': K2:='K2'
#       Computing the wall shear stress coefficient
        print(evalf(sub(y=S,ω[n]*S^2)));
        od:
        quit();
```

References

[1] G.A. Baker Jr. *Essentials of Padé Approximants.* Academic Press, New York, 1975.

[2] G.K. Batchelor. *An Introduction to Fluid Dynamics.* Cambridge University Press, 1967.

[3] H. Blasius. Laminare strdmung in kanalen wechselnder. *Breite, Z. Math. Phys.*, **58**: 225–233, 1910.

[4] M.S. Borgas, T.J. Pedley. Non-uniqueness and bifurcation in annular antl planar channel flows. *J. Fluid Mech.*, **214**: 229–250, 1990.

[5] A.K. Common. Applications of Hermite–Padé approximants to water waves and the harmonic oscillator on a lattice. *J. Phys.*, **A 15**: 3665–3677, 1982.

[6] C. Domb, M.F. Sykes. On the susceptibility of a ferromagnetic above the curie point. *Proc. R. Soc. London, Ser. A*, **240**: 214–228, 1957.

[7] P.G. Drazin, Y. Tourigny. Numerical study of bifurcation by analytic continuation of a function defined by a power series. *SIAM Journal of Appl. Math.*, **56**: 1–18, 1996.

[8] P.G. Drazin. *Nonlinear Systems.* Cambridge University Press, 1992.

[9] L.E. Fraenkel. Laminar flow in symmetrical channels with slightly curved walls. I: On the Jeffery–Hamel solutions for flow between plane walls. *Proc. R. Soc. Lond.*, **A267**: 119–138, 1962.
[10] D.S. Gaunt, A.J. Gutmann. Phase transitions and critical phenomena. 3 ed. Domb C., and Green, M S., 1974.
[11] G. Hamel. Spiralförmige Bewgungen Zäher Flüssigkeiten. *Jahresbericht der Deutschen Math. Vereinigung*, **25**: 34–60, 1916.
[12] E. Hille. *Analytic Function Theory, Vol. I.* Chelsea, New York, 2nd ed., 1976.
[13] G.B. Jeffery. The two-dimensional steady motion of a viscous fluid. *Phil. Mag.*, **6**: 455–465, 1915.
[14] J.N. Lyness. Differentiation formulas for analytic functions. *Math. Comput.*, **21**: 352–362, 1967.
[15] E.H. Neville. Iterative interpolation. *J. Indian Math. Soc.*, **20**: 87–120, 1934.
[16] T.J. Pedley. *The Fluid Mechanics of Large Blood Vessels.* Cambridge University Press, Cambridge, 1980.
[17] R.E. Shafer. On quadratic approximation. *SIAM J. Numer. Anal.*, **11**: 447–460, 1974.
[18] I.J. Sobey, P.G. Drazin. Bifurcation of two-dimensional channel flows. *J. Fluid Mech.*, **171**: 263–287, 1986.
[19] M. Van Dyke. Analysis and improvement of perturbation series. *Quart. J. Mech. Appl. Math.*, **27**: 423–449, 1974.
[20] M.M. Vainberg, V.A. Trenogin. *Theory of branching of solutions of nonlinear equations.* Noordoff Leyden, 1974.