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Solving systems of polynomial equations using Gröbner basis calculations with applications to mechanics

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Solving systems of algebraic equations is presented using the *Gröbner Basis Package* of the computer algebra system MAPLE V. The Gröbner basis computations allow exact conclusions on the solutions of sets of polynomial equations, such as to decide if the given set is solvable, if the set has (at most) finitely many solutions, to determine the exact number of solutions in case there are finitely many, and their actual computation with arbitrary precision. The Gröbner basis computations are illustrated by two examples: computing the global equilibrium paths of a propped cantilever and of a simple arch.

1. Introduction

In mechanics there are problems which lead to solving systems of polynomial equations of several variables. In the computer algebra developed Gröbner basis theory is very helpful in solving such algebraic equations. Moreover, the most modern computer algebra systems (e.g. MAPLE or MATHEMATICA) include a *Gröbner Basis Package*, which engineers can very easily use to prepare their algebraic systems for solving.

Consider the problem of solving a system of multivariate polynomial equations

$$p_1(x_1,\dots,x_n)=0, \quad \dots, \quad p_k(x_1,\dots,x_n)=0. \tag{1}$$

If you add to this set of equations a further equation $p_{k+1}(x_1,\dots,x_n)=0$, where p_{k+1} is a linear combination of the polynomials $p_i(x_1,\dots,x_n)=0, i=1,\dots,k$ with polynomial coefficients $a_i(x_1,\dots,x_n)$, then you get an *equivalent* set of equations, i.e. one with all of the original common solutions. Similarly, you can omit one of the equations if it is a linear combination of the others.

Consequently, every calculation of solutions must take place in the set

$$\langle p_1\dots p_k\rangle=\left\{\sum_{i=1}^{k}a_ip_i : a_i\in K[x_1,\dots,x_n]\right\}, \tag{2}$$

where $K[x_1,\dots,x_n]$ denote the set (ring) of all polynomials in variables $x_1,\dots,x_n$ over a field K (e.g. the complex numbers).

The set $\langle p_1\dots p_k\rangle$ is called (polynomial) *ideal* generated by the (finite) set of polynomials. The set $\{p_1\dots p_k\}\subset K[x_1,\dots,x_n]$ is said to form a *basis* for this ideal.

For finding the solutions of the system of algebraic equations (1) you can use the method of *repeated elimination* [13]. The basic idea is to try transform the set of polynomials $\{p_1\dots p_k\}$ into an equivalent set from which the zeros could be more easily obtained.

Example 1. The nonlinear set of equations

$$\{p_1\equiv x_1^2+x_2^2-1=0, \quad p_2\equiv 3x_1x_2-1=0\} \tag{3}$$

may be transformed into the system

$$\{p_3 \equiv x_1 + 3x_2^3 - 3x_2 = 0, \quad 9x_2^4 - 9x_2^2 + 1 = 0\} \tag{4}$$

(to eliminate x_1 first compute $3x_2p_1 - x_1p_2$, then $3x_2p_3 - p_2$), which is then solved. In general the elimination may give not only the true solutions, but also some *parasitic solutions*, that is solutions of the transformed system which are not solutions of the given system. (This is a consequence of the fact, that the elimination is not free from omitting equations which are not linear combinations of the others.)

Example 2. Consider the simple set of equations (taken from [9]):

$$\{p_1 \equiv (y-1)x + y - 1 = 0, \quad p_2 \equiv (y+1)x = 0\}. \tag{5}$$

The solutions are the two points: $x = 0, y = 1$ and $x = -1, y = -1$. If you eliminate x from the second equation, you find the system

$$\{(y-1)x + y - 1 = 0, \quad y^2 - 1 = 0\}, \tag{6}$$

the solutions of which are the point $x = -1, y = -1$ and the line $y = 1$ with x undetermined. On the contrary, if you eliminate x from the first equation, you find the system

$$\{(y+1)x = 0, \quad y^2 - 1 = 0\}, \tag{7}$$

the solutions of which are the line $y = -1$ with x undetermined, and the point $x = 0, y = 1$. Thus, one sees that elimination may even give parasitic results, depending on the order of elimination. To make the elimination correct, you must check that all the solutions satisfy all the given equations. If the equations are not satisfied, it is still possible that a subset of the solution holds, such as the subset $x = -1, y = -1$ of the "solution" $y = -1$ and x undetermined, which you have found before.

In Example 1 the repeated elimination has been used to transform a set of polynomial equations into an equivalent set from which the zeros could be more easily obtained. Similarly the nonlinear system of equations

$$\{x^2 + y^2 + z^2 - 1 = 0, \quad 2x^2 + y^2 - 4z = 0, \quad 3x^2 - 4y + z^2 = 0\} \tag{8}$$

may be transformed into the equivalent system

$$\{3x^2 - 4y + z^2 = 0, \quad -8y + 8z^2 + 24z - 6 = 0,$$
$$192z^4 + 1152z^3 + 1824z^2 - 96z - 276 = 0\}, \tag{9}$$

which you can solve easier than the original one.

However, such a transformed system will not always exist; moreover one cannot always tell from a transformed system whether a given system of equations is solvable or not. On the other hand, it should be clear that a transformed system for $\{p_1, \dots p_k\}$ is simply an alternate (but more useful) basis for the ideal $\langle p_1, \dots p_k \rangle$.

What you would like, however, is an alternate ideal basis, which *always exists* and from which the existence and uniqueness of solutions (as well as the solutions themselves) may easily be determined. In fact, such bases exist and are called *Gröbner bases*. These bases were introduced by Buchberger [2]. Buchberger presented an algorithm to perform the required transformation in the context of polynomial ideals. Today, most modern computer algebra systems (e.g. MAPLE [8] or MATHEMATICA) include a Gröbner basis package based on the implementation of variants of Buchberger's algorithm.

2. TERM ORDERINGS AND REDUCTIONS OF POLYNOMIALS

The set of *terms* in variables $x_1, \ldots, x_n$ is defined by

$$T(x_1, \ldots, x_n) = \{x_1^{i_1}, \ldots, x_n^{i_n} : i_1, \ldots, i_n \in N\}, \tag{10}$$

where N is the set of non-negative integers. Note that this mapping assigns a (vector space) basis for $K[x_1, \ldots, x_n]$ over the field K (complex coefficients).

Definition 1 *An* admissible *term ordering* $<$ *for the set* $T(x_1, \ldots, x_n)$ *is one which satisfies the following two conditions:*

1. $1 \leq t$ *for all* $t \in T$,

2. $s < t$ *implies* $sv < tv$ *for all* $s, t, v \in T$, *where* $1 = x_1^0 \ldots x_n^0$.

Each of the following is an admissible term ordering on T.

Definition 2 *The (pure)* lexicographic *term ordering is defined by*

$$s = x_1^{i_1}, \ldots, x_n^{i_n} <_L x_1^{j_1}, \ldots, x_n^{j_n} = t \tag{11}$$

if and only if there exists l *such that* $i_l < j_l$ *and* $i_k = j_k$, $1 \leq k < l$.

Note that specifying the polynomial set as $K[x_1, \ldots, x_n]$, the precedence

$$x_1 >_L x_2 >_L \ldots >_L x_n \tag{12}$$

is implied.

Example 3. The trivariate terms in (x, y, z) are lexicographically ordered

$$1 <_L z <_L z^2 <_L \ldots <_L y <_L yz <_L yz^2 <_L \ldots$$

$$\ldots <_L y^2 <_L y^2z <_L \ldots <_L x <_L xz <_L \ldots <_L xy <_L \ldots \tag{13}$$

Definition 3 *The* (total) degree *term ordering is defined by*

$$s = x_1^{i_1}, \ldots, x_n^{i_n} <_D x_1^{j_1}, \ldots, x_n^{j_n} = t \tag{14}$$

if and only if

1. $\deg(s) \equiv i_1 + \ldots + i_n < j_1 + \ldots + j_n \equiv \deg(t)$, *or*

2. $\deg(s) = \deg(t)$ *and there exists* l *such that* $i_l > j_l$ *and* $i_k = j_k$, $l < k \leq n$.

Note that the terms of equal total degree are ordered using an *inverse lexicographic* ordering, which is admissible within these graduations. A different term ordering results from using the regular lexicographic ordering for this purpose. Both types are referred to as "total degree" orderings in the literature.

Example 4. The trivariate terms in (x, y, z) are degree ordered.

$$1 <_D z <_D y <_D x <_D$$

$$<_D z^2 <_D yz <_D xz <_D y^2 <_D xy <_D x^2 <_D$$

$$<_D z^3 <_D yz^2 <_D xz^2 <_D y^2z <_D xyz <_D \ldots \tag{15}$$

Suppose that every (non-zero) polynomial is written in decreasing ordering (according to $<$) of its monomials: $\sum_{i=1}^{s} c_i t_i$ with $c_i \neq 0$ and $t_i > t_{i+1}$ for every i. We call $c_1 t_1$ the *leading monomial* and t_1 the *leading term* of the polynomial.

Let G be a finite set of polynomials, and $>$ a fixed term ordering satisfying the two conditions in Definition 1.

Definition 4 *A polynomial p is* reduced *with respect to G if no leading term of an element of G divides the leading term of p.*

In other words, no combination $p - hg_i$ of p and an element of G can have a leading term less (for the ordering $<$) than the leading term of p. If p is not reduced, you can subtract from it a multiple of an element of G in order to eliminate its leading term (and to get a new leading term less than the leading term of p). This process is called a *reduction* of p with respect to G. Note that a polynomial can have several reductions with respect to G (one for each element of G the leading term of which divides the leading term of p). For example, let $G = \{g_1 = x - 1, g_2 = y - 2\}$ and $p = xy$. Then there are two possible reductions of p: by g_1, which gives $p - yg_1 = y$, and by g_2, which gives $p - xg_2 = 2x$.

A polynomial p cannot have an infinite chain of reductions: you have to terminate with a reduced polynomial.

The definition of "reduced" involves the leading term of p, and implies that there is no linear combination $p - hg_i$ which has a leading term less than that of p. It is possible that there are other terms of p which can be eliminated to make the linear combination "smaller". For example, suppose that the variables are x and y, with respect to the lexicographic term ordering $y < x$, and that $G = \{y - 1\}$. The polynomial $x + y^2 + y$ is reduced, for its leading term is x. Nevertheless, you can eliminate the terms y^2 and y with respect to G. This fact leads to the following definition, which is stronger than that of "reduced".

Definition 5 *A polynomial p is* completely reduced *with respect to G if no term of p is divisible by the leading term of an element of G.*

3. THE GRÖBNER BASES AND BUCHBERGER'S ALGORITHM

Definition 6 *A basis G of an ideal I is called a* Gröbner basis *(with respect to a fixed term ordering $<$) if every reduction of a p of I to a reduced polynomial (with respect to G) always gives zero.*

We now state two — and in the next section also two — theorems on Gröbner bases. We shall not prove them; the reader who is interested in the proofs is referred to the papers by Buchberger [2-7]. We have slightly reformulated the theorems in order to avoid the usage of some concepts from modern algebra.

Theorem 1 *Every ideal has a Gröbner basis with respect to any admissible term ordering.*

In the following we shall explain the Buchberger's algorithm for computation of Gröbner bases. Suppose you have chosen once and for all an admissible term ordering.

Definition 7 *Let p and g be two non-zero polynomials, with leading monomials p_l and g_l. Let h denote the least common multiple of p_l and g_l. The S-polynomial of p and g, $S(p, g)$, is defined by*

$$S(p,g) = \frac{h}{p_l}p - \frac{h}{g_l}g. \tag{16}$$

The least common multiple of two terms or monomials is simply the product of all the variables, each to a power which is the maximum of its powers in the two monomials. h/g_l and h/p_l are monomials, therefore $S(p, g)$ is a linear combination with polynomial coefficients p and g, and belongs to the ideal generated by p and g. Moreover, the leading monomials of the two components of $S(p, g)$ are equal to h, and therefore cancel each other. Note, that $S(p,p) = 0$ and that $S(p,g) = -S(g,p)$.

Theorem 2 *A basis G is a Gröbner basis, if and only if for every pair of polynomials p and g of G, $S(p, g)$ reduces to zero with respect to G.*

This theorem gives you a *criterion* for deciding whether a basis is Gröbner basis or not. It is enough to calculate all the S-polynomials and to check that they do reduce to zero. But if you do not have a Gröbner basis, it is precisely because one of these S-polynomials (say $S(p,g)$) does not reduce to zero. Then, as its reduction is a linear combination of the elements of G, you can add it ($g_k = S(p,g)$) to G without changing the ideal generated. After this addition, $S(p,g)$ reduces to zero (because $S(p,g) - g_k \equiv 0$), but there rise new S-polynomials to be considered. Buchberger also proved a remarkable fact, that *this process always comes to an end* (and therefore gives a Gröbner basis of the ideal).

Example 5. Apply the Buchberger's algorithm to the basis

$$G = \{g_1, g_2\} = \{x^2 + y^2 - 1,\ 3xy - 1\} \tag{17}$$

for the ideal from Example 1 and choose the pure lexicographic term ordering $x > y$. The leading monomials of g_1 and g_2 are x^2 and $3xy$, respectively, so that the least common multiple of g_1 and g_2 is $3x^2y$. Therefore

$$S(g_1, g_2) = 3yg_1 - xg_2 = x + 3y^3 - 3y. \tag{18}$$

This polynomial is non-zero and is reduced with respect to G (the leading monomial of $S(g_1,g_2)$, that is the x is divisible neither by x^2 nor by $3xy$), and hence G is not a Gröbner basis. The polynomial $S(g_1, g_2)$ is a linear combination of the elements of G, so you can add it to G without changing the ideal generated by G. Now G consists of

$$\begin{aligned} g_1 &= x^2 + y^2 - 1, \\ g_2 &= 3xy - 1, \\ g_3 &= x + 3y^3 - 3y. \end{aligned} \tag{19}$$

After this enlargement the polynomial $S(g_1, g_2)$ is not reduced with respect to $G = \{g_1, g_2, g_3\}$ (its leading monomial x is divisible by the leading monomial of g_3), but now you easily can reduce the $S(g_1, g_2)$ to zero: really, $S(g_1, g_2) - g_3 \equiv 0$. The new S-polynomials to be considered are: $S(g_1, g_3)$ and $S(g_2, g_3)$. The polynomial

$$S(g_1, g_3) = g_1 - xg_3 = -3xy^3 + 3xy + y^2 - 1 \tag{20}$$

can be reduced to zero as follows:

$$S(g_1, g_3) \to S(g_1, g_3) - (-y^2)g_2 = 3xy - 1 \to 3xy - 1 - g_2 = 0. \tag{21}$$

The polynomial

$$S(g_2, g_3) = g_2 - 3yg_3 = -9y^4 + 9y^2 - 1 \tag{22}$$

is non-zero and reduced with respect to G; hence G is not a Gröbner basis. If you add this polynomial to the polynomials in G then the new basis consists of

$$\begin{aligned} g_1 &= x^2 + y^2 - 1, \\ g_2 &= 3xy - 1, \\ g_3 &= x + 3y^3 - 3y, \\ g_4 &= -9y^4 + 9y^2 - 1. \end{aligned} \tag{23}$$

The S-polynomials to be considered are: $S(g_1, g_4)$, $S(g_2, g_4)$ and $S(g_3, g_4)$. The polynomial

$$S(g_1, g_4) = -9y^4 g_1 - x^2 g_4 = -9x^2y^2 + x^2 - 9y^6 + 9y^4 \tag{24}$$

is not reduced with respect to G (because you can divide its leading monomial $-9x^2y^2$ by the leading monomials of g_1, g_2 and g_3). The reduction:

$$S(g_1, g_4) \to S(g_1, g_4) - (-9y^2 + 1)g_1 = -9y^6 + 18y^4 - 10y^2 + 1$$
$$\to -9y^6 + 18y^4\ 10y^2 + 1 - (y^2 - 1)g_4 = 0. \tag{25}$$

The polynomial

$$S(g_2, g_4) = -3y^3g_2 - xg_4 = -9xy^2 + x + 3y^3 \tag{26}$$

and its reduction:

$$S(g_2, g_4) \to S(g_2, g_4) - (-3y)g_2 = x + 3y^3 - 3y$$
$$\to x + 3y^3 - 3y - g_3 = 0. \tag{27}$$

Similarly

$$S(g_3, g_4) = -9y^4g_3 - xg_4 = -9xy^2 + x - 27y^7 + 27y^5 \tag{28}$$

and its reduction:

$$S(g_3, g_4) \to S(g_3, g_4) - (-3y)g_2 = x - 27y^7 + 27y^5 - 3y$$
$$\to x - 27y^7 + 27y^5 - 3y - g_3 = -27y^7 + 27y^5 - 3y^3$$
$$\to -27y^7 + 27y^5 - 3y^3 - 3y^3g_4 = 0. \tag{29}$$

You see, that G reduces all these S-polynomials to zero, and hence G is a Gröbner basis for the ideal.

Gröbner bases are by no means unique. Fortunately, the problem of non-uniqueness is very easily avoided if using reduced Gröbner bases.

Definition 8 *A basis G is said to be a* reduced basis *if every polynomial $g \in G$ is completely reduced with respect to all the others (i.e. to $G - \{g\}$).*

In other words, no monomial of a polynomial $g \in G$ is divisible by the leading monomial of an other element of G.

The Gröbner basis $G = \{g_1, g_2, g_3, g_4\}$ is not reduced, because g_1 and g_2 contain monomials, (now it is of no importance that these are also leading monomials), which are divisible by the leading monomial of g_3. The reduction

$$g_1 - xg_3 = -3xy^3 + 3xy + y^2 - 1 = (1 - y^2)(3xy - 1) \tag{30}$$

implies that the generator $g_1 = x^2 + y^2 - 1$ is a linear combination of the generators g_2 and g_3 and hence you can omit g_1 and so suppress G.

Similarly from the reduction

$$g_2 - 3yg_3 = g_4 \tag{31}$$

you can see that the generator g_2 is a linear combination of the generators g_3 and g_4 thus g_2 also can be omited from G.

Since the monomials of g_3 and g_4 are not divisible by the leading monomials of g_4 and g_3, respectively, the set

$$G = \{x + 3y^3 - 3y,\ \ -9y^4 + 9y^2 - 1\} \tag{32}$$

is the *reduced Gröbner basis* for the ideal generated by the set $\{x^2 + y^2 - 1, 3xy - 1\}$.

Excellent references on Gröbner bases and variants of Buchberger's algorithm are [1, 9, 10].

4. Solution of systems of polynomial equations

Let G be a Gröbner basis for the ideal

$$\langle p_1, \dots, p_k \rangle \subset K[x_1, \dots, x_n]. \quad (33)$$

Theorem 3 *The system of algebraic equations (1) is* unsolvable *(has no solutions) if and only if G contains a constant (polynomial).*

Theorem 4 *Let L be the set of leading terms of polynomials in G. Then the system of equations (1) has (at most)* finitely many solutions *if and only if for every variable x_i, $i = 1, \dots, n$ there is a positive integer m such that $x_i^m \in L$.*

In other words, a system of polynomial equations has *finitely many solutions* over the complex numbers if and only if each variable x_i appears alone (such as x_i^m) in one of the leading terms of the corresponding Gröbner basis.

It must be noted that these powerful results do not depend on the term ordering chosen to construct the Gröbner basis. Neither do they require that the solutions themselves be produced.

If this basis is computed with respect to a *lexicographic ordering*, you can determine all the solutions by the following method. (Lexicographic Gröbner bases are not always the easiest to compute, but their use offers more direct insight into the solution of a system of algebraic equations than the total degree bases.)

Suppose, that the variables are $x_1, \dots, x_n$ with $x_1 > \dots > x_n$. The variable x_n appears alone in the leading term of a generator of the Gröbner basis. But all the other terms of this generator are less (in the sense of $<$) than this term, and therefore can contain only x_n, for you are using the lexicographic ordering. Thus you have a polynomial in x_n (and only one, because with two polynomials, you can always reduce one with respect to the other), which has only a finite number of roots. x_{n-1} appears by itself (to the power k, for example) in the leading term of a generator of the Gröbner basis. But the other terms of this generator are less (in the sense of $<$) than this term, and can therefore contain only x_{n-1} (to a power less than k) and, possibly, x_n, for you are using the lexicographic ordering. For every possible value of x_n , you have k possible values of x_{n-1}: the roots of this polynomial in x_{n-1} (with respect to their multiplicity). It is possible that there are other polynomials in x_{n-1} and x_n, and that certain combinations of values of x_{n-1} and x_n do not satisfy these polynomials, and must therefore be deleted, but you are certain of having only a finite number of possibilities for x_{n-1} and x_n. You determine $x_{n-2}, \dots, x_1$ in the same way.

In fact, there is a much simpler algorithm to find these finite solutions. This algorithm, suggested independently by Gianni [11] and Kalkbrener [12], will be illustrated on the following simple example.

Example 6. Consider the following Gröbner basis:

$$\{x^2 + y^2 - 1,\ xy - Fx,\ y^3 - Fy^2 - y + F\}, \quad (34)$$

where $x = x_1$, $y = x_2$ are variables (indeterminates) and F is a parameter. The Gianni–Kalkbrener algorithm tells you to take the unique polynomial in x_2 and to solve it: there are three solutions $y = F$ and $y = \pm 1$. For each of these solutions you must solve the other equations.

1. $y = F$: Here the other equations become $x^2 + F^2 - 1$ and 0, and you have two solutions for x: $x = \pm\sqrt{1 - F^2}$.

2. $y = \pm 1$: Here the other equations become x^2 and $(F \pm 1)x$ and now there is only one solution: $x = 0$.

The Gianni–Kalkbrener algorithm states that, for each variable x_k, it is both necessary and sufficient to consider the polynomial of the lowest degree in that variable, such that its leading coefficient does not vanish for the values of $x_{k+1}, \ldots, x_n$ being considered. In our example, when $y = F$ the leading coefficient of $(y - F)x$ vanishes, and you have to take $x^2 + y^2 - 1$, whereas when $y = \pm 1$, the leading coefficient does not vanish, and it is sufficient to take this polynomial, and to ignore $x^2 + y^2 - 1$.

Example 7. Stability of a propped cantilever [14]. Consider the propped cantilever of Fig. 1, comprising a rigid link of length $l = 1$ m, pinned to a rigid foundation and supported by a linear extensible spring of stiffness $c = 1$ N/m. The spring is assumed to be capable of resisting both tension and compression and retains its horizontal orientation as the system deflects. Assume that the system is perfect in the sense that the spring is unstrained when the link is vertical. What are the equilibrium paths of the structure loaded by a dead vertical force of magnitude F?

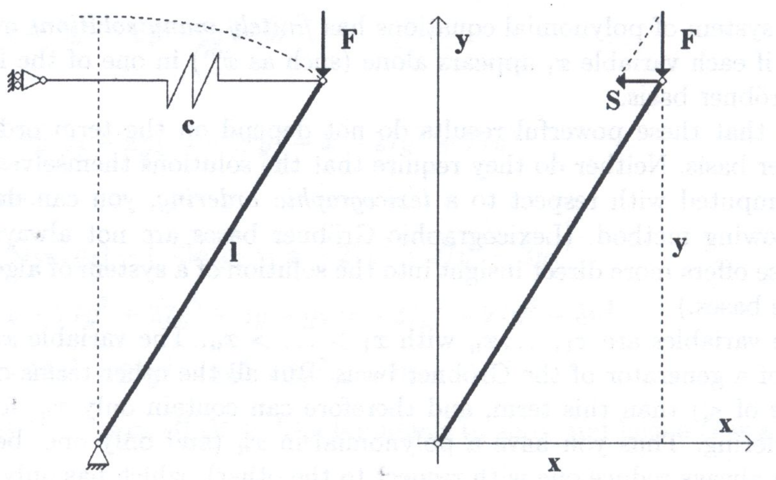

Fig. 1. Layout of the propped cantilever.

The equilibrium equation (moments around the origin) $Sy = Fx$, with spring force $S = cx$, and the geometrical equation $x^2 + y^2 = 1$ imply a set of algebraic equations, which corresponds to an ideal generated by the basis

$$\{xy - Fx,\ x^2 + y^2 - 1\}. \tag{35}$$

To compute the reduced Gröbner basis of this set of polynomials, you can apply e.g. the command

$$grobner[gbasis](\{x * y - F * x, x^2 + y^2 - 1\}, [x, y, F], plex); \tag{36}$$

in the *Gröbner Basis Package* of the symbolic system MAPLE V, where $[x, y, F]$ is the list of indeterminates (not including parameters) which induces the ordering $x > y > F$, and *plex* means that the pure lexicographic term ordering is used.

The resulting reduced Gröbner basis is

$$[x^2 + y^2 - 1,\ xy - Fx,\ y^3 - Fy^2 - y + F]. \tag{37}$$

First of all you see that the Gröbner basis does not contain a constant (polynomial) and hence (by Theorem 3) the corresponding set of algebraic equations is solvable.

If F is considered to be a parameter, (only x and y are indeterminates) then x^2 and y^3 are the leading terms of the Gröbner basis (with respect not only to *plex*, but to any admissible term ordering) which contain the variables x and y alone. Consequently (by Theorem 4) the corresponding system of polynomial equations has a finite number of solutions, i.e. for any prescribed load there are finitely many equilibrium positions. For solving, see Example 6.

If F is considered to be indeterminate too, then the variable F does not appear alone in one of the leading terms of the corresponding Gröbner basis and hence (by Theorem 4) there are infinitely many solutions.

To prepare the given system of algebraic equations for solving, it is more advantageous to use the following command in the *Gröbner Basis Package*:

$$grobner[gsolve](\{x*y-F*x, x^2+y^2-1\}, [x,y,F]). \tag{38}$$

The result is the following list of reduced subsystems whose roots are those of the original system, but whose variables have been successively eliminated and separated as far as possible:

$$[[x^2+y^2-1, y-F], [x, y-1], [x, y+1]]. \tag{39}$$

Fig. 2 illustrates the solutions (in the interval $-2 \le F \le 2$) and its bifurcations.

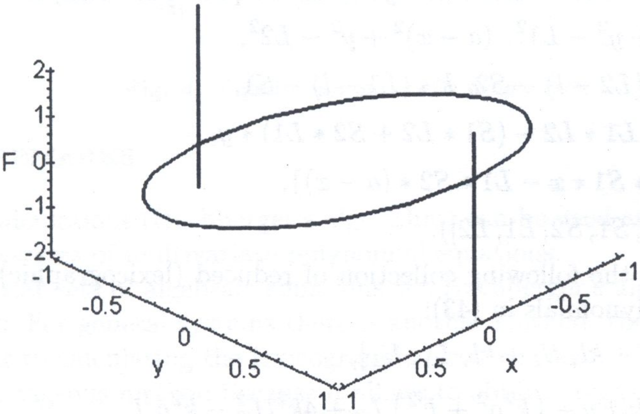

Fig. 2. Equilibrium paths of the propped cantilever.

Example 8. Model of a Simple Arch [14]. Consider the structural system shown in Fig. 3 consisting of two elastic bars with equal initial length $l = 5$ and equal stiffness $k = 1$, pinned to each other and to rigid supports by ideal hinges. The distance between the abutments is taken as

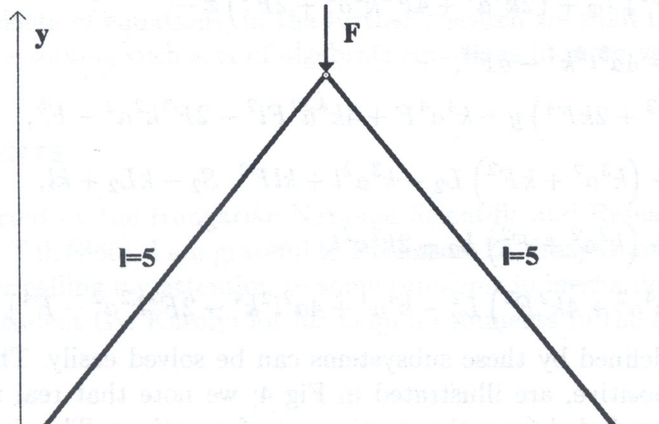

Fig. 3. Model of the simple arch.

$a = 6$. We want to find all the possible positions of the central hinge subjected to a vertical dead load of arbitrary magnitude F. Denote the horizontal position of the hinge by x and the vertical position by y.

Denoting the new lengths of the beams after loading by L_1 and L_2, the geometrical equations are

$$x^2 + y^2 = L_1^2, \quad (a-x)^2 + y^2 = L_2^2; \tag{40}$$

the material equations are

$$k(L_1 - l) = S_1, \quad k(L_2 - l) = S_2, \tag{41}$$

(where S_1 and S_2 are the inner forces of the bars, and Hooke's law is assumed to hold for arbitrarily large deflections); and the equilibrium equations are

$$FL_1L_2 = (S_1L_2 + S_2L_1)y, \quad L_2S_1x = L_1S_2(a-x). \tag{42}$$

To prepare this system of six algebraic equations of six unknowns for solving, you can use the command *gsolve* of the *Gröbner Basis Package* as in the previous example:

$$\begin{aligned} &grobner[gsolve]\,(\{x^2 + y^2 - L1^2,\ (a-x)^2 + y^2 - L2^2,\\ &\qquad k*(L2-l) - S2,\ k*(L1-l) - S1,\\ &\qquad F*L1*L2 - (S1*L2 + S2*L1)*y,\\ &\qquad L2*S1*x - L1*S2*(a-x)\},\\ &\qquad [x, y, S1, S2, L1, L2]). \end{aligned} \tag{43}$$

This command computes the following collection of reduced (lexicographic) Gröbner basis corresponding to the set of polynomials in (43):

$$\begin{aligned} &[\,[\,2x - a,\ 4y^2 + a^2,\ S_1 + kl,\ S_2 + kl,\ L_1,\ L_2\,],\\ &[\,2x - a,\ -4k^2L_2^3 + 2klFy + \left(k^2a^2 + F^2\right)L_2 + 4k^2lL_2^2 - k^2a^2l,\\ &\quad S_1 - kL_2 + kl,\ S_2 - kL_2 + kl,\ -L_2 + L_1,\\ &\quad -8k^2lL_2^3 + 4k^2L_2^4 + 2k^2a^2lL_2 + \left(-k^2a^2 + 4k^2l^2 - F^2\right)L_2^2 - k^2a^2l^2\,],\\ &[\,x - a,\ y,\ S_1 - kL_1 + kl,\ S_2 + kl,\ L_1^2 - a^2,\ L_2\,],\\ &[\,x,\ y,\ S_1 + kl,\ S_2 + kl - ka,\ L_1,\ L_2 - a\,],\\ &[\,x,\ y,\ S_1 + kl,\ S_2 + kl + ka,\ L_1,\ L_2 + a\,],\\ &[\,\left(4lk^4a^3 + 4ak^2lF^2\right)L_2 + \left(2k^4a^4 + 4F^2k^2a^2 + 2F^4\right)x -\\ &\quad k^4a^5 - 2F^2k^2a^3 - 4a^3l^2k^4 - aF^4,\\ &\quad \left(2k^5a^4 + 4k^3a^2F^2 + 2kF^4\right)y - k^4a^4F + 4k^4a^2Fl^2 - 2F^3k^2a^2 - F^5,\\ &\quad \left(k^2a^2 + F^2\right)S_1 + \left(k^3a^2 + kF^2\right)L_2 - k^3a^2l + klF^2,\ S_2 - kL_2 + kl,\\ &\quad \left(k^2a^2 + F^2\right)L_1 + \left(k^2a^2 + F^2\right)L_2 - 2k^2a^2l,\\ &\quad -8a^2L_2lk^4 + \left(4k^4a^2 + 4k^2F^2\right)L_2^2 - k^4a^4 + 4a^2l^2k^4 - 2F^2k^2a^2 - F^4\,]\,]. \end{aligned} \tag{44}$$

The set of equations defined by these subsystems can be solved easily. The results, for which L_1 and L_2 are real and positive, are illustrated in Fig 4; we note that real negative solutions arise because radicals were excluded from the starting set of equations. The simplest equilibrium path corresponds to the symmetric shape of the loaded simple arch, when the central hinge remains on the $x = 3$ line during the loading process. The other two curves correspond to more complicated cases: the curves with $x < 0$ or $6 < x$ correspond to other possible shapes of the loaded simple arch, when the originally central hinge moves sideways far away from its starting position.

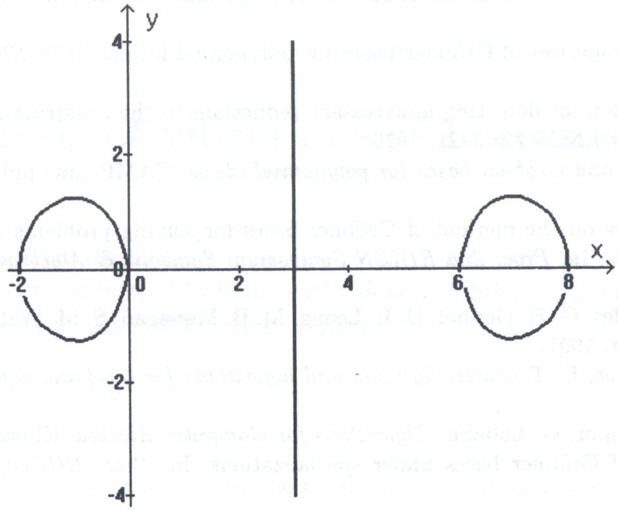

Fig. 4. Equilibrium paths of a simple arch.

5. Concluding Remarks

The Gröbner basis calculations (Buchberger's algorithm) can be used as a method for determining *all the solutions* of systems of multivariate polynomial equations.

For the case of linear sets of algebraic equations the Buchberger's algorithm corresponds to the Gaussian elimination. For general systems there is another method, the repeated elimination [13], which is quite similar to calculating the lexicographic Gröbner bases. However, Buchberger's algorithm is a more advantageous method because it allows to obtain *exact conclusions* on the existence and the number of solutions of sets of algebraic equations.

The fact that we have an algorithm does not mean that every problem can be solved easily. Although Buchberger has proved that his algorithm terminates, he has not given any limit for the calculating time or for the number of polynomials in the Gröbner basis.

The application of the Gröbner basis method to nonlinear mechanical systems has confirmed the known fact of computer algebra, that calculating a Gröbner basis requires, in general, memory space exponential in the number of variables. Some examples has been solved very easily by this method, but there are problems which use several megabytes of memory without reaching the solution.

Taking into account such special properties of the problems arising in mechanics as the symmetry and sparsity of coefficients of equations, in the further research we shall try to make the Gröbner basis method useful for solving such sets of algebraic equations in more variables.

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References

[1] T. Becker, V. Weispfenning. *Gröbner Bases. A Computational Approach to Commutative Algebra.* Springer-Verlag, 1993.

[2] B. Buchberger. Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems. *Aequationes Mathematicæ*, 4: 374–383, 1970.

[3] B. Buchberger. A theoretical basis for the reduction of polynomials to canonical form. *ACM SIGSAM Bulletin*, **10/3**: 19–29, 1976.

[4] B. Buchberger. Some properties of Gröbner bases for polynomial ideals. *ACM SIGSAM Bulletin*, **10/4**: 19–24, 1976.

[5] B. Buchberger. A criterion for detecting unnecessary reductions in the construction of Gröbner bases. In: *Proc. EUROSAM 79*, Springer LNCS **72**: 3–21, 1979.

[6] B. Buchberger. *H-bases and Gröbner bases for polynomial ideals.* CAMP-Linz publication 81-2.0, Univ. of Linz, 1981.

[7] B. Buchberger. A survey on the method of Gröbner bases for solving problems in connection with systems of multivariate polynomials. In: *Proc. 2nd RIKEN Symposium Symbolic & Algebraic Computation*, 69–83. World Scientific Publ., 1985.

[8] B. W. Char, K. O. Geddes, G. H. Gonnet, B. L. Leong, M. B. Monagan, S. M. Watt. *Maple V, Library Reference Manual.* Springer-Verlag, 1991.

[9] J. H. Davenport, Y. Siret, E. Tournier. *Systems and algorithms for algebraic computation. Computer Algebra.* Academic Press, 1993.

[10] K. O. Geddes, S. R. Czapor, G. Labahn. *Algorithms for Computer Algebra.* Kluwer Academic Publishers, 1992.

[11] P. Gianni. Properties of Gröbner bases under specializations. In: *Proc. EUROCAL 87*, Springer LNCS **378**: 293–297, 1989.

[12] M. Kalkbrener. Solving systems of algebraic equations by using Gröbner bases. In: *Proc. EUROCAL 87*, Springer LNCS **378**: 282–292, 1989.

[13] J. Moses. Solution of a System of Polynomial Equations by Elimination. *Comm. ACM*, **9**: 634–637, 1966.

[14] J. M. T. Thompson, G. W. Hunt. *A General Theory of Elastic Stability.* John Wiley & Sons, 1973.